SABA Publishing

**Journal of Mathematical Analysis and Modeling**
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2024)5(1) : 41-59
doi:10.48185/jmam.v4i3.970

# One-step three-parameter optimized hybrid block method for solving first order initial value problems of ODEs

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• *Received: 15 January 2024* • *Accepted: 26 January 2024* • *Published Online: 20 March 2024*

## Abstract

A three-parameter optimized hybrid block method and second derivative hybrid block method with optimized points were proposed for the solution of first-order ordinary differential equations. The techniques of interpolation and collocation were adopted for the derivation of the methods using a three-parameter approximation. The hybrid points were obtained by optimizing the local truncation error of the method. The schemes obtained were reformulated to reduce the number of occurrences of the source term. The hybrid points were used in the derivation of the second derivative hybrid block method. The discrete schemes were produced as a by-product of the continuous scheme and used to simultaneously solve initial value problems (IVPs) in block mode. The resulting schemes are self-starting, do not require the creation of individual predictors, consistent, zero-stable, and convergent. The accuracy and efficiency of the methods were ascertained using several numerical test problems. The numerical results were favorably compared to some techniques from the cited literature.

Keywords: Linear stability, Second derivative, Local truncation error, Parameter approximations, Initial Value Problems (IVPs).
*2010 MSC:* 65L05, 65L20.

## 1. Introduction

A system of differential equations is produced through the mathematical modelling of physical processes in the scientific and engineering fields, particularly in epidemiological systems with numerous interactions among various compartments. The analytical solutions to the majority of differential equations are typically difficult to find. This demanded the use of numerical techniques to provide an approximate solution. Numerous methodologies, including collocation, interpolation, integration, and interpolation polynomials,

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have been extensively examined in scholarly works to develop continuous linear multistep methods (LMMs) for directly solving initial value problems in ordinary differential equations. Trigonometric functions, the monomial, power series, Chebyshev polynomials, [1], [2], and [3] are a few examples of basis functions that have been used for this purpose.

Similarly, other types of differential equations such as fractional-order differential equation, Caputo fractional-order derivative, Fuzzy Volterra integral equation have been treated using diverse approach such as establishing the existence and stability of solution [4, 5, 6, 7, 8]; Laplace Adomain Decomposition method [9]; Haar Wavelets method [10]; and finite difference method [11, 12, 13, 14] among others.

Furthermore, a wide range of numerical integrators can be found in the scientific literature for solving differential systems. For instance, two clearly defined types of integrators that have been extensively utilized for this purpose are the Runge-Kutta (RK) and multistep approaches. The backward differentiation formulae (BDFs) are among the last ones, and they are employed to resolve numerically challenging issues. In addition to them, investigators have developed some fresh strategies like the hybrid and block methods, which have recently been applied to solve these issues. Milne initially introduced block approaches as a means of resolving some challenges associated with predictor-corrector methods. Subsequently, these methods were further developed to encompass versatile codes, as highlighted by [15]. Block techniques have been demonstrated to be more accurate and need fewer function evaluations, resulting in cheaper computational costs.

Nevertheless, the limitations inherent in block methods, including their low order of accuracy, error term, and subpar performance, prompted the emergence of hybrid approaches. Hybrid techniques were initially proposed as a solution to address the issue of zero-stability encountered in block methods [16]. In addition to the capacity to modify step size, these approaches possess another advantage, namely the utilization of data at intra-step moments, which serves to boost the accuracy of the methods. In their study, [17] developed a set of three-step hybrid block methods for solving first-order systems. These methods were generated using collocation and interpolation techniques, along with the use of power series as basis functions. The initial and subsequent three-step hybrid block approaches are formulated by include one off-grid points within the three-step integration period, respectively. In their study, [16] introduced a novel approach consisting of a three-step optimized block backward differentiation formula. This method was specifically designed to address the numerical solution of stiff first-order differential equations.

In the study conducted by [18], a two-step technique was devised, wherein two intermediate locations were selected based on the optimization of the LTEs. The method underwent reformulation as an R-K method, but, its execution necessitated a higher computing expenditure. Nevertheless, the most effective formulation was achieved by simply reformulating the approach in a manner that reduces the frequency of occurrences of the source term f. A comparison between the proposed economic reformulation and existing methods from the literature revealed that the former exhibited superior performance. In their study, [19] introduced an optimized one-step hybrid block technique designed specifically for the optimization of first-order initial value problems (IVPs). The approach involved the selection of three hybrid points in a judicious manner to maximize the LTEs of the primary equations for the block. The approach exhibited zero stability, demonstrating a fifth-order algebraic accuracy. The efficacy and precision of the approach were validated by numerical illustrations. In addition, a novel one-step implicit block method with three intra-step grid points was proposed by [20]. The primary objective of the LTE was to minimize the main term in order to achieve one of the three optimal intra-step points. A modification of the approach resulted in a notable reduction in computational expenses while preserving identical levels of consistency, zero-stability, A-stability, and convergence. The methodology was employed to address practical issues, and a comparative analysis was conducted with existing approaches in the literature to ascertain the superiority of the novel approach. Additional studies that have focused on the optimization of hybrid points through the minimization of the LTE include the works of [21, 22, 23, 24].

Moreover, multi-derivative multistep hybrid block approaches encompass the integration of higher order derivatives into the construction of approximate solutions, with the aim of enhancing the accuracy and stability characteristics of these methods. [25] presented multivalve multistep implicit methods that incorporate second derivative evaluations for solution of stiff systems. The approaches involve the introduction of intermediate off-step points that are strategically placed between the conventional step-points. Significant improvements in efficiency, accuracy, and stability were attained through these modifications. The authors [26] devised a method that utilizes second derivatives through an interpolation-collocation technique. This method is known for its A-stability. The continuous method obtained is utilized to construct the primary method and supplementary techniques for solving initial value issues of ordinary differential equations using the boundary value technique.

The novel class of hybrid block approaches employed in our study incorporates three off-step points and utilizes three-parameter approximations. By optimizing the LTE, one can achieve optimal hybrid points. The primary objective of this study is to propose a very effective approach for solving initial value problems that conform to the following form.

$$v' = f(t, v), v(t_0) = v_0 \tag{1.1}$$

where $t \in [t_0, T]$, $f : [t_0, T] \times \mathfrak{R} \to \mathfrak{R}$. To start with, we assume that equation (1.1) satisfies the conditions of the existence and uniqueness theorem for the initial value problems.

## 2. Derivation of the method

The exact solution $v(t)$ of equation (1.1) is approximated by the polynomial $p(t)$ of the form

$$p(t) = \sum_{j=0}^{d} w_j t^j, \tag{2.1}$$

where $w_j \in R$ are real unknown coefficients to be determined. $d = (C + I) - 1$, $I$ is the number of interpolation points and $C$ denotes the number of collocation points. The first derivative of (2.1) is obtained

$$p'(t) = \sum_{j=0}^{d} j w_j t^{j-1}, \tag{2.2}$$

while the second derivative of (2.2) is given as

$$p''(t) = \sum_{j=0}^{d} j(j-1)w_j t^{j-2}. \tag{2.3}$$

To obtain the coefficients of equation (2.1), interpolation is done at $t_{n+j}, j = 0$, collocations are done at $t_{n+j}, j = 0, u_1, \ldots u_2, \ldots u_3, 1$. Note that $u_1, \ldots u_2, \ldots u_3$ are the hybrid points within the interval $[0, 1]$ whose values are determined by the number of discretization/partition. This leads to system of equations with unknowns $w_j, j = 0, 1, \ldots, d-1$ given by

$$\begin{cases} v_{n+j} = p(t_{n+j}); j = 0, \\ \quad\vdots \\ f_{n+j} = p'(t_{n+j}); j = 0, u_1, \ldots u_2, \ldots u_3, 1. \end{cases} \tag{2.4}$$

which yield the one-step three-parameter optimized hybrid block method.

Similarly, the conditions

$$\begin{cases} v_{n+j} &= p(t_{n+j}); j = 0, \\ f_{n+j} &= p'(t_{n+j}); j = 0, u_1, u_2, u_3, 1, \\ g_{n+j} &= p''(t_{n+j}); j = 0, 1. \end{cases} \tag{2.5}$$

gives the second derivative hybrid block method.

*2.1. One-step optimal hybrid block method (OSOHBM)*

Equation (2.4) is expressed explicitly in a matrix representation as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 1 & 2t_{n+u_1} & 3t_{n+u_1}^2 & 4t_{n+u_1}^3 & 5t_{n+u_1}^4 \\ 0 & 1 & 2t_{n+u_2} & 3t_{n+u_2}^2 & 4t_{n+u_2}^3 & 5t_{n+u_2}^4 \\ 0 & 1 & 2t_{n+u_3} & 3t_{n+u_3}^2 & 4t_{n+u_3}^3 & 5t_{n+u_3}^4 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} v_n \\ f_n \\ f_{n+u_1} \\ f_{n+u_2} \\ f_{n+u_3} \\ f_{n+1} \end{pmatrix}, \tag{2.6}$$

Solving equation (2.6) by Gaussian Elimination method to obtain the coefficients $w_j$'s, $j = 0, 1, \ldots, 5$ and substituting back into equation (2.2) yield the implicit continuous one-step hybrid method of the form

$$v(t) = \alpha(t)v_n + h\sum_{j=0}^{1} \beta_j(t) f_{n+1} + h \sum_{j=u_1,u_2,u_3} \beta_j(t) f_{n+j}. \tag{2.7}$$

where $f_{n+j} = f(t_{n+j}, v_{n+j})$, for $j = u_1, u_2, u_3, 1$, and $v_{n+j} \approx v(t_n + jh)$ are approximations of the true solution. Evaluating equation (2.7) at the points $t = t_{n+u_1}, t_{n+u_2}$,

$t_{n+u_3}, t_{n+1}$, yield the following

$$\begin{aligned} v_{n+u_1} &= v_n + \frac{hu_1\left(-3u_1^3+30u_2u_3+5u_1^2(1+u_2+u_3)-10u_1(u_2+u_3+u_2u_3)\right)f_n}{60u_2u_3} \\ &+\frac{hu_1\left(12u_1^3-30u_2u_3-15r^2(1+u_2+u_3)+20u_1(u_2+u_3+u_2u_3)\right)f_{n+u_1}}{60(-1+u_1)(u_1-u_2)(u_1-u_3)} \\ &+\frac{hu_1^3\left(3u_1^2+10u_2u_3-5u_1(u_2+u_3)\right)f_{n+1}}{60(-1+u_1)(-1+u_2)(-1+u_3)} + \frac{hu_1^3\left(3u_1^2+10u_3-5u_1(1+u_3)\right)f_{n+u_2}}{60(u_1-u_2)(-1+u_2)u_2(u_2-u_3)} \\ &+\frac{hu_1^3\left(3u_1^2+10u_2-5u_1(1+u_2)\right)f_{n+u_3}}{60(u_1-u_3)(-1+u_3)u_3(-u_2+u_3)}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} v_{n+u_2} &= v_n + \frac{hu_2\left(5u_1\left(u_2^2+6u_3-2u_2(1+u_3)\right)+u_2\left(-3u_2^2-10u_3+5u_2(1+u_3)\right)\right)f_n}{60u_1u_3} \\ &+\frac{hu_2^3(u_2(3u_2-5u_3)-5u_1(u_2-2u_3))f_{n+1}}{60(-1+u_1)(-1+u_2)(-1+u_3)} - \frac{hu_2^3\left(3u_2^2+10u_3-5u_2(1+u_3)\right)f_{n+u_1}}{60(-1+u_1)u_1(u_1-u_2)(u_1-u_3)} \\ &+\frac{hu_2\left(5u_1\left(3u_2^2+6u_3-4u_2(1+u_3)\right)+s\left(-12u_2^2-20u_3+15u_2(1+u_3)\right)\right)f_{n+u_2}}{60(u_1-u_2)(-1+u_2)(u_2-u_3)} \\ &-\frac{hu_2^3(5u_1(-2+u_2)+(5-3u_2)u_2)f_{n+u_3}}{60(u_1-u_3)(-1+u_3)u_3(-u_2+u_3)}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} v_{n+u_3} &= v_n + \frac{hu_3(u_3(5u_2(-2+u_3)+(5-3u_3)u_3)+5u_1(-2u_2(-3+u_3)+(-2+u_3)u_3))f_n}{60u_1u_2} \\ &+\frac{hu_3^3\left(10u_1u_2-5u_1u_3-5u_2u_3+3u_3^2\right)f_{n+1}}{60(-1+u_1)(-1+u_2)(-1+u_3)} \\ &+\frac{hu_3^3(5u_2(-2+u_3)+(5-3u_3)u_3)f_{n+u_1}}{60(-1+u_1)u_1(u_1-u_2)(u_1-u_3)} - \frac{hu_3^3(5u_1(-2+u_3)+(5-3u_3)u_3)f_{n+u_2}}{60(u_1-u_2)(-1+u_2)u_2(u_2-u_3)} \\ &+\frac{hu_3(u_3(3(5-4u_3)u_3+5u_2(-4+3u_3))+5u_1(u_2(6-4u_3)+u_3(-4+3u_3)))f_{n+u_3}}{60(u_1-u_3)(-1+u_3)(-u_2+u_3)}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} v_{n+1} &= v_n + \frac{h(-3+u_2(5-10u_3)+5u_3+5u_1(1-2u_3+u_2(-2+6u_3)))f_n}{60u_1u_2u_3} \\ &+\frac{h(-12+15u_2+15u_3-20u_2u_3+5u_1(3-4u_3+u_2(-4+6u_3)))f_{n+1}}{60(-1+u_1)(-1+u_2)(-1+u_3)} \\ &-\frac{h(3-5u_3+5u_2(-1+2u_3))f_{n+u_1}}{60(-1+u_1)u_1(u_1-u_2)(u_1-u_3)} + \frac{h(3-5u_3+5u_1(-1+2u_3))f_{n+u_2}}{60(u_1-u_2)(-1+u_2)u_2(u_2-u3)} \\ &+\frac{h(3-5u_2+5u_1(-1+2u_2))f_{n+u_3}}{60(u_1-u_3)(-1+u_3)u_3(-u_2+u_3)}. \end{aligned} \tag{2.11}$$

The principal term of the LTE in $v_{n+1}$ is computed and used to calculate the appropriate values for unknown parameters.

$$\begin{aligned} L(v(t_{n+1});h) &= \frac{1}{7200}\left(-2+3u_1+3u_2-5u_1u_2+3u_3-5u_1u_3-5u_2u_3\right. \\ &\left.+10u_1u_2u_3\right)v^6[t_n]h^6 \\ &+\frac{1}{302400}\left(-24+21u_1+21u_1^2+21u_2-14u_1u_2-35u_1^2u_2+21u_2^2\right. \\ &\left.-35u_1u_2^2+21u_3\right)v^7[t_n]h^7 \\ &+\frac{1}{302400}\left(-14u_1u_3-35u_1^2u_3-14u_2u_3+70u_1^2u_2u_3-35u_2^2u_3\right. \\ &\left.+70u_1u_2^2u_3\right)v^7[t_n]h^7 \\ &+\frac{1}{302400}\left(21u_3^2-35u_1u_3^2-35u_2u_3^2+70u_1u_2u_3^2\right)v^7[t_n]h^7+O[h]^8. \end{aligned} \tag{2.12}$$

Setting the principal term of the LTE in (2.12) to zero yield the following one equation in three unknown parameters:

$$\frac{(-2+3u_1+3u_2-5u_1u_2+3u_3-5u_1u_3-5u_2u_3+10u_1u_2u_3)}{7200}=0. \tag{2.13}$$

There are an infinite number of solutions for $t_{u_1}, t_{u_2}, t_{u_3}$ since there are more unknowns than equations, $u_2$ is optimized when $u_1$ and $u_3$ are treated as free parameters yielding:

$$u_2 = \frac{2-3u_1-3u_3+5u_1u_3}{3-5u_1-5u_3+10u_1u_3}, \tag{2.14}$$

while the other two parameters are given as

$$u_1 = \frac{1}{10}\left(5-\sqrt{5}\right); u_3 = \frac{1}{10}\left(5+\sqrt{5}\right). \tag{2.15}$$

Substituting equation (2.15) into equation (2.14), we get

$$u_2 = \frac{1}{2}. \tag{2.16}$$

The LTE of the main formula in (2.11) are computed by substituting the values of the parameters $u_1, u_2, u_3$ into (2.12) to obtain

$$L(v(t_{n+1}); h) = -\frac{y^{(7)}[t_n]h^7}{1512000} + O[h]^8. \tag{2.17}$$

Lastly, putting the values of the parameters $u_1, u_2, u_3$ into equations (2.8)-(2.11) we get the following one-step three-parameter optimized hybrid block method:

$$\begin{aligned}
v_{n+u_1} &= v_n + \tfrac{h}{3000}\left((275+\sqrt{5})f_n + (625+95\sqrt{5})f_{n+u_1} - 192\sqrt{5}f_{n+u_2}\right.\\
&\quad \left.+(625-205\sqrt{5})f_{n+u_3} + (-25+\sqrt{5})f_{n+1}\right),\\
v_{n+u_2} &= v_n + \tfrac{h}{192}\left(17f_n + (40+15\sqrt{5})f_{n+u_1} + (40-15\sqrt{5})f_{n+u_3} - f_{n+1}\right),\\
v_{n+u_3} &= v_n + \tfrac{h}{3000}\left((275-\sqrt{5})f_n + (625+205\sqrt{5})f_{n+u_1} + 192\sqrt{5}f_{n+u_2}\right.\\
&\quad \left.+(625-95\sqrt{5})f_{n+u_3} - (25+\sqrt{5})f_{n+u_1}\right),\\
v_{n+1} &= v_n + \tfrac{h}{12}\left(f_n + 5f_{n+u_1} + 5f_{n+u_3} + f_{n+1}\right).
\end{aligned} \tag{2.18}$$

It is clear from the proposed block method's structure (2.18) that the source term $f$ appears four times in $v_{n+u_1}$, $v_{n+u_2}$, and three times in $v_{n+u_3}$, $v_{n+1}$ without taking $f_n$ into account because it is merely a numerical number. When $f$ is sophisticated, calculation costs increase. Nevertheless, the occurrence of $f$ in each equation of the aforementioned formulas can be mitigated by decreasing its frequency. To do this, we solve the set of equations (2.18) simultaneously for $f_{n+u_1}, f_{n+u_2}, f_{n+u_3}$, and $f_{n+1}$. This leads to an analogous version of (2.18) that eliminates all but one instance of $f$ in each equation. This turns out to be computationally efficient, especially when $f$ cannot be easily differentiated. The obtained reformulation is composed of the following elements:

$$\begin{aligned}
hf_{n+u_1} &= -\tfrac{1}{10}\left(2hf_n + (21+\sqrt{5})v_n + (-25+15\sqrt{5})(v_{n+u_1} + (32-32\sqrt{5})v_{n+u_2}\right.\\
&\quad \left.+(-25+15\sqrt{5})v_{n+u_3} + (-3+\sqrt{5})v_{n+1}\right),\\
hf_{n+u_2} &= \tfrac{1}{16}\left(2hf_n + 20v_n + (-25-25\sqrt{5})v_{n+u_1} + 32v_{n+u_2} + (-25+25\sqrt{5})v_{n+u_3}\right.\\
&\quad \left.-2v_{n+1}\right),\\
hf_{n+u_3} &= \tfrac{1}{10}\left(-2hf_n + (-21+\sqrt{5})v_n + (25+15\sqrt{5})v_{n+u_1} - (32+32\sqrt{5})v_{n+u_2}\right.\\
&\quad \left.+(25+15\sqrt{5})v_{n+u_3} - (3+\sqrt{5})v_{n+1}\right),\\
hf_{n+1} &= hf_n + 9v_n - 25v_{n+u_1} + 32v_{n+u_2} - 25v_{n+u_2} + 9v_{n+1}.
\end{aligned} \tag{2.19}$$

where the method (2.19) is termed the Modified One-step three-parameter optimized hybrid block method (MOSOHBM).

*2.2. Second derivative method with optimized hybrid point (SDMWOHP)*

When equation (2.5) is expressed explicitly, it results in a matrix representation of a system of linear equations given as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 1 & 2t_{n+u_1} & 3t_{n+u_1}^2 & 4t_{n+u_1}^3 & 5t_{n+u_1}^4 & 6t_{n+u_1}^5 & 7t_{n+u_1}^6 \\ 0 & 1 & 2t_{n+u_2} & 3t_{n+u_2}^2 & 4t_{n+u_2}^3 & 5t_{n+u_2}^4 & 6t_{n+u_2}^5 & 7t_{n+u_2}^6 \\ 0 & 1 & 2t_{n+u_3} & 3t_{n+u_3}^2 & 4t_{n+u_3}^3 & 5t_{n+u_3}^4 & 6t_{n+u_3}^5 & 7t_{n+u_3}^6 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 & 30t_{n+1}^4 & 42t_{n+1}^5 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{pmatrix} = \begin{pmatrix} v_n \\ f_n \\ f_{n+u_1} \\ f_{n+u_2} \\ f_{n+u_3} \\ f_{n+1} \\ g_n \\ g_{n+1} \end{pmatrix}, \tag{2.20}$$

Here, the parameters $u_1, u_2, u_3$ are given in equations (2.15) and (2.16). Solving the system (2.20) by Gaussian Elimination method, the coefficients $w_j$'s, $j = 0, 1, \ldots, 7$ are obtained and substituted into equation (2.2) to get the implicit continuous one-step hybrid method of the form yield

$$\begin{aligned} p(t) &= \alpha_0(t)v_n + h(\beta_0(t)f_n + \beta_{u_1}(t)f_{n+u_1} + \beta_{u_2}(t)f_{n+u_2} \\ &\quad + \beta_{u_3}(t)f_{n+u_3} + \beta_1(t)f_{n+1}) + h^2(\gamma_0 g_n + \gamma_1 g_{n+1}), \end{aligned} \tag{2.21}$$

Evaluating (2.21) at the points $t = t_{n+u_1}, t_{n+u_2}, t_{n+u_3}, t_{n+1}$, yield the following

$$\begin{aligned} v_{n+u_1} &= v_n + \tfrac{1}{105000}\Big(h[(14325+107\sqrt{5})f_n + (15625+1375\sqrt{5})f_{n+u_1} \\ &\quad +(8000-6464\sqrt{5})f_{n+u_2} + (15625-5625\sqrt{5})f_{n+u_3} + (-1075+107\sqrt{5})f_{n+1}] \\ &\quad +h^2[(600+8\sqrt{5})g_n + (100-8\sqrt{5})g_{n+1}]\Big), \\ v_{n+u_2} &= v_n + \tfrac{1}{13440}\Big(h[1723f_n + (2000+875\sqrt{5})f_{n+u_1} + 1024f_{n+u_2} \\ &\quad +(2000-875\sqrt{5})f_{n+u_3} - 27f_{n+1} + h^2[67g_n + 3g_{n+1}]\Big), \\ v_{n+u_3} &= v_n + \tfrac{1}{105000}\Big(h[(14325-107\sqrt{5})f_n + (15625+5625\sqrt{5})f_{n+u_1} \\ &\quad +(8000+6464\sqrt{5})f_{n+u_2} + (15625-1375\sqrt{5})f_{n+u_3} - (1075+107\sqrt{5})f_{n+1}] \\ &\quad +h^2[(600g_n - 8\sqrt{5})g_n + (100+8\sqrt{5})g_{n+1}]\Big), \\ v_{n+1} &= v_n + \tfrac{1}{420}\Big(h[53f_n + 125f_{n+u_1} + 64f_{n+u_2} + 125f_{n+u_3} + 53f_{n+1}] \\ &\quad +h^2[2g_n - 2g_{n+1}]\Big). \end{aligned} \tag{2.22}$$

## 3. Analysis of basic properties of the hybrid methods

In what follows, the basic theoretical properties of the proposed OSOHBM (2.18) (or equivalently MOSOHBM (2.19)) and the SDMWOHP (2.22) including accuracy, consistency, zero-stablity, convergence, linear stability and A-stability are investigated.

### 3.1. *Order of accuracy and consistency*

#### 3.1.1. *OSOHBM*

Rewriting the OSOHBM (2.18) in the matrix difference form yields

$$W_1 V_{n+1} = W_0 V_{n-1} + h(Z_0 F_{n-1} + Z_1 F_n), \tag{3.1}$$

where $W_0, W_1, Z_0$ and $Z_1$ are $4 \times 4$ matrices given by

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \tag{3.2}$$

$$Z_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{275+\sqrt{5}}{3000} \\ 0 & 0 & 0 & \frac{17}{192} \\ 0 & 0 & 0 & \frac{275-\sqrt{5}}{3000} \\ 0 & 0 & 0 & \frac{1}{12} \end{bmatrix}; Z_1 = \begin{bmatrix} \frac{625+95\sqrt{5}}{3000} & \frac{-192\sqrt{5}}{3000} & \frac{625-205\sqrt{5}}{3000} & \frac{-25+\sqrt{5}}{3000} \\ \frac{40+15\sqrt{5}}{192} & 0 & \frac{40-15\sqrt{5}}{192} & -\frac{1}{192} \\ \frac{625+205\sqrt{5}}{3000} & \frac{192\sqrt{5}}{3000} & \frac{625-205\sqrt{5}}{3000} & -\frac{(25+\sqrt{5})}{3000} \\ \frac{5}{12} & 0 & \frac{5}{12} & \frac{1}{12} \end{bmatrix} ; \tag{3.3}$$

$$\begin{aligned} V_n &= (v_{n+u_1}, v_{n+u_2}, v_{n+u_3}, v_{n+1})^T, \\ V_{n-1} &= (v_{n-1+u_1}, v_{n-1+u_2}, v_{n-1+u_3}, v_n)^T, \\ F_n &= (f_{n+u_1}, f_{n+u_2}, f_{n+u_3}, f_{n+1})^T, \\ F_{n-1} &= (f_{n-1+u_1}, f_{n-1+u_2}, f_{n-1+u_3}, f_n)^T. \end{aligned} \tag{3.4}$$

For a sufficiently differentiable test function $m(t_n)$ in the interval $[0, T]$, let the difference operator $\bar{D}$ for OSOHBM in (3.8) be given as

$$\bar{D}\left[m(t_n); h\right] = \sum_{j=0,u_1,u_2,u_3,1} \left[\bar{\xi}_j (t_n + jh) - h\bar{\mu}_j m'(t_n + jh)\right], \tag{3.5}$$

where $\bar{\xi}_j$ and $\bar{\mu}_j$ are the column vectors of the matrices $W_1$ and $W_0$, respectively. The Taylor's series expansion for $v(t_n + jh)$ and $v'(t_n + jh)$ about $t_n$ yield

$$\bar{D}\left[m(t_n); h\right] = c_0 v(t_n) + c_1 h v(t_n) + c_2 h^2 v^{(2)}(t_n) + \cdots + c_p h^p v^p(t_n) + \cdots \tag{3.6}$$

where $c_i, i = 0, 1, 2, \ldots$ are vectors. From equation (3.6), the order of the OSOHBM is $p = (5, 5, 5, 6)^T$ with error constant

$$c_{p+1} = \left(\frac{1}{180000}, \frac{1}{180000}, \frac{1}{230400}, -\frac{1}{1512000}\right)^T. \tag{3.7}$$

Hence, the OSOHBM (or equivalently the MOSOHBN) has at least fifth order accuracy.

### *3.1.2. SDMWOHP*

Rewriting the SDMWOHP (2.22) in the matrix difference form yields

$$W_1 V_{n+1} = W_0 V_{n-1} + h(Z_0 F_{n-1} + Z_1 F_n) + h^2(Q_0 G_{n-1} + Q_1 G_n), \tag{3.8}$$

where $W_0, W_1, Z_0$, $Z_1$ $G_0$ and $G_1$ are $4 \times 4$ matrices given by

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; Z_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{14326+107\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{1723}{13440} \\ 0 & 0 & 0 & \frac{14326-107\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{53}{420} \end{bmatrix} \tag{3.9}$$

$$Z_1 = \begin{bmatrix} \frac{15625+5625\sqrt{5}}{105000} & \frac{8000-6464\sqrt{5}}{105000} & \frac{15625+5625\sqrt{5}}{105000} & \frac{-1075+107\sqrt{5}}{105000} \\ \frac{2000+8757\sqrt{5}}{13440} & \frac{1024}{13440} & \frac{2000-8757\sqrt{5}}{13440} & \frac{-27}{13440} \\ \frac{-(15625+5625\sqrt{5})}{105000} & \frac{8000-6464\sqrt{5}}{105000} & \frac{15625-5625\sqrt{5}}{105000} & \frac{-(1075+107\sqrt{5})}{105000} \\ \frac{125}{420} & \frac{64}{420} & \frac{125}{420} & \frac{53}{420} \end{bmatrix}; \tag{3.10}$$

$$G_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{600+8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{67}{13440} \\ 0 & 0 & 0 & \frac{600-8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{1}{210} \end{bmatrix}; G_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{100-8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{3}{13440} \\ 0 & 0 & 0 & \frac{100+8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{-1}{210} \end{bmatrix}; \tag{3.11}$$

$$\begin{aligned} V_n &= (y_{n+u_1}, y_{n+u_2}, y_{n+u_3}, y_{n+1})^T, \\ V_{n-1} &= (y_{n-1+u_1}, y_{n-1+u_2}, y_{n-1+u_3}, y_n)^T, \\ F_n &= (f_{n+u_1}, f_{n+u_2}, f_{n+u_3}, f_{n+1})^T, \\ F_{n-1} &= (f_{n-1+u_1}, f_{n-1+u_2}, f_{n-1+u_3}, f_n)^T, \\ G_n &= (g_{n+u_1}, g_{n+u_2}, g_{n+u_3}, g_n)^T, \\ G_{n-1} &= (g_{n-1+u_1}, g_{n-1+u_2}, g_{n-1+u_3}, g_n)^T. \end{aligned} \tag{3.12}$$

The SDMWOHP is of order $p = (7, 7, 7, 8)^T$ with the error constant

$$c_{p+1} = \left(-\frac{1}{75600000}, -\frac{1}{154828800}, -\frac{1}{75600000}, -\frac{1}{1016064000}\right)^T. \tag{3.13}$$

Indicating that the SDMWOHP has at least seventh order accuracy.

### *3.2. Zero-stability and convergence*

The zero-stability of OSOHBM can be established from the difference equation (3.8) as $h \to 0$ given as

$$W_1 V_n - W_0 V_{n-1} = 0, \tag{3.14}$$

where $W_0$ and $W_1$ are given in equation (3.2). The first characteristic polynomial $\rho(\sigma) = \det(\sigma W_1 - W_0) = \sigma^3(\sigma - 1) = 0$. This implies that $\sigma_1 = \sigma_2 = \sigma_3 = 0, \sigma_4 = 1$. Therefore, both OSOHBM and SDMWOHP exhibit zero-stability.

Since both OSOHBM and SDMWOHP satisfy the properties of consistency and zero-stability, then the methods are convergent according to [27].

### 3.3. Linear stability

To obtain the stability region, consider a trial problem

$$v(t) = \sigma v(t), Re(\sigma) < 0. \tag{3.15}$$

Applying the OSOHBM on equation (3.15), we obtain the recurrence relation

$$V_n = H(\hbar)V_{n-1}, \hbar = \sigma h. \tag{3.16}$$

where the matrix $H(\hbar)$ is given by $(W_1 - rZ_1)^{-1}(W_0 + rZ_0)$. The stability property of this matrix's eigenvalues, which governs how the numerical solution behaves, is the spectral radius, $H(\hbar)$, which is used in the method to define the region of absolute stability $S$. The method is said to be A-stable if

$$S = \{\hbar \in \mathbb{C} : |\rho[H(\hbar)]| < 1\} \tag{3.17}$$

After various computations, the dominant eigenvalue can be expressed as a quotient function.

$$\rho[H(\hbar)] = \frac{1200 + 600\hbar + 132\hbar^2 + 16\hbar^3 + \hbar^4}{1200 - 600\hbar + 132\hbar^2 - 16\hbar^3 + \hbar^4}, \tag{3.18}$$

which has modulus less than one in $\mathbb{C}^-$ (see Figure 1). Hence, the OSOHBM (2.18) is A-stable.

Similarly, for SDMWOHP has the dominant eigenvalue expressed as a quotient function.

$$\rho[H(\hbar)] = \frac{h^5 + 27h^4 + 360h^3 + 2820h^2 + 12600h + 25200}{-h^5 + 27h^4 - 360h^3 + 2820h^2 - 12600h + 25200}, \tag{3.19}$$

which has modulus less than one in $\mathbb{C}^-$ (see Figure 2). Hence, the SDMWOHP (2.22) is A-stable.

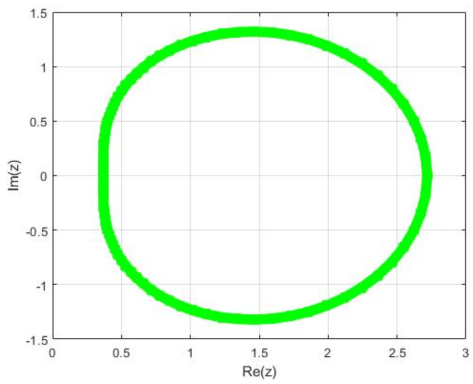


Figure 1: Region of absolute stability for OSOHBM

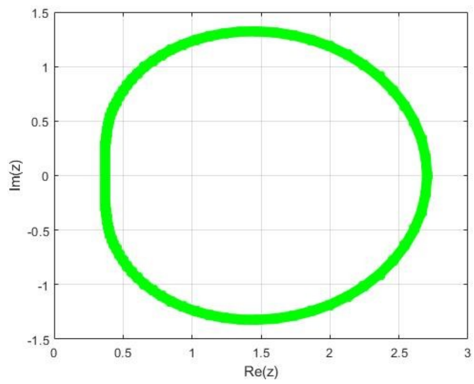


Figure 2: Region of absolute stability for SDMWOHP

Figure 1 shows the region of absolute stability of the OSOHBM. The figure indicates that the stability region contains the entire left half complex plane and thus, the method is A-stable.

Similarly, Figure 2 shows the region of absolute stability of the SDMWOHP. The figure indicates that the stability region contains the entire left half complex plane and thus, the method is A-stable.

## 4. Numerical Experiments and Results

In the sequel, the accuracy of the proposed methods will be demonstrated by implementation in solving some popular applied problems of the form (1.1) in literature. The methods being compared are the OSOHBM (2.14), the MOSOHBM (2.15), the SDMWOHP (2.22), the two-step block optimized hybrid method (BHMO) and reformulated two-step block optimized hybrid method (RBHMO) in [18].

Different types of errors are calculated in measuring the performance of each of the aforementioned methods. These include maximum global absolute error (AbErr), absolute error at the final grid point (FErr), mean absolute error (MErr), norm, and the CPU time measured in seconds. The problem used to test the efficiency of the schemes are the linear stiff IVP considered in [18], system of linear equations in [25] and system of equations in [18] which had also appeared in the work of [28] among others.

*Problem 4.1*

Given the linear stiff IVP

$$v'(t) = -1000v + +e^{-2t}, \quad v(0) = 0. \tag{4.1}$$

The exact solution is $v(t) = \frac{1}{998}(e^{-2t} - e^{-1000})$. This equation has been subject to several numerical investigations in the literature such as [18]. The problem is solved in the interval $[0, 1]$ taking $n = 3750, 7500, 15000$. The MErr, FErr, AbErr, and CPU time are computed using the methods OSOHBM, MOSOHBM, BHMO, and RBHMO, and the results are presented in Table 1. The efficiency curves of AbErr and CPU time are represented in Figure 3 .

*Problem 4.2*

Given the system of linear IVP

$$\begin{aligned} v_1'(t) &= v_1(t) + v_2(t), \quad v_1(0) = 0 \\ v_2'(t) &= -v_1(t) + v_2(t), \quad v_2(0) = 1. \end{aligned} \tag{4.2}$$

The exact solution: $v_1(t) = \exp t \sin t, v_2(t) = \exp t \cos t$. This problem was numerically investigated by [25]. The problem is solved for step sizes $n = 20, 40, 80$ with the MErr, FErr, AbErr, and CPU time computed using the methods OSOHBM, MOSOHBM, SDMWOHP, and RBHMO and results presented in Table 2. The efficiency curves of AbErr and CPU time are represented in Figure 4 . The figure reveals that the OSOHBM, MOSOHBM, and SDMWOHP outperform existing methods with respect to accuracy and computing time.

*Problem 4.3*

Given the linear system which has been investigated by [18] among others

$$\begin{aligned} v_1' &= -21v_1 + 19v_2 - 20v_3, \quad v(0) = 1. \\ v_2' &= 19v_1 - 21v_2 + 20v_3, \quad v(0) = 0. \\ v_3' &= 40v_1 - 40v_2 - 40v_3, \quad v(0) = -1. \end{aligned} \tag{4.3}$$

with exact solution

$$\begin{aligned} v_1(t) &= \tfrac{1}{2}(e^{-2t} + e^{-40t}(\cos(40t) + \sin(40t))). \\ v_2(t) &= \tfrac{1}{2}(e^{-2t} - e^{-40t}(\cos(40t) + \sin(40t))). \\ v_3(t) &= e^{-40t}(\cos(40t) + \sin(40t)). \end{aligned} \tag{4.4}$$

The problem is solved in the interval $[0, 1]$ taking $n = 50, 100, 200, 400$. The AbErr, and CPU time are computed using the methods OSOHBM, MOSOHBM, and RBHMO, and results are presented in Table 3. The efficiency curves of AbErr and CPU time are represented in Figure 5. As revealed by the figure, the OSOHBM and MOSOHBM outperform existing methods with respect to accuracy and computing time.

*Problem 4.4*

We consider the system of differential equations explored in [29] given by

$$\begin{aligned} \frac{dS_h}{dt} &= \Lambda_h - \frac{\alpha_1 S_h V_2}{N_h} - (\nu + \mu_h)S_h \\ \frac{dI_h}{dt} &= \frac{\alpha_1 S_h V_2}{N_h} - (\gamma_h + \mu_h + \delta_h)I_h \\ \frac{dR_h}{dt} &= \nu S_h + \gamma_h I_h - \mu_h R_h \\ \frac{dV_1}{dt} &= \Lambda_v - \frac{\alpha_2 V_1 I_h}{N_h} - \frac{\alpha_3 V_1 I_m}{N_m} - (\mu_v + \delta_v)V_1 \\ \frac{dV_2}{dt} &= \frac{\alpha_2 V_1 I_h}{N_h} - \frac{\alpha_3 V_1 I_m}{N_m} - (\mu_v + \delta_v)V_2 \\ \frac{dS_m}{dt} &= \Lambda_m - \frac{\alpha_4 S_m V_2}{N_m} - \mu_m S_m \\ \frac{dI_m}{dt} &= \frac{\alpha_4 S_m V_2}{N_m} - -(\mu_m + \delta_m)I_m \end{aligned} \tag{4.5}$$

which modelled the dynamics of yellow fever with the incorporation of a secondary host (see Table 6 in appendix for details of the parameters). This problem is numerically solved using the new OSOHBM and Mathematica's in-built NDSolve. The solution curves presented in Figures 6, 7, 8, 9, 10, and 11 show that the graphs of the two solvers coincide.

Figure 3 shows the efficiency curves of the absolute global errors AbErr (in logarithmic scale) versus CPU time. The figure indicates that the OSOHBM and MOSOHBM outperform existing methods (BHMO, RBHMO) with respect to accuracy and computing time. It is also observed that the modified scheme (MOSOHBM) performs a bit better than the initial formulation (OSOHBM).

Figure 4 shows the efficiency curves of the absolute global errors AbErr (in logarithmic scale) versus CPU time. The figure indicates that the OSOHBM and MOSOHBM outperform the RBHMO with respect to accuracy and computing time. It is also observed that the OSOHBM and MOSOHBM outperform the SDMWOHP with respect to CPU time. However, the SDMWOHP is more accurate than the OSOHBM, MOSOHBM and RBHMO.

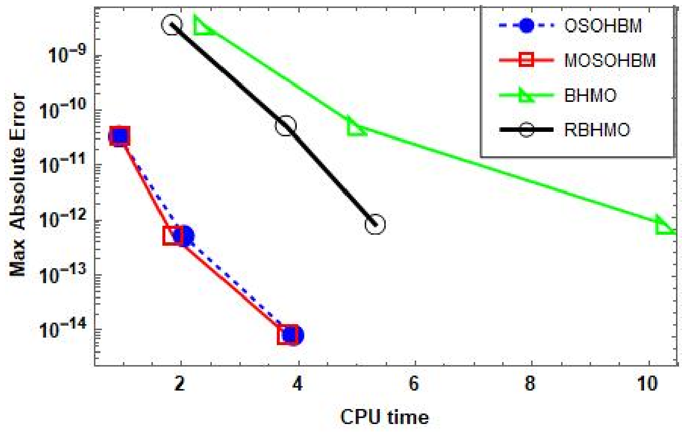


Figure 3: Efficiency plot for problem 4.1

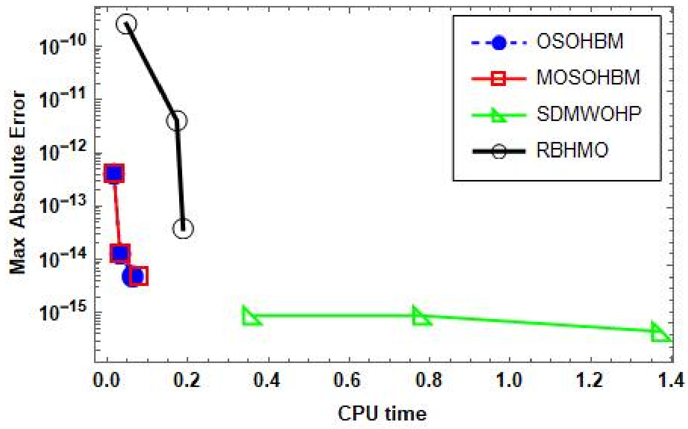


Figure 4: Efficiency plot for problem 4.2

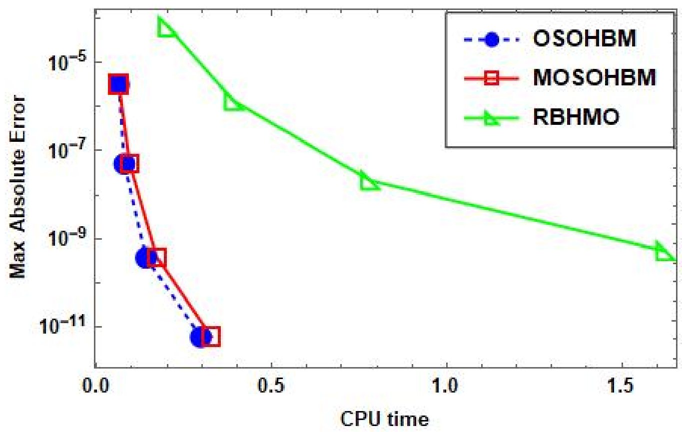


Figure 5: Efficiency plot for problem 4.3

Figure 5 shows the efficiency curves of the absolute global errors AbErr (in logarithmic

Table 1: Comparison of the methods under consideration on the basis of absolute errors and CPU time for Problem 1 with different number of steps (n)

| $n$ | Method | ME | LE | AE | Norm | CPU time |
|---|---|---|---|---|---|---|
| 3750 | OSOBHM | 9.04148E-14 | 2.71051E-20 | 3.40034E-11 | 6.32612E-11 | 9.219E-01 |
| | MOSOBHM | 9.04148E-14 | 5.42101E-20 | 3.40034E-11 | 6.32612E-11 | 9.375E-01 |
| | BHMO | 1.82222E-12 | 2.71051E-20 | 3.6235E-09 | 1.27497E-09 | 2.375E+00 |
| | RBHMO | 1.82222E-12 | 2.71051E-20 | 3.6235E-09 | 1.27497E-09 | 1.828E+00 |
| 7500 | OSOBHM | 1.42542E-15 | 5.42101E-20 | 5.26482E-13 | 1.3884E-12 | 2.031E+00 |
| | MOSOBHM | 1.42542E-15 | 0.0000E+00 | 5.26482E-13 | 1.3884E-12 | 1.843E+00 |
| | BHMO | 3.29471E-14 | 2.71051E-20 | 5.2457E-11 | 3.20935E-11 | 5.000E+00 |
| | RBHMO | 3.29471E-14 | 2.71051E-20 | 5.2457E-11 | 3.20935E-11 | 3.781E+00 |
| 15000 | OSOBHM | 2.25155E-17 | 8.13152E-20 | 8.20774E-15 | 3.06156E-14 | 3.906E+00 |
| | MOSOBHM | 2.25916E-17 | 2.71051E-20 | 8.20774E-15 | 3.06156E-14 | 3.8125E+00 |
| | BHMO | 5.53901E-16 | 2.71051E-20 | 8.4148E-13 | 7.59532E-13 | 10.281E+00 |
| | RBHMO | 5.53901E-16 | 2.71051E-20 | 8.4148E-13 | 7.59532E-13 | 6.937E+00 |

Table 2: Comparison of the methods under consideration on the basis of absolute errors and CPU time for Problem 2 with different number of steps (n)

| $n$ | Method | AE-$y_1$ | AE-$y_2$ | LE-$y_1$ | LE-$y_2$ | CPU time |
|---|---|---|---|---|---|---|
| 20 | OSOBHM | 4.09894E-13 | 1.86406E-12 | 4.09894E-13 | 1.86406E-12 | 1.562E-02 |
| | MOSOBHM | 4.09894E-13 | 1.86406E-12 | 4.09894E-13 | 1.86406E-12 | 1.562E-02 |
| | SDMWOHP | 8.88178E-16 | 8.88178E-16 | 4.44089E-16 | 4.44089E-16 | 3.594E-01 |
| | RBHMO | 2.5921E-10 | 1.07341E-10 | 2.5921E-10 | 9.60987E-11 | 4.687E-02 |
| 40 | OSOBHM | 1.28786E-14 | 3.30846E-14 | 1.28786E-14 | 3.30846E-14 | 3.125E-02 |
| | MOSOBHM | 3.9968E-15 | 2.68674E-14 | 3.9968E-15 | 2.68674E-14 | 3.125E-02 |
| | SDMWOHP | 8.88178E-16 | 1.55431E-15 | 8.88178E-16 | 0.0000E+00 | 7.8125E-01 |
| | RBHMO | 4.03499E-12 | 1.68487E-12 | 4.03499E-12 | 1.51301E-12 | 1.719E-01 |
| 80 | OSOBHM | 4.88498E-15 | 1.11022E-14 | 4.44089E-15 | 1.11022E-14 | 6.25E-02 |
| | MOSOBHM | 7.32747E-15 | 6.43929E-15 | 6.21725E-15 | 5.32907E-15 | 7.812E-02 |
| | SDMWOHP | 4.44089E-16 | 1.33227E-15 | 5.32907E-15 | 1.9984E-15 | 3.75E-01 |
| | RBHMO | 3.81917E-14 | 2.66454E-14 | 3.81917E-14 | 2.66454E-14 | 1.875E-01 |

Table 3: Comparison of the methods under consideration on the basis of absolute errors and CPU time for Problem 1 with different number of steps (n)

| $n$ | Method | AE-$y_1$ | AE$y_2$ | AE-$y_3$ | LE-$y_1$ |
|---|---|---|---|---|---|
| 50 | OSOBHM | 2.19901E-06 | 2.19901E-06 | 5.37612E-06 | 2.04003E-15 |
| | MOSOBHM | 2.19901E-06 | 2.19901E-06 | 5.37612E-06 | 2.47025E-15 |
| | RBHMO | 2.8506E-05 | 2.8506E-05 | 1.40091E-04 | 5.87308E-14 |
| 100 | OSOBHM | 3.30216E-08 | 3.30216E-08 | 8.71343E-08 | 1.38778E-16 |
| | MOSOBHM | 3.30216E-08 | 3.30216E-08 | 8.71343E-08 | 1.80411E-16 |
| | RBHMO | 9.33449E-07 | 9.33449E-07 | 8.74301E-16 | 8.74301E-16 |
| 200 | OSOBHM | 5.53462E-10 | 5.53462E-10 | 1.38163E-09 | 7.49401E-16 |
| | MOSOBHM | 5.53462E-10 | 5.53462E-10 | 1.38163E-09 | 2.91434E-16 |
| | RBHMO | 1.40086E-08 | 1.40086E-08 | 3.62727E-08 | 1.11022E-16 |
| 400 | OSOBHM | 8.63265E-12 | 8.63276E-12 | 2.16697E-11 | 8.63265E-12 |
| | MOSOBHM | 8.63265E-12 | 8.63266E-12 | 2.16697E-11 | 1.97065E-15 |
| | RBHMO | 2.30804E-10 | 2.30804E-10 | 5.75553E-10 | 2.88658E-15 |

scale) versus CPU time. The figure indicates that the OSOHBM and MOSOHBM outperform the RBHMO with respect to accuracy and computing time. It is also noted that the performance of OSOHBM and MOSOHBM are quite similar for this problem.

Figure 6 shows the comparison of the impact of high vaccination rate on human population (Susceptible Human (SH), Infected Human (IH), Recovered Human (RH)) using the OSOHBM and the Mathematica NDSolve methods. It is observed that the performances of the two methods are quite similar.

Figure 7 shows the comparison of the impact of moderate vaccination rate on human

Table 4: Continuation of Table 3

| $n$ | Method | LE-$y_2$ | LE-$y_3$ | CPU time |
|---|---|---|---|---|
| 50 | OSOBHM | 2.04003E-15 | 4.93182E-18 | 6.25E-02 |
| | MOSOBHM | 2.51188E-15 | 8.1838E-18 | 6.25E-02 |
| | RBHMO | 5.87308E-14 | 1.11081E-18 | 2.031E-01 |
| 100 | OSOBHM | 1.52656E-16 | 1.4137E-17 | 7.8125E-02 |
| | MOSOBHM | 2.35922E-16 | 3.08061E-18 | 9.375E-02 |
| | RBHMO | 8.74301E-16 | 8.59621E-18 | 3.9060E-01 |
| 200 | OSOBHM | 7.49401E-16 | 2.96144E-11 | 1.406E-01 |
| | MOSOBHM | 2.77556E-16 | 9.79858E-18 | 1,719E-01 |
| | RBHMO | 1.38778E-16 | 1.11574E-17 | 7.812E-01 |
| 400 | OSOBHM | 8.63265E-12 | 2.16697E-11 | 2.969E-01 |
| | MOSOBHM | 2.01228E-15 | 1.5598E-17 | 3.281E-01 |
| | RBHMO | 2.88658E-15 | 1.92975E-18 | 1.625E+00 |

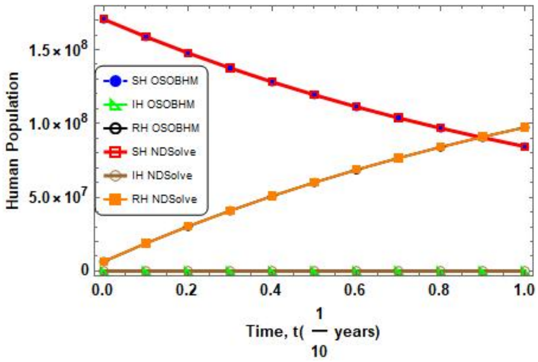


Figure 6: Impact of high vaccination rate on human population

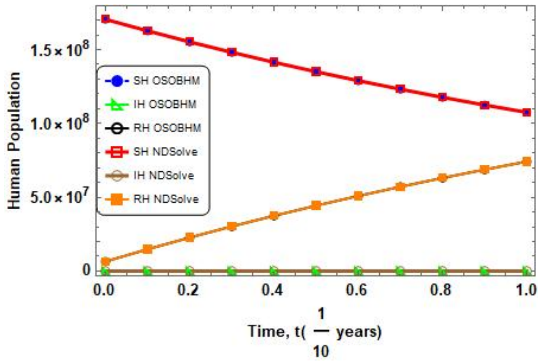


Figure 7: Impact of moderate vaccination rate on human population

population (Susceptible Human (SH), Infected Human (IH), Recovered Human (RH)) using the OSOHBM and the Mathematica NDSolve methods. It is observed that the performances of the two methods are quite similar.

Figure 8 shows the comparison of the impact of low vaccination rate on human population (Susceptible Human (SH), Infected Human (IH), Recovered Human (RH)) using the OSOHBM and the Mathematica NDSolve methods. It is observed that the two methods perform very closely.

Figure 9 shows the comparison of the impact of different vaccination rate on vulnerable human (High vaccination (v1), Moderate vaccination (v2), Low vaccination (v3)) using the OSOHBM and the Mathematica NDSolve methods. It is observed that the performances of the two methods are quite similar.

Figure 10 shows the comparison of the impact of different recovery rate on infected human (High recovery ($\gamma 1$), Moderate recovery ($\gamma 2$), Low recovery ($\gamma 3$)) using the OSOHBM and the Mathematica NDSolve methods. It is observed that two methods perform in a similar manner.

Figure 11 shows the comparison of the impact of different vaccination rate on recovered human (High vaccination (v1), Moderate vaccination (v2), Low vaccination (v3)) using the OSOHBM and the Mathematica NDSolve methods. It is observed that the per-

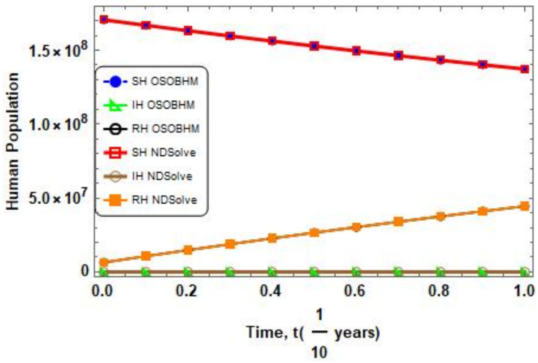


Figure 8: Impact of low vaccination rate on human population

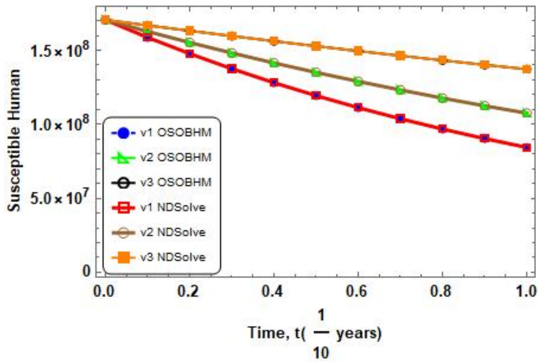


Figure 9: Impact of different vaccination rate on vulnerable human

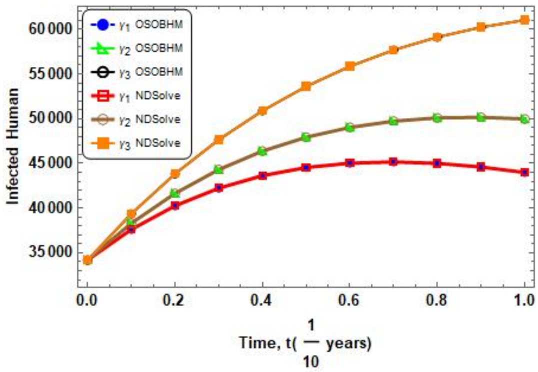


Figure 10: Impact of different recovery rate on infected human

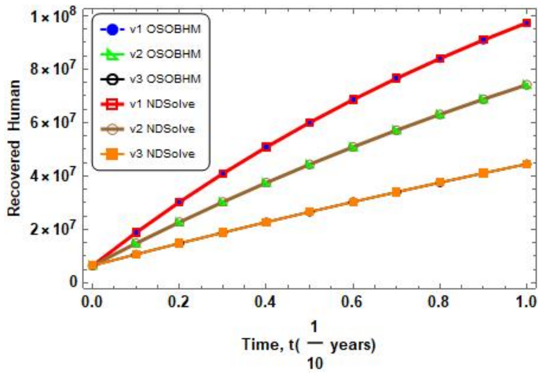


Figure 11: Impact of different vaccination rate on recovered human

formances of the two methods are quite similar.

## 5. Discussion

The solutions obtained using the OSOBHM and NDSolve techniques for various compartments of the model were plotted, and the resulting figures demonstrated a convergence between the two methods. The impact of varying vaccination rates on the human population is illustrated in Figures 6, 7, and 8, which correspond to high, moderate, and low vaccination rates, respectively. The data demonstrates a direct correlation between the increase in vaccination rate and the subsequent drop in the vulnerable human population, while concurrently observing an increase in the recovered human population. This phenomenon occurs because individuals who are vulnerable to infection receive vaccinations, hence transitioning into the recovered category. Additionally, it has been demonstrated that a higher vaccination rate leads to a greater increase in the number of individuals who have recovered from the infection compared to those who are still vulnerable. Additionally, as the vaccination rate declines, there is a slight decrease in the vulnerable population and a corresponding slight increase in the recovered population. The impact of varying vaccination rates on the vulnerable human population is depicted in Figure 9. A positive correlation exists between the vaccination rate and the susceptibility of the population, whereby an increase in the former leads to a decrease in the latter. The proportion reached its lowest point, nearly approaching zero. This observation demonstrates that the vaccination of individuals in the vulnerable group leads to their transition into the recovered population. In Figure 10, the impact of various recovery rates on the population of infected individuals is depicted. The prevalence of infection among the human population has a positive correlation with a lower rate of recovery, whereas conversely, it demonstrates a negative correlation with a higher rate of recovery. The population affected by the infection exhibits an initial increase, but thereafter experiences a decline due to the implementation of treatment interventions and the occurrence of natural healing processes. In Figure 11, the impact of varying vaccination rates on the community of individuals who have recovered from a particular disease is depicted. The population that experienced recovery exhibited a positive correlation with a high rate of vaccination, while a negative correlation was observed between the population that experienced recovery and a low rate of vaccination. Individuals who have received the vaccination have transitioned from the susceptible category to the recovered category.

## 6. Conclusion

We have presented the optimal hybrid block method, modified optimized hybrid method, and second derivative method with optimized hybrid points for solving first-order initial value problems of ODEs. The results in Tables 1, 2, 3, and 4 reveal that the methods OSOHBM (2.18), MOSOHBM (2.19), and SDMWOHP (2.22) are highly efficient with minimal errors. Furthermore, the modified method (2.19) apart from having minor errors also reduced the computational time which is an added advantage over the proposed method. However, in some instances, there is a trade-off between the error and the CPU time for the two methods (2.18) and (2.19). In comparison with another popular method from available literature, our methods produced minor errors and faster CPU time. Although the method (2.22) produced a much smaller error in implementation since higher derivatives were included in it, it required more computational time. The derived methods were implemented in block modes with the merit of being self-starting and thus required no starting values. The methods have good accuracy properties and are indeed of the higher order of accuracy at the final grid point where the LTEs were optimized, a major advantage of the method.

Also, the methods do not require the creation of separate predictors. The MOSOHBN showed that the efficiency of the method is dependent on the implementation strategies. The methods are more advantageous when economic computations in terms of the number of function evaluations and computing times are of major concern. Hence, the techniques are strongly suggested for general use. The Mathematica software package was used to develop the schemes, the plots and the results.

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## Appendix

Table 5: Definition and values of parameters (source [29])

| **Variables** | **Description** | **Value per year** |
|---|---|---|
| $S_h(0)$ | Number of susceptible humans at initial time | 177092454 |
| $T_h(0)$ | Number of infectious humans at initial time | 34200 |
| $R_h(0)$ | Number of recovered/Immune human at initial time | 29070 |
| $V_1(0)$ | Number of non-carrier vectors at initial time | 35000000 |
| $V_2(0)$ | Number of carrier vectors at initial time | 15000000 |
| $S_m(0)$ | Number of susceptible secondary host at initial time | 35000 |
| $I_m(0)$ | Number of infectious secondary host at initial time | 1500 |
| $N_h$ | Total human population at time t | 177155754 |
| $N_v$ | Total vector population at time t | 50000000 |
| $N_m$ | Total secondary vector population at time t | 50000 |
| $\alpha_1$ | Effective virus Transmission rate from mosquito to humans | 0.05 |
| $\alpha_2$ | Effective virus Transmission rate from humans to mosquito | 0.48 |
| $\alpha_3$ | Effective virus Transmission rate from secondary host to mosquito | 0.042 |
| $\alpha_4$ | Effective virus Transmission rate from mosquito to secondary host | 0.001 |
| $\lambda_h$ | Recruitment number of human population | 6865728 |
| $\lambda_v$ | Recruitment number of mosquito population | 2000000 |
| $\lambda_m$ | Recruitment number of secondary vector population | 5000 |
| $\delta_h$ | Disease-induced death rate of humans | 0.15 |
| $\delta_v$ | Death rate of mosquito due to application of insecticide | 0.001 |
| $\delta_m$ | Disease-induced death rate of secondary host | 0.002 |
| $\mu_h$ | Natural death rate of human population | 0.012 |
| $\mu_v$ | Natural death rate of mosquito population | 0.02 |
| $\mu_m$ | Natural death rate of secondary host population | 0.005 |
| $\gamma_h$ | Recovery rate of human population due to drug administration | 0.85 |
| $v$ | vaccination rate for the human population | 0.75 |