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A Modified Fourth Derivative Block Method and its direct applications to third-order initial value problems

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Abstract

A Modified Fourth Derivative four-step block method (MFDFBM) of theoretical order eight has been developed. It has been analyzed and numerically tested to solve a range problems in Fluid Dynamics, engineering, and other sciences. The MFDFBM was derived using a combination of collocation and interpolation techniques applied to a power series approximation. By introducing fourth derivative terms at each of the collocating points, a block method possessing improved order of accuracy was obtained. It was observed that the order of the block method increased with the number of fourth derivative terms introduced into the integration interval. Numerical experiments were conducted to test MFDFBM on various numerical examples, including non-linear homogeneous thin film flow (NHTFF) problems and two non-linear initial value problems(IVPs). The results of the experiments confirmed the effectiveness of adding the fourth derivative terms, which helped improve the order of accuracy of the derived MFDFBM. The experiments demonstrated that the MFDFBM method minimized error and agreed with analytical solutions up to at least seven decimal places, thereby highlighting the importance of using this method in non-linear numerical computations.

Keywords: Uniform order, Theoretical order, test problem, Linear multistep method, Jerk equation, Genesio equation .
2010 MSC: 65L05, 65L06.

1. Introduction

The sole aim of the article is to make available approximate solution of third-order initial value problem of ordinary differential equation:

$$v'''(x) = w(x, v(x), v'(x), v''(x)), \tag{1.1}$$

$$v(x_0) = \alpha_0,\ v'(x_0) = \beta_0,\ v''(x_0) = \gamma_0,\ x \in [x_0,\ x_N] \subset \mathbb{R},\ \alpha_0,\ \beta_0 \text{ and } \gamma_0 \in \mathbb{R}$$

where x_0 stand for the initial point and u is believed to be a continuous function that satisfies Lipchitz's condition that guarantee the existence and uniqueness solution of equation (1.1).

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In the literature, many scholars have done much work on solving this problem in (1.1) because it depicts numerous mathematical models for the applied sciences and engineering. Problems of form (1.1) emerge in the study of the thin film flow, fluid dynamics and mechanics, entry-flow phenomenon, hydrodynamics, the uniform flow of water in a long rectangular tank, and many others. We suggest the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the sources cited therein for further details on the application of the model in(1.1) and associated issues.

The exact solution of (1.1) can only be determined in some cases ([12]). Therefore, we depend on numerical techniques to obtain approximate solutions when the problem in (1.1) cannot be solved analytically. The traditional methodology to tackle a higher-order problem is to convert it into a first-order system with initial conditions and address the obtained system utilizing specific first-order techniques, such as linear multistep strategies and Runge-Kutta techniques [12].

Numerous researchers have presented various numerical techniques for resolving problems of this nature in (1.1). These techniques include, among others, the harmonic balance method in Ref. [13], the Jacobi-Gauss collocation method in Ref. [14], the multi-derivative method from Shokri [15], the P-stable method in Ref. [16], the embedded explicit strategy in Ref. [17], the direct integration method in Ref. You and Chen [18], the mesh-free approach in Ref. [19], or the numerical method presented in Tuck and Schwartz [20].

In this work, the solution of problem (1.1) is considered in the interval $[0,\ 4]$. Ordinarily, the research would have produced a method of uniform order four. However, by introducing the fourth derivative at all collocation points, the order of the method increased by four, thereby improving the properties of the proposed MFDFBM. The need to achieve better accuracy when using the numerical method with step length $k = 4$ has driven the development of a new method, which is the primary objective of this work. This method is expected to deliver significantly improved results and resolve the current issues of accuracy with existing methods. The success of this approach will result in a significant advancement in the field of numerical methods, offering a more reliable and precise solution to complex problems.

2. Mathematical formulation and Derivation of the MFDFBM

This section focused on the derivation of MFDFBM through the procedure of collocation and interpolation. The derivation is achieved by approximating $v(x)$ by a polynomial $q(x)$, given as

$$v(x) \approx q(x) = \sum_{z=0}^{10} a_z x^z, \tag{2.1}$$

where $a_z \in \mathbb{R}$ are parameters that are obtained uniquely by applying collocation and interpolation at proper points. In the estimation in (2.1), its first and second derivatives are evaluated at x_n and its third and fourth derivatives at x_{n+z}. A total of eleven equations with eleven unknowns are obtained, which satisfy the following:

$$q(x_n) = v_n,\ q'(x_n) = v'_n,\ q''(x_n) = v''_n, \tag{2.2}$$

$$v'''(x_{n+k}) = w_{n+k},\ \ v^{(4)}(x_{n+k}) = w'_{n+k},\ k = 1,\ 2, ..., 4 \tag{2.3}$$

where, v_n, v'_n, v''_n, w_{n+k}, w'_{n+k} are regarded as the approximate solution of $v(x_n)$, $v'(x_n)$, $v''(x_n)$, $w(x_{n+kh}, v_{n+kh}, v'_{n+kh}, v''_{n+kh})$, $w'(x_{n+kh}, v_{n+kh}, v'_{n+kh}, v''_{n+kh})$. The system of eleven algebraic equations is solved for the parameter's a_z's and is substituted into equation (2.1), with $x_{n+ih} = x_n + ih$, yields the continuous scheme

$$v(x_n + ih) = \alpha_0(i)v_n + \alpha_1(i)hv'_n + \alpha_2(i)h^2v''_n + h^3\sum_{z=1}^{4}\beta_z(i)w_{n+z} + h^4\sum_{z=1}^{4}\gamma_z(i)w'_{n+z}, \tag{2.4}$$

where for each z, $\alpha_0(i)$, $\alpha_1(i)$, $\alpha_2(i)$, $\beta_z(i)$ and $\gamma_z(i)$ are the continuous coefficients that determines the method.

2.1. *The formulas of the MFDFBM*

The twelve formulas that constituted the block method MFDFBM are obtained by substituting the above coefficients into (2.4) and evaluating $q(x)$, $q'(x)$, $q''(x)$ at the points $x_{n+i} = x_n + i\,h$, $i = 1, 2, .., 4$, to obtain the estimation for $v(x_{n+i})$, $v'(x_{n+i})$, $v''(x_{n+i})$, $i = 1, 2, .., 4$. The coefficients of the formulas are presented in the Table 1.

3. Analysis of fundamental properties of MFDFBM

This section reveals the basic properties of the derived MFDFBM using an extension of Dahlquist-Henrici theory on multistep methods.

3.1. *Accuracy and consistency of the MFDFBM*

The derived MFDFDM can be written as a block matrix formula as

$$M_1V_n = h\,M_2V'_n + h^2\,M_3V''_n + h^3\,M_4W_n + h^4\,M_5W'_n, \tag{3.1}$$

note that M_1, M_2, M_3, M_4, and M_5 are the matrices of the coefficients of MFDFBM presented in Table 1, and

$$\begin{aligned}
V_n &= (v_n,\ v_{n+1},\ v_{n+2},\ v_{n+3},\ v_{n+4})^T,\\
V'_n &= (v'_n,\ v'_{n+1},\ v'_{n+2},\ v'_{n+3},\ v'_{n+4})^T,\\
V''_n &= (v''_n,\ v''_{n+1},\ v''_{n+2},\ v''_{n+3},\ v''_{n+4})^T,\\
W_n &= (w_n,\ w_{n+1},\ w_{n+2},\ w_{n+3},\ w_{n+4})^T,\\
W'_n &= (w'_n,\ w'_{n+1},\ w'_{n+2},\ w'_{n+3},\ w'_{n+4})^T.
\end{aligned}$$

Using the procedure in [21, 22, 23], and assuming that $v(x)$ is a sufficiently differentiable function, we define the operator $\mathbf{L}$ as

$$\begin{aligned}
\mathbf{L}\,[v(x);h] &= \sum_j \alpha_j v(x_n + jh) - h\beta_j v'(x_n + jh) - h^2\gamma_j v''(x_n + jh)\\
&\quad - h^3\mu_j v''(x_n + jh) - h^4\nu_j v''(x_n + jh),
\end{aligned} \tag{3.2}$$

where α_j, β_j, γ_j, μ_j and ν_j are the column vectors of M_1, M_2, M_3, M_4, and M_5 respectively, $j = 1, 2, ..., 4$. Taylor series expansion of the formulas in (3.2) about x_n yields the following

$$\mathbf{L}\,[v(x);h] = c_0v(x_n) + c_1hv'(x_n) + c_2h^2v''(x_n) + ... + c_ph^pv^p(x_n) + o(h^{p+1}). \tag{3.3}$$

Table 1: The coefficients of the Linear Multistep Formulas

	i	α_0	α_1	α_2	β_1	β_2	β_3	β_3	γ_1	γ_2	γ_3	γ_4
y_{n+i}	1	1	1	$\frac{1}{2}$	$-\frac{299269}{90720}$	$-\frac{51427}{20160}$	$\frac{9785}{2016}$	$\frac{210971}{181440}$	$-\frac{2711}{2016}$	$-\frac{111521}{20160}$	$-\frac{36257}{10080}$	$-\frac{3583}{12096}$
	2	1	2	2	$-\frac{26596}{1701}$	$-\frac{1268}{105}$	$\frac{7388}{315}$	$\frac{47552}{8505}$	$-\frac{18964}{2835}$	$-\frac{1690}{63}$	$-\frac{52}{3}$	$-\frac{4036}{2835}$
	3	1	3	$\frac{9}{2}$	$-\frac{41373}{1120}$	$-\frac{63423}{2240}$	$\frac{63117}{1120}$	$\frac{6003}{448}$	$-\frac{18117}{1120}$	$-\frac{4131}{64}$	$-\frac{9315}{224}$	$-\frac{7641}{2240}$
	4	1	4	8	$-\frac{63488}{945}$	$-\frac{3200}{63}$	$\frac{32768}{315}$	$\frac{23264}{945}$	$-\frac{5632}{189}$	$-\frac{37376}{315}$	$-\frac{24064}{315}$	$-\frac{1184}{189}$
	i	α'_0	α'_1	α'_2	β'_1	β'_2	β'_3	β'_3	γ'_1	γ'_2	γ'_3	γ'_4
y'_{n+i}	1	0	1	1	$-\frac{302687}{38880}$	$-\frac{30047}{5040}$	$\frac{115939}{10080}$	$\frac{373537}{136080}$	$-\frac{74023}{22680}$	$-\frac{132469}{10080}$	$-\frac{42869}{5040}$	$-\frac{12683}{18144}$
	2	0	1	2	$-\frac{143144}{8505}$	$-\frac{4099}{315}$	$\frac{8104}{315}$	$\frac{52019}{8505}$	$-\frac{3004}{405}$	$-\frac{9292}{315}$	$-\frac{1196}{63}$	$-\frac{4414}{2835}$
	3	0	1	3	$-\frac{28869}{1120}$	$-\frac{5427}{280}$	$\frac{44973}{1120}$	$\frac{2661}{280}$	$-\frac{6471}{560}$	$-\frac{10287}{224}$	$-\frac{8271}{280}$	$-\frac{387}{160}$
	4	0	1	4	$-\frac{295168}{8505}$	$-\frac{8048}{315}$	$\frac{17408}{315}$	$\frac{15784}{1215}$	$-\frac{8896}{567}$	$-\frac{19616}{315}$	$-\frac{12608}{315}$	$-\frac{9344}{2835}$
	i	α''_0	α''_1	α''_2	β''_1	β''_2	β''_3	β''_3	γ''_1	γ''_2	γ''_3	γ''_4
y''_{n+i}	1	0	0	1	$-\frac{169325}{18144}$	$-\frac{4825}{672}$	$\frac{9511}{672}$	$\frac{60947}{18144}$	$-\frac{126467}{30240}$	$-\frac{54541}{3360}$	$-\frac{35059}{3360}$	$-\frac{25853}{30240}$
	2	0	0	1	$-\frac{188}{21}$	$-\frac{47}{7}$	$\frac{100}{7}$	$\frac{71}{21}$	$-\frac{1304}{315}$	$-\frac{577}{35}$	$-\frac{368}{35}$	$-\frac{271}{315}$
	3	0	0	1	$-\frac{6007}{672}$	$-\frac{1395}{224}$	$\frac{3309}{224}$	$\frac{2281}{672}$	$-\frac{4633}{1120}$	$-\frac{18351}{1120}$	$-\frac{11889}{1120}$	$-\frac{967}{1120}$
	4	0	0	1	$-\frac{5056}{567}$	$-\frac{128}{21}$	$\frac{320}{21}$	$\frac{2140}{567}$	$-\frac{3904}{945}$	$-\frac{1712}{105}$	$-\frac{1088}{105}$	$-\frac{856}{945}$

Adopting the procedure outlined in [21, 22, 23], the **L** operator in (3.3) and the associated formulas of MFDFBM in Table 1 are said to of algebraic order p if $c_0 = c_1 =, ..., = c_{p+2} = 0$, $c_{p+3} \neq 0$. Note that terms $c_0 = c_1, ..., c_{p+3}$ are column vectors and c_{p+3} represents vector of the error constant. Hence, each formula of MFDFBM has eight algebraic order of convergence and consistent with the following error constant

$$c_{p+3} = \left(\frac{142943}{139708800}, \frac{1523}{311850}, \frac{5751}{492800}, \frac{23344}{1091475}, \frac{2711}{112890}, \frac{169}{31752}, \frac{207}{25088}, \frac{74}{6615}, \frac{74023}{25401600}, \frac{2323}{793800}, \frac{919}{313600}, \frac{292}{99225} \right)^T .$$

3.2. Zero-stability and convergence of the MFDFBM

The zero-stability discussed the possible behaviours of a numerical method as h approaches zero. As for the derived MFDFBM, when $h \to 0$, the formulas reduce to the one that can be written separately as

$$\bar{M}_{11}\bar{V}_{11} - \bar{M}_{12}\bar{V}_{12} = 0, \quad (3.4)$$
$$\bar{M}_{21}\bar{V}'_{21} - \bar{M}_{22}\bar{V}'_{22} = 0, \quad (3.5)$$
$$\bar{M}_{31}\bar{V}''_{31} - \bar{M}_{32}\bar{V}''_{32} = 0, \quad (3.6)$$

where

$$\bar{V}_{11} = (u_{n+4}, u_{n+3}, u_{n+2}, u_{n+1}), \bar{V}_{12} = (u_n, u_{n-3}, u_{n-2}, u_{n-1}),$$
$$\bar{V}'_{21} = (u'_{n+4}, u'_{n+3}, u'_{n+2}, u'_{n+1}), \bar{V}'_{22} = (u'_n, u'_{n-3}, u'_{n-2}, u'_{n-1}),$$
$$\bar{V}'_{31} = (u''_{n+4}, u''_{n+3}, u''_{n+2}, u''_{n+1}), \bar{V}''_{32} = (u''_n, u''_{n-3}, u''_{n-2}, u''_{n-1}),$$

$\bar{M}_{11} = \bar{M}_{21} = \bar{M}_{31}$ and $\bar{M}_{12} = \bar{M}_{22} = \bar{M}_{32}$ are non empty matrices defined respectively as

$$\bar{M}_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \bar{M}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let us consider the first characteristic polynomial $q(t)$ given as

$$q(y) = |\bar{M}_{11}\, t - \bar{M}_{12}| \quad (3.7)$$

and satisfies $|t_i| \leqslant 1$, if the roots for which $|t_i| = 1$ is not repeated more than three, then MFDFBM is said to be zero-stable (see [24, 25]). Applying (3.7) to (3.4), (3.5) and (3.6) yields $q(t_i) = (t_i - 1)t_i^3$, $i = 1, 2, 3$ hence the MFDFBM is zero-stable. According to Henrici [26], for a multistep method to be convergent, it has to be consistent and zero-stable. Since MFDFBM satisfies the two conditions, it is convergent.

3.3. Region of Absolute Stability of the MFDFBM

As previously stated, the zero-stability of a numerical technique refers to how the numerical method behaves when $h \to 0$. $h > 0$ is a common occurrence in practice (see Refs. [22, 25, 27]). We need a definition of stability other than zero-stability to assess whether a numerical technique will generate acceptable results for a given value of $h > 0$. Since the proposed methods are designed for solving general third-order differential equations, The analysis of the linear stability is studied by considering the test problems

$$y''' = -\left(3y''\lambda + 3y'\lambda^2 + y\lambda^3\right) \tag{3.8}$$

which includes first, second, and third derivatives since the scheme is designed to general third order ODEs. The test problem has a bounded solution for $\lambda > 0$ which tend to be zero for $x \to \infty$, and thus the numerical solution is also bounded. The region in the complex λh-plane in which the numerical method imitates the behavior of the true solutions is known as the region of absolute stability. We start the linear stability analysis by applying formulas presented in Table 1 to the test problem (3.8). This implies f_{n+i} and g_{n+i} are equated to

$$f_{n+i} = -\left(3y''_{n+i}\lambda + 3y'_{n+i}\lambda^2 + y_{n+i}\lambda^3\right) \tag{3.9}$$

and

$$g_{n+i} = \left(6y''_{n+i}\lambda^2 + 3y'_{n+i}\lambda^3 + 3y_{n+i}\lambda^4\right). \tag{3.10}$$

The procedure yields the system of equations that can be written as:

$$\Gamma\, Y_m = \Pi\, Y_n \tag{3.11}$$

where Γ and Π are matrices of dimension 14 whose entries are too large to be documented in this report, $z = \lambda h$,

$$Y_m = \left(y_{n+1},\, y_{n+2}, \dots, y_{n+k}, y'_{n+1},\, y'_{n+2}, \dots, y'_{n+k},\, y''_{n+1}\; y''_{n+2}, \dots, y''_{n+k}\right)^T,$$

and

$$Y_n = \left(y_{n-1},\, y_{n-2}, \dots, y_n, y'_{n-1},\, y'_{n-2}, \dots, y'_n,\, y''_{n-1}\; y''_{n-2}, \dots, y''_n\right)^T.$$

We study the boundedness of their solutions through the eigenvalues of the stability matrix

$$M(z) = \Gamma^{-1}\Pi. \tag{3.12}$$

The absolute values of these eigenvalues must be less than 1 for the method to be stable. If $\lambda \in R$, then the absolute stability region is reduced to a real interval called an interval of stability. The above procedures are applied to the proposed methods with step lengths four. The absolute stability region of MFDFBM is presented in the Figure 1 below. Figure 1 shows the stability region for the proposed method, being the primary stability interval $(0, \infty)$.

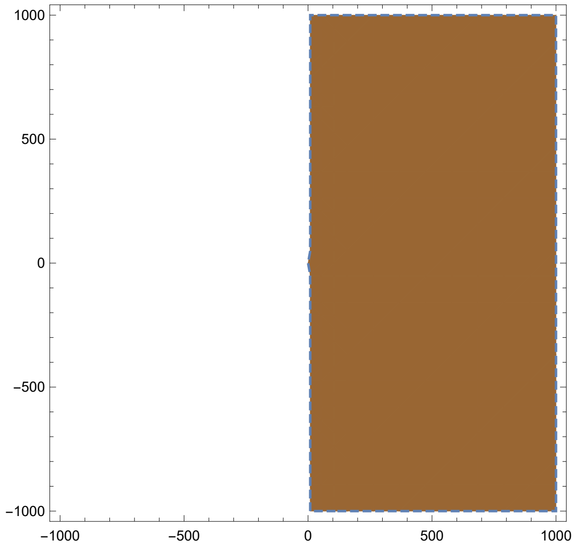

Figure 1: Region of absolute stability of the four step method

4. Detail implementation strategies of MFDFBM

The MFDFBM is applied to solve third-order initial value problem of ordinary differential equation in a block version. This is achieved by writing the formalas of MFDFBM in Table 1 in form of $W(v) = 0$ on each interval $[x_n,\ x_{n+4}]$, $n = 0, 4, ..., N-2$ with N divisible by four. The following are solution values that are obtained uniquely

$$\bar{\mathbf{V}} = \left(v_0,\, v_1,\, v_2, \ldots,\, v_N,\, v_0',\, v_2',\, v_1', \ldots,\, v_N',\, v_0'',\, v_2'',\, v_1'', \ldots,\, v_N''\right).$$

Since the result system is implicit, the Newton's algorithm is given as

$$\bar{\mathbf{V}}^{i+1} = \bar{\mathbf{V}}^{i} - \frac{W^i}{\mathbf{J}^i},$$

where $\mathbf{J}$ is the Jacobian matrix of W. To kick start the Newton's process, approximations provided by Taylor series expansion up to fourth terms are utilized:

$$\begin{aligned} v_{n+j} &= v_n + (jh)v_n' + \frac{(jh)^2}{2}v_n'' + \frac{(jh)^3}{6}w_n, \\ v_{n+j}' &= v_n' + (jh)v_n'' + \frac{(jh)^2}{2}w_n, \\ v_{n+j}'' &= v_n'' + (jh)w_n \quad n = 0, 1,\ 2, ..., N. \end{aligned}$$

5. Numerical application of MFDFBM to test problems and results

This section made available numerical experiment with application of MFDFBM to some test problems and presentation of our findings.

- FDTHBS: The fourth-derivative two-step hybrid block strategy in [27].
- ITBDM: An implicit three-point block direct method of order $p = 9$ in [28].

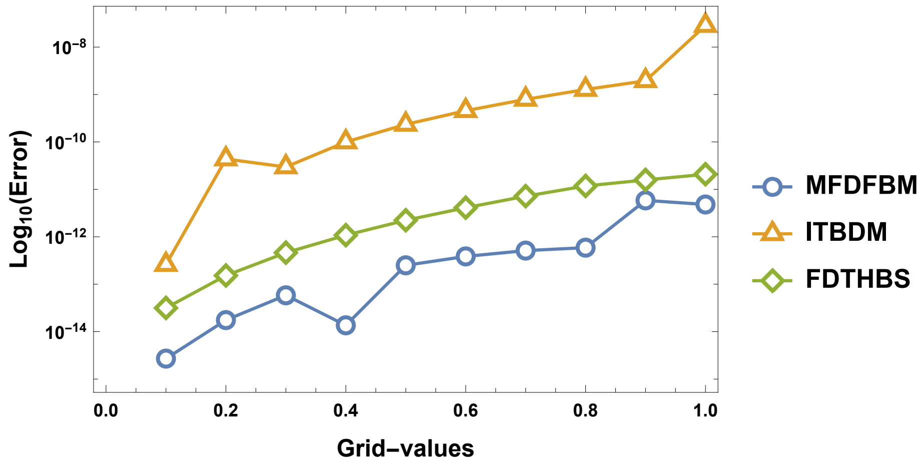

Figure 2: The plot of the absolute errors on numerical test problem 1

5.1. *Numerical test problem 1*

Consider the non-homogeneous linear problem

$$v'''(x) = 34\,x\,e^{(-2x)} - 16\,e^{(-2x)} + 2v''(x) + 3v'(x) - 10u(x) - 10x^2 + 6x + 34$$

$u(0) = 0, v'(0) = 3, v''(0) = 0$, whose exact solution is given as $u(x) = x^2e^{(-2x)} - x^2 + 3$. This problem has recently appeared [27] and [28]. The solution of numerical test problem 1 is considered within $[0,\ 1]$ for $h = 0.1$. The results are presented in Table 2, showing the exact and approximate solution obtained using MEDEBM. It is evident in Table 2 that the approximate and exact solutions agree up to at least twelve decimal places. The plot

Table 2: Results of numerical test problem 1 using MFDFBM

x-value	y-exact	y-computed	Error with MFDFBM
0.1	2.9981925591438	2.99819113196516	2.69880 e-15
0.2	2.9868155914388	2.98681799068347	1.75158 e-14
0.3	2.9593933746054	2.95939053369829	5.83707 e-14
0.4	2.9118955353240	2.91189596122658	1.35992 e-14
0.5	2.8419794877870	2.84197579912396	2.50212 e-13
0.6	2.7484389638190	2.74843979893671	3.85408 e-13
0.7	2.6308315502050	2.63083831025026	5.11291 e-13
0.8	2.4892180198840	2.48921570587050	5.89641 e-13
0.9	2.3238977064480	2.32389180435244	5.85475 e-12
1.0	2.1353481847970	2.13534353317941	4.79478 e-12

of the absolute errors with MFDFBM on the logarithm scale are compared with those of FDTHBS in Ref. [27] and ITBDM in Ref. [28] are shown in Figure 2. MFDFBM showed good accuracy over the other two methods when applied to numerical test 1.

5.2. Numerical test problem 2

Nonlinear third-order initial value problem

$$v = \frac{1+2\sin}{{}^{2}(v)\cos^{5}(v)},$$

$v(0) = v''(0) = 0,\ v'(0) = 1$ is also considered over the interval $0 \leqslant x \leqslant \frac{\pi}{4}$ for eighty iterations. Table 3 shows the comparison of the absolute errors generated by solving the

Table 3: Results of numerical test problem 2 using MFDFBM

x-value	Error with MFDFBM	Error with ITPBO9	Error With [29]
0.1	2.77556×10^{-17}	0	5.5511×10^{-17}
0.2	0	5.551115×10^{-17}	8.3266×10^{-17}
0.3	2.22045×10^{-16}	1.110223×10^{-16}	5.5511×10^{-17}
0.4	2.77556×10^{-16}	3.330669×10^{-16}	2.7755×10^{-16}
0.5	5.55112×10^{-16}	4.440892×10^{-16}	2.2204×10^{-16}
0.6	7.77156×10^{-16}	4.440892×10^{-16}	2.2204×10^{-16}
0.7	8.88178×10^{-16}	5.551115×10^{-16}	6.6613×10^{-16}
0.8	1.66533×10^{-15}	8.881784×10^{-16}	1.6653×10^{-15}

test problem 2 with MFDFBM, ITPBO9 and ninth order method in [29]. The results of the methods agreed with the exact solution up to fifteen decimal digit indicating their good performance to solve non-linear problems. In order to perform a thorough analysis of the

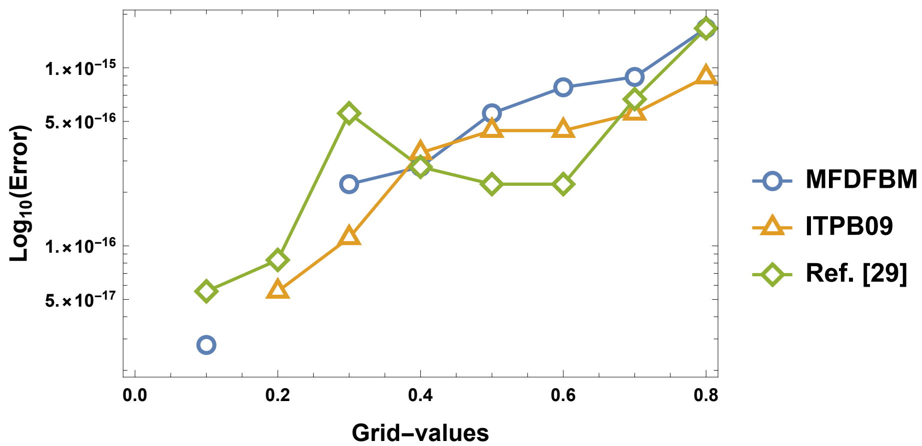

Figure 3: The plot of the absolute errors on numerical test problem 2

data presented in Table 3, a graphical representation of the results has been created. The plotted data can be found in Figure3 and used to gain a more comprehensive understanding of the findings. By visually representing the data in this way, it is easier to see that the suggested method compared well with both methods in ITPB09 and Ref. [29].

5.3. Numerical test problem 3

Another problem considered is initial value problem

$$v'''(x) = 3\sin(x),$$

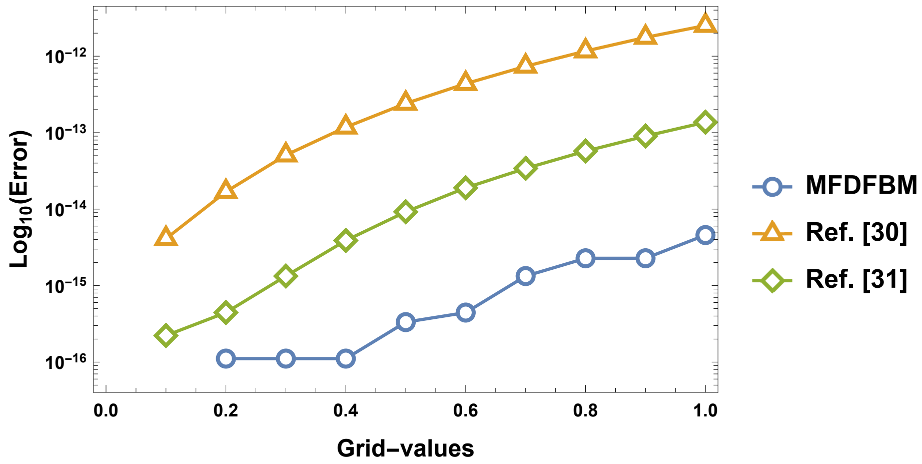

Figure 4: The plot of the absolute errors on numerical test problem 3

with initial conditions $\nu(0) = 1, \nu'(0) = 0, \nu''(0) = -2$. The exact solution is obtained as $\nu(x) = 3\ \cos(x) + \frac{x^2}{2} - 2$. Authors such as [30] and [31] have previously considered numerical test problem 3. The numerical results of test problem are as presented in Table 4. The numerical results coincide with the analytical solutions up to fifteen decimal places indicating the good performance of the present method. The comparison of the absolute errors of the proposed method MFDFBM with two cited methods [30, 31] presented in Figure 4 clearly showed it good performance.

Table 4: Numerical results of test problem 3 with $h = 0.1$

x-value	y-exact-solution	y-computed solution	Error with MFDFBM
0.1	0.990012375742659	0.990012837769113	0
0.2	0.960200167955914	0.960200731599065	1.11022 e-16
0.3	0.911009818705337	0.911009818705336	1.11022 e-16
0.4	0.843183320159553	0.843183320596122	1.11022 e-16
0.5	0.757748531057948	0.757748315799123	3.33067 e-16
0.6	0.656007812197896	0.656007129798936	4.44089 e-16
0.7	0.539527585583155	0.539527858310250	1.33227 e-15
0.8	0.410120329057801	0.410120295705870	2.27596 e-15
0.9	0.269830570018770	0.269830701804352	2.77556 e-15
1.0	0.120907879935818	0.120907793533179	4.57967 e-15

5.4. Numerical test problem 4

The initial value problem of the form

$$\nu'''(x) = -6y^4(x),$$

with initial conditions $\nu(0) = -\frac{1}{2}, \nu'(0) = 0, \nu''(0) = -\frac{21}{4}$. The exact solution is obtained as $\nu(x) = \frac{1}{x-2}$. The results of the numerical experiment on test problem 4 are presented in Table 5, indicating that the numerical solution by MFDFBM agrees well with the exact

Table 5: Numerical results of test problem 4 with $h = 0.1$

x-value	y-computed solution	y-exact-solution	Error with MFDFBM
0.	-0.5	-0.5	0.
0.1	-0.5263157894736835	-0.5263157894736842	7.00331×10^{-16}
0.2	-0.5555555555555513	, -0.5555555555555556	4.25842×10^{-15}
0.3	-0.5882352941176322	-0.5882352941176471	1.48448×10^{-14}
0.4	-0.6249999999999591	-0.625	4.08330×10^{-14}
0.5	-0.6666666666665669	-0.6666666666666666	9.97854×10^{-14}
0.6	-0.7142857142854839	, -0.7142857142857143	2.30379×10^{-13}
0.7	-0.7692307692302466	-0.7692307692307693	5.22644×10^{-13}
0.8	-0.8333333333321333	-0.8333333333333334	1.19998×10^{-12}
0.9	-0.9090909090880492	-0.9090909090909091	2.85981×10^{-12}
1.	-0.9999999999927517	-1.	7.24829×10^{-12}

solution up to twelve decimal digits. This further shows the MFDFBM's capability to solve non-linear problems with higher accuracy.

6. Further application of MFDFBM

MFDFBM is applied in this section to solve two well known nonlinear physical problems which are playing important roles in applications.

6.1. Nonlinear Jerk Equation

The nonlinear third-order initial value problem

$$v''' = -v' + v v' v'', \quad v(0) = v''(0), v'(0) = \mathbf{D},$$

known as Jerk equation is considered as one of the applied test problems. The exact solution for this test problem , according to [32], is given as

$$v(x) = \frac{\mathbf{D}}{\Pi}\sin(\Pi x) + \frac{\mathbf{D}}{96\Pi^3}\left(\left(-9\mathbf{D}^2\Pi^2 - 48 + 48\Pi^2\right)\sin(\Pi x) - \mathbf{D}^2\sin(3\pi x)\right),$$

with $\Pi = \frac{1}{2}\sqrt{\mathbf{D}^2 + 4}$. The problem is integrated in the interval $[0, 10]$ with $h = 0.1$ and $\mathbf{D} = 0.2$ over 25 iteration. The results of the Jerk equation are presented in Table 6. The errors generated by MFDFBM are presented in column four of Table 6, which indicates that the results agree with the exact solution up to at least seven decimal places. The solutions curve in Figure 5 represents the results of the Jerk equation in Table 6. The dots represent the solution given by MFDFBM, while the dashed line denotes the results by NSolve. Figure 6 shows the behaviour of the absolute errors over the integration interval.

6.2. Nonlinear Genesio equation

The initial value problem of ordinary differential equation termed the Genesio equation

$$v''' = -\mu_a v'' - \mu_b v' - \mu_c v(x) + v^2(x), \quad v(0) = 0.2, \ v'(0) = -0.3, \ v''(0) = 0.1,$$

is also considered as test problem. As shown in Refs. [28, 33], the exact solution to this problem is not yet available. The results are compared in Table 7 with $h = 0.1$.

Table 6: Numerical results of Jerk equation with $h = 0.1$

x	y-approx	y-Exact	y-Error
0.	0	0.	0.
0.1	0.019966681999368886	0.01996668198459112	1.4778×10^{-11}
0.2	0.03973382393021961	0.03973382346610275	4.6412×10^{-10}
0.3	0.05910372477557912	0.059103721390005665	3.3856×10^{-09}
0.4	0.07788235842629682	0.07788234495321575	1.3473×10^{-08}
0.5	0.09588120188856142	0.09588116370825571	3.8180×10^{-08}
0.6	0.11291904897565924	0.11291896224434265	8.6731×10^{-08}
0.7	0.12882379926084436	0.128823631059993	1.6820×10^{-07}
0.8	0.14343420806992968	0.1434339189703123	2.8910×10^{-07}
0.9	0.15660157903117822	0.15660112796047343	4.5107×10^{-07}
1.	0.16819137660821634	0.16819072728452114	6.4932×10^{-07}

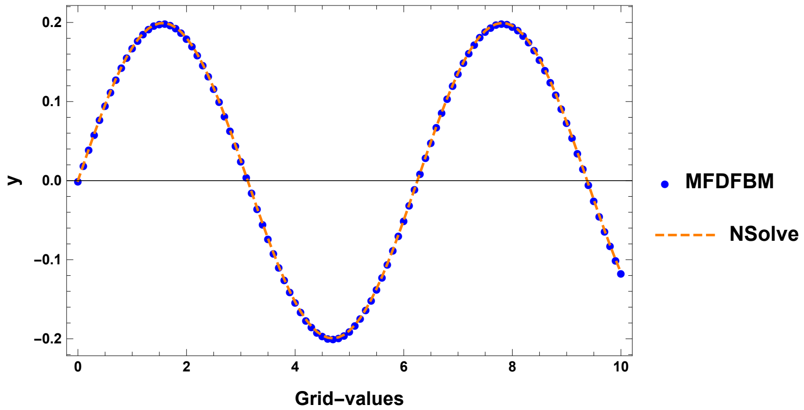

Figure 5: Solution curve of the Jerk eqaution as compared with MFDFBM

6.3. Non-linear Homogeneous Thin Film Flow (NHTFF) problem

Lastly, we consider the initial value problem

$$v''' = \frac{1 + v_a + v_a^2}{v(x)^2} - \frac{v_a + v_a^2}{v(x)^3} - 1,$$

which represents the well-known fluid wet surface drainage problem according to [27] and $v_a > 0$ denotes the small film thickness. The solution obtained for $v_a = 0.1$ in the interval $[0, 5]$ over ten iteration. Figure 8 compares the results of MFDFBM and `Nsolve`. As shown in the plot legend, the dots represent the numerical solutions by MFDFBM, while the dashed line represents the solution of the curve obtained using `Nsolve`. The results agree up to at least eight decimal places as the difference between the results at $b = 5$ is 8.49347×10^{-8}.

7. Conclusion

In this article, a block technique called MFDFBM is introduced and used to provide numerical solutions to third-order IVPs. MFDFBM is p-stable, has order eight, zero-stable,

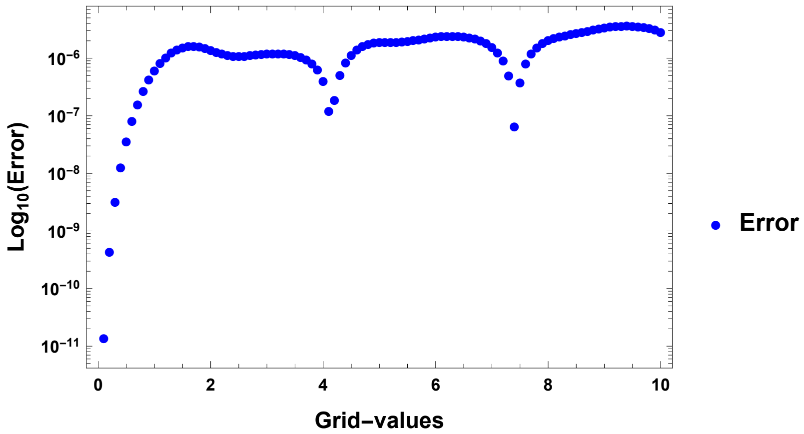

Figure 6: Logarithm representation of the absolute error of MFDFBM on Jerk equation

Table 7: Numerical results Genesio equation

b	h	Method	Step	Computed solution	ΔB
1	0.1	MFDFBM	4	-0.05400408355609352‘	3.64961×10^{-9}
		ITPBO9	4	-0.0540040835391235	3.63398×10^{-9}
		NDSolve	10	-0.0540040799051468	0
4	0.1	MFDFBM	10	-0.06763060470524687‘	7.0805×10^{-9}
		ITPBO9	13	-0.0676306051287455	7.5040×10^{-9}
		NDSolve	40	-0.0676380593281975	0

consistent and convergent. The Genesio equation, Jerk equation and kinds of non-linear homogeneous thin film flow problems are among the seven benchmark problems that have been solved to show the effectiveness and applicability of the MFDFBM in addressing practical issues in Engineering and Applied Sciences. The numerical results in Tables 2-7 and Figures 2-8 provide evidence that the suggested approach effectively resolves the problem of the kind in (1.1).

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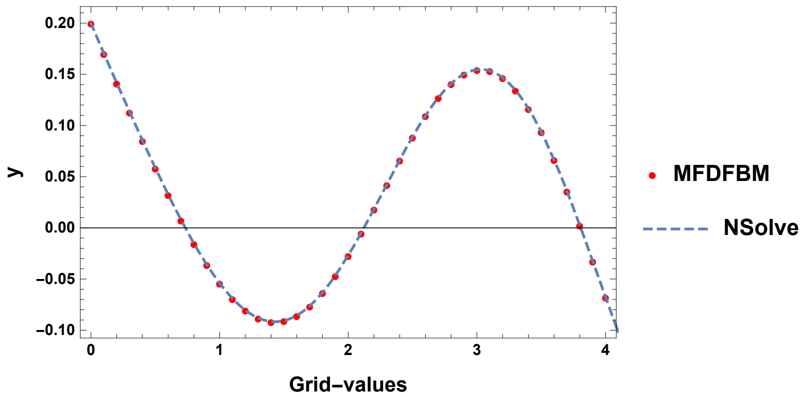

Figure 7: Comparison of the solution by MFDFBM wuth those of NDSolve Genesio equation

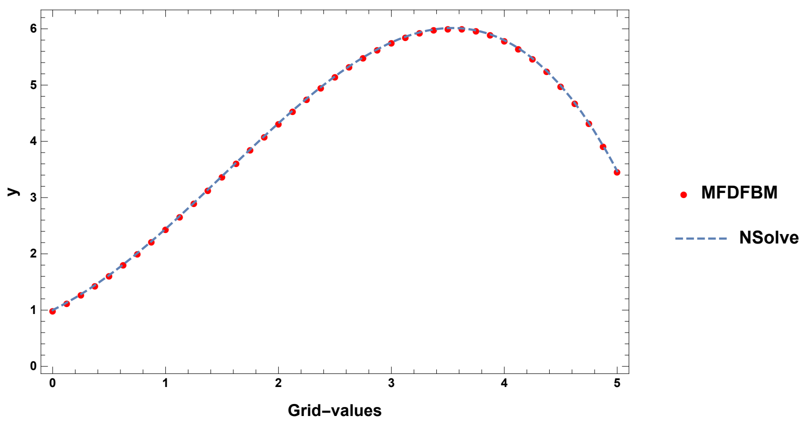

Figure 8: Solution curve of the test problem NHTFF using MFDFBM and NDSolve on Thin Film Flow (NHTFF) problem

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