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Study on the Relationship Between Two Curvature Tensors in Finsler spaces

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Abstract

The generalized $\mathfrak{B}P-$recurrent space and generalized $\mathfrak{B}P-$ bi-recurrent space are introduced by [3, 5]. In this paper, we establish two theorems that clarify the relationship between P^i_{jkh} and R^i_{jkh}, and we prove them in the spaces mentioned above. Moreover, the necessary and sufficient conditions for R^i_{jkh}, satisfying the generalized recurrence and bi-recurrence property, are obtained.

Keywords: Cartan's second curvature tensor P^i_{jkh}, Cartan's third curvature tensor R^i_{jkh}, Berwald covariant derivative $\mathfrak{B}_k$.
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1. Introduction

The Finslerian geometrics has studied the generalized recurrence and bi-recurrence properties. Pandey et al. [19], Qasem and Abdallah [8, 9], Qasem and Baleedi [12] and Alaa et al. [4] introduced the generalized recurrent Finsler spaces for H^i_{jkh}, R^i_{jkh}, K^i_{jkh} and P^i_{jkh}, respectively. Also, Alaa et al. [6] studied the generalized $\mathfrak{B}R-$recurrent space and obtained certain identities belonging to it. The generalized property for normal projective curvature tensor N^i_{jkh} in the sense of Berwald has been introduced by [11].

In the same regard, Alaa et al. [2] introduced the generalized birecurrent Finsler spaces for P^i_{jkh}. Qasem [7] studied a generalized $H-$birecurrent space, and Qasem and Hadi [13] introduced a generalized $\mathfrak{B}R-$birecurrent space. In addition, Qasem and Saleem [10] studied the projective curvature tensor W^i_{jkh}, which satisfies the generalized birecurrence property. Further, Zlatanovic and Mincic [16] introduced several identities for some curvature tensors in generalized Finsler space. This paper aims to discuss Cartan's third curvature tensor R^i_{jkh} in two Finsler spaces.

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2. Preliminaries

This section will provide some equations and definitions for this paper. Let F_n be an nˇdimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [1, 18, 20]. The vector y_i is defined by

$$y_i = g_{ij}(x,y)y^j. \tag{2.1}$$

Two sets of quantities g_{ij} and its associative g^{ij} are connected by

$$g_{ij}g^{ik} = \delta_j^k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{2.2}$$

Given Eqs. (2.1) and (2.2), we have

$$a)\ \delta_k^i y^k = y^i, \quad b)\ \delta_j^i g_{ir} = g_{jr}, \quad c)\ \delta_k^i y_i = y_k \quad \text{and} \quad d)\ \delta_j^i y_i = y_j. \tag{2.3}$$

Matsumoto [17] introduced the $(h)hv$–torsion tensor C_{ijk} that is positively homogeneous of degree -1 in y^i and symmetric in all its indices and defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2.$$

The above tensor satisfies the following

$$a)\ C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0 \quad \text{and} \quad b)\ \delta_k^i C_{jil} = C_{jkl}. \tag{2.4}$$

Berwald covariant derivative $\mathfrak{B}_k$ of an arbitrary tensor field T_j^i with respect to x^k is given by [14]

$$\mathfrak{B}_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i)G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r.$$

Berwald covariant derivative of the vector y^i and metric tensor g_{ij} satisfy

$$a)\ \mathfrak{B}_k y^i = 0 \quad \text{and} \quad b)\ \mathfrak{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathfrak{B}_h C_{ijk}. \tag{2.5}$$

The h–curvature tensor (Cartan's third curvature tensor) is defined by

$$R_{jkh}^i = \partial_h\Gamma_{jk}^{*i} + \left(\dot{\partial}_\ell\Gamma_{jh}^{*i}\right)G_k^\ell + G_{jm}^i\left(\partial_h G_k^m - G_{h\ell}^m G_k^\ell\right) + \Gamma_{mh}^{*i}\Gamma_{jk}^{*m} - h/k.$$

This tensor satisfies the following relations

$$R_{jki}^i = R_{jk}. \tag{2.6}$$

The curvature tensor R_{jkh}^i, its associative R_{rjkh}, RˇRicci tensor R_{jk} and curvature vector R_k satisfy

$$a)\ R_{rjkh} = g_{ri}R_{jkh}^i, \quad b)\ R_{jk}y^j = R_k, \quad \text{and} \quad c)\ R_{jkh}^i y^j = H_{kh}^i = K_{jkh}^i y^j. \tag{2.7}$$

The tensor P_{jkh}^i called hv–curvature tensor (Cartan's second curvature tensor) is positively homogeneous of degree in and y^i defined by [14]

$$P_{jkh}^i = \dot{\partial}_h\Gamma_{jk}^{*i} + C_{jr}^i P_{kh}^r - C_{jh|k}^i.$$

The associate tensor P_{ijkh}, torsion tensor P^i_{kh} and P−Ricci tensor P_{jk} of $h\nu$−curvature tensor P^i_{jkh} satisfies the relations

$$a)\ P_{ijkh} = g_{ir}P^r_{jkh}, \qquad b)\ P^i_{jkh}y^j = \Gamma^{*i}_{jkh}y^j = P^i_{kh} = C^i_{kh|r}y^r, \tag{2.8}$$
$$c)\ P^i_{jki} = P_{jk}, \quad \text{and} \quad d)\ P^i_{jk}y_i = 0.$$

Alaa et al. [2, 3] introduced the generalized $\mathfrak{B}P$− recurrent space and generalized $\mathfrak{B}P$−bi-recurrent space which are characterized by the conditions

$$\mathfrak{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right), \tag{2.9}$$

and

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} = a_{lm}P^i_{jkh} + b_{lm}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2y^t\mu_m\mathfrak{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right), \tag{2.10}$$

respectively. These spaces denoted them by $G(\mathfrak{B}P) - RF_n$ and $G(\mathfrak{B}P) - BRF_n$, respectively.

Transvecting the condition (2.9) by g_{il}, using Eqs. (2.8), (2.5) and (2.3), we get

$$\mathfrak{B}_m P_{ljkh} = \lambda_m P_{ljkh} + \mu_m\left(g_{lj}g_{kh} - g_{lk}g_{jh}\right) + 2P^i_{jkh}y^t\mathfrak{B}_t C_{ilm}. \tag{2.11}$$

Transvecting the condition (2.9), by y^j, using Eqs. (2.8), (2.5), (2.3) and (2.1), we get

$$\mathfrak{B}_m P^i_{kh} = \lambda_m P^i_{kh} + \mu_m\left(y^i g_{kh} - \delta^i_k y_h\right). \tag{2.12}$$

Contracting the indices i and h in the condition (2.9), using Eqs. (2.8) and (2.3), we get

$$\mathfrak{B}_m P_{jk} = \lambda_m P_{jk}. \tag{2.13}$$

Transvecting the condition (2.10) by y^j, using Eqs. (2.8), (2.5), (2.3), (2.1) and (2.4), we get

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = a_{lm}P^i_{kh} + b_{lm}\left(y^i g_{kh} - \delta^i_k y_h\right) - 2y^t\mu_m\mathfrak{B}_t\left(y^i C_{khl}\right). \tag{2.14}$$

Contracting the indices i and h in the condition (2.10), using Eqs. (2.8), (2.3) and (2.4), we get

$$\mathfrak{B}_l\mathfrak{B}_m P_{jk} = a_{lm}P_{jk}. \tag{2.15}$$

3. Main Result

In this section, we focus on the conditions for Cartan's third curvature tensor R^i_{jkh} that satisfies the generalized recurrence and bi-recurrence property in generalized $\mathfrak{B}P$−recurrent space and generalized $\mathfrak{B}P$−bi-recurrent space, respectively. The holomorphically projective curvature tensor P^h_{ijk} is defined by [15]

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right), \tag{3.1}$$

where $S_{ij} = F^a_i R_{aj}$. Transvecting Eq. (3.1) by g_{hl}, using Eqs. (2.8), (2.7) and put $(g_{hl}F^h_j = g_{jl})$, we get

$$P_{lijk} = R_{lijk} + \frac{1}{n+2}\left(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl}\right) + 2P^h_{ijk}y^t\mathfrak{B}_t C_{hlm}. \tag{3.2}$$

Transvecting Eq. (3.1) by y^i, using Eqs. (2.8), (2.5), (2.7) and (2.1), we get

$$P^h_{jk} = H^h_{jk} + \frac{1}{n+2}\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right). \tag{3.3}$$

Contracting the indices i and h in Eq. (3.1), using Eqs. (2.8), (2.6), (2.3) and put $(S_{ik}F^i_j = S_{jk})$, we get

$$P_{jk} = R_{jk} + \frac{1}{n+2}\left[(1-n)R_{jk} + (3-n)S_{jk}\right]. \tag{3.4}$$

In the next theorem, we obtain the condition for R^i_{jkh} that satisfies the generalized recurrence property.

Theorem 3.1. *In $G(\mathfrak{B}P) - RF_n$, Cartan's third curvature tensor R^h_{ijk} is a generalized $\mathfrak{B}-$recurrent if and only if the tensor $\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)$ behaves as recurrent.*

Proof. Let us consider a $G(\mathfrak{B}P) - RF_n$, i.e, characterized by the condition (2.9). Taking $\mathfrak{B}-$covariant derivative for Eq. (3.1) with respect to x^m and using the condition (2.9), we get

$$\mathfrak{B}_m R^h_{ijk} = \lambda_m P^h_{ijk} + \mu_m\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) - \frac{1}{n+2}\mathfrak{B}_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right).$$

Using Eq. (3.1) in above equation, we get

$$\begin{aligned}\mathfrak{B}_m R^h_{ijk} &= \lambda_m R^h_{ijk} + \mu_m\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right)\\ &+\frac{1}{n+2}\lambda_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)\\ &-\frac{1}{n+2}\mathfrak{B}_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right).\end{aligned}$$

This shows that

$$\mathfrak{B}_m R^h_{ijk} = \lambda_m R^h_{ijk} + \mu_m\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right)$$

if and only if

$$\begin{aligned}&\mathfrak{B}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)\\ &= \lambda_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right).\end{aligned}$$

The above equation means that the behavior of $\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)$ is recurrent. The proof for this theorem is completed.

Now, we have a result related to the last theorem. Taking $\mathfrak{B}-$covariant derivative for Eq. (3.2) with respect to x^m and using Eq. (2.11), we get

$$\mathfrak{B}_m R_{lijk} = \lambda_m P_{ljkh} + \mu_m\left(g_{jl}g_{kh} - g_{kl}g_{jh}\right) + 2P^i_{jkh}y^t\mathfrak{B}_t C_{ilm}$$
$$-\frac{1}{n+2}\mathfrak{B}_m\left(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl}\right) - 2P^h_{ijk}y^t\mathfrak{B}_t C_{hlm}.$$

Using Eq. (3.2) in above equation, we get

$$\mathfrak{B}_m R_{lijk} = \lambda_m R_{ljkh} + \mu_m\left(g_{jl}g_{kh} - g_{kl}g_{jh}\right) + 2P^i_{jkh}y^t\mathfrak{B}_t C_{ilm}$$
$$+\frac{1}{n+2}\lambda_m\left(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl}\right)$$
$$-\frac{1}{n+2}\mathfrak{B}_m\left(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl}\right).$$

This shows that

$$\mathfrak{B}_m R_{lijk} = \lambda_m R_{ljkh} + \mu_m\left(g_{jl}g_{kh} - g_{kl}g_{jh}\right) + 2P^i_{jkh}y^t\mathfrak{B}_t C_{ilm} \tag{3.5}$$

if and only if

$$\mathfrak{B}_m\left(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl}\right) \tag{3.6}$$
$$= \lambda_m\left(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl}\right).$$

Taking $\mathfrak{B}-$covariant derivative for Eq. (3.3) with respect to x^m and using Eq. (2.12), we get

$$\mathfrak{B}_m H^h_{jk} = \lambda_m P^h_{jk} + \mu_m\left(y^h g_{jk} - \delta^h_k y_j\right)$$
$$-\frac{1}{n+2}\mathfrak{B}_m\left(R_k\delta^h_j - R_{jk}y^h + S_k F^h_j - S_{jk}y^h + 2S_j F^h_k\right).$$

Using Eq. (3.3) in above equation, we get

$$\mathfrak{B}_m H^h_{jk} = \lambda_m H^h_{jk} + \mu_m\left(y^h g_{jk} - \delta^h_k y_j\right)$$
$$+\frac{1}{n+2}\lambda_m\left(R_k\delta^h_j - R_{jk}y^h + S_k F^h_j - S_{jk}y^h + 2S_j F^h_k\right)$$
$$-\frac{1}{n+2}\mathfrak{B}_m\left(R_k\delta^h_j - R_{jk}y^h + S_k F^h_j - S_{jk}y^h + 2S_j F^h_k\right).$$

This shows that

$$\mathfrak{B}_m H^h_{jk} = \lambda_m H^h_{jk} + \mu_m\left(y^h g_{jk} - \delta^h_k y_j\right) \tag{3.7}$$

if and only if

$$\mathfrak{B}_m\left(R_k\delta^h_j - R_{jk}y^h + S_k F^h_j - S_{jk}y^h + 2S_j F^h_k\right) \tag{3.8}$$
$$= \lambda_m\left(R_k\delta^h_j - R_{jk}y^h + S_k F^h_j - S_{jk}y^h + 2S_j F^h_k\right).$$

Taking $\mathfrak{B}-$covariant derivative for Eq. (3.4) with respect to x^m and using Eq. (2.13), we get

$$\mathfrak{B}_m R_{jk} = \lambda_m P_{jk} - \frac{1}{n+2}\mathfrak{B}_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right].$$

Using Eq. (3.4) in above equation, we get

$$\begin{aligned}\mathfrak{B}_m R_{jk} &= \lambda_m R_{jk} + \frac{1}{n+2}\lambda_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right]\\ &\quad -\frac{1}{n+2}\mathfrak{B}_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right].\end{aligned}$$

This shows that

$$\mathfrak{B}_m R_{jk} = \lambda_m R_{jk} \tag{3.9}$$

if and only if

$$\mathfrak{B}_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right] = \lambda_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right]. \tag{3.10}$$

The equations (3.6), (3.8) and (3.10) refer to the behavior of $(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl})$, $\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right)$ and $\left[(1-n)R_{jk} + (3-n)S_{jk}\right]$ are recurrent, respectively. Thus, we conclude the following corollary:

Corollary 3.2. *In* $G(\mathfrak{B}P) - RF_n$*, Berwald's covariant derivative of first order for* R_{lijk}*,* H^h_{jk} *and* R_{jk} *are given by Eqs. (3.5), (3.7) and (3.9) if and only if the tensors* $(R_{ik}g_{jl} - R_{jk}g_{il} + S_{ik}g_{jl} - S_{jk}g_{il} + 2S_{ij}g_{kl})$, $\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right)$ *and* $\left[(1-n)R_{jk} + (3-n)S_{jk}\right]$ *behave as recurrent, respectively.*

In the next theorem, we obtain the condition R^i_{jkh} that satisfies the generalized bi-recurrence property.

Theorem 3.3. *In* $G(\mathfrak{B}P) - BRF_n$*, Cartan's third curvature tensor* R^i_{jkh} *is a generalized* $\mathfrak{B}-$*bi-recurrent if and only if the tensor* $\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)$ *behaves as bi-recurrent.*

Proof. Let us consider a $G(\mathfrak{B}P) - BRF_n$, i.e, characterized by the condition (2.10). Taking $\mathfrak{B}-$covariant derivative for Eq. (3.1) twice with respect to x^m and x^l, using the condition (2.10), we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m R^h_{ijk} &= a_{lm}P^h_{ijk} + b_{lm}\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) - 2y^t\mu_m\mathfrak{B}_t\left(\delta^h_i C_{jkl} - \delta^h_j C_{ikl}\right)\\ &\quad -\frac{1}{n+2}\mathfrak{B}_l\mathfrak{B}_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right).\end{aligned}$$

Using Eq. (3.1) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m R^h_{ijk} &= a_{lm}R^h_{ijk} + b_{lm}\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) - 2y^t\mu_m\mathfrak{B}_t\left(\delta^h_i C_{jkl} - \delta^h_j C_{ikl}\right)\\ &\quad +\frac{1}{n+2}a_{lm}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)\\ &\quad -\frac{1}{n+2}\mathfrak{B}_l\mathfrak{B}_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right).\end{aligned}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m R^h_{ijk} = a_{lm} R^h_{ijk} + b_{lm}\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) - 2y^t \mu_m \mathfrak{B}_t\left(\delta^h_i C_{jkl} - \delta^h_j C_{ikl}\right)$$

if and only if

$$\begin{aligned}&\mathfrak{B}_l\mathfrak{B}_m\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)\\ &= a_{lm}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right).\end{aligned}$$

The above equation refers to the behavior of $\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right)$ is bi-recurrent. The proof for this theorem is completed.

Now, we have a result related to the last theorem. Taking $\mathfrak{B}-$covariant derivative for Eq. (3.3) twice with respect to x^m and x^l, using Eq. (2.14), we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m H^h_{jk} &= a_{lm} P^h_{jk} + b_{lm}\left(y^h g_{jk} - \delta^h_k y_j\right) - 2y^t \mu_m \mathfrak{B}_t\left(y^h C_{jkl}\right)\\ &-\frac{1}{n+2}\mathfrak{B}_l\mathfrak{B}_m\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right).\end{aligned}$$

Using Eq. (3.3) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m H^h_{jk} &= a_{lm} H^h_{jk} + b_{lm}\left(y^h g_{jk} - \delta^h_k y_j\right) - 2y^t \mu_m \mathfrak{B}_t\left(y^h C_{jkl}\right)\\ &+\frac{1}{n+2}a_{lm}\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right)\\ &-\frac{1}{n+2}\mathfrak{B}_l\mathfrak{B}_m\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right).\end{aligned}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m H^h_{jk} = a_{lm} H^h_{jk} + b_{lm}\left(y^h g_{jk} - \delta^h_k y_j\right) - 2y^t \mu_m \mathfrak{B}_t\left(y^h C_{jkl}\right) \tag{3.11}$$

if and only if

$$\begin{aligned}&\mathfrak{B}_l\mathfrak{B}_m\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right)\\ &= a_{lm}\left(R_k\delta^h_j - R_{jk}y^h + S_kF^h_j - S_{jk}y^h + 2S_jF^h_k\right).\end{aligned} \tag{3.12}$$

Taking $\mathfrak{B}-$covariant derivative for Eq. (3.4) twice with respect to x^m and x^l, using Eq. (2.15), we get

$$\mathfrak{B}_l\mathfrak{B}_m R_{jk} = a_{lm} P_{jk} - \frac{1}{n+2}\mathfrak{B}_l\mathfrak{B}_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right].$$

Using Eq. (3.4) in above equation, we get

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m R_{jk} &= a_{lm} R_{jk} + \frac{1}{n+2}a_{lm}\left[(1-n)R_{jk} + (3-n)S_{jk}\right]\\ &-\frac{1}{n+2}\mathfrak{B}_l\mathfrak{B}_m\left[(1-n)R_{jk} + (3-n)S_{jk}\right].\end{aligned}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m R_{jk} = a_{lm} R_{jk} \tag{3.13}$$

if and only if

$$\mathfrak{B}_l\mathfrak{B}_m\left[(1-n)\,R_{jk}+(3-n)\,S_{jk}\right]=a_{lm}\left[(1-n)\,R_{jk}+(3-n)\,S_{jk}\right]. \tag{3.14}$$

The equations (3.12) and (3.14) mean that the behavior of $\left(R_k\delta^h_j-R_{jk}y^h+S_kF^h_j-S_{jk}y^h+2S_jF^h_k\right)$ and $\left[(1-n)\,R_{jk}+(3-n)\,S_{jk}\right]$ are bi-recurrent, respectively. Thus, we conclude the following corollary:

Corollary 3.4. *In* $G(\mathfrak{B}P)-BRF_n$*, Berwald's covariant derivative of second order for* H^h_{jk} *and* R_{jk} *give Eqs.* (3.11) *and* (3.13) *if and only if the tensors* $\left(R_k\delta^h_j-R_{jk}y^h+S_kF^h_j-S_{jk}y^h+2S_jF^h_k\right)$ *and* $\left[(1-n)\,R_{jk}+(3-n)\,S_{jk}\right]$ *behave as bi-recurrent, respectively.*

4. conclusion

We obtained the necessary and sufficient condition for R^i_{jkh}, which be generalized recurrent and bi-recurrent in $G(\mathfrak{B}P)-RF_n$ and $G(\mathfrak{B}P)-BRF_n$, respectively.

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