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Deterministic Model and Analysis of Fuel Subsidy in Nigeria Commodity Market Dynamics

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Abstract

The Nigerian populace's anxieties, pessimism, and optimism over removing fuel subsidies have brought economic tension. The conflicting views are centered on the overbearing effect of subsidy removal on the cost of living and the commodity market. The federal government has hinged the decision on the excessive revenue leakages due to the huge subsidy and the need to re-channel the subsidy fund to more developmental projects impacting the Nigerian economy's growth. Trade union and civil society hinge their opposition to the removal of fuel subsidies based on their overbearing effect on the livelihood of citizens in terms of the subsidy-induced high cost of living and production. In light of this commotion, there is a need to study Nigeria's fuel subsidy, consumer purchasing power, and commodity market dynamics using mathematical modeling and analysis. This paper proposes a deterministic model to study the dynamics of fuel subsidy, consumer purchasing power, oil-pirating groups, and commodity markets. To gain insight into the impact of oil leaks on the government oil revenue, the time delay is used to depict the oil theft control by the Nigerian government. Analytically, three steady-states, namely subsidy-free, pirate-free, and critical steady-states, are obtained, and the conditions for their existence are determined and analyzed. The findings of this work highlight veritable conditions for acceptable implementation of these states. The analytical results were numerically verified, and the dynamics under these states were demonstrated graphically. The work further recommends the conditions for the acceptable implementation of fuel subsidy removal and the blockage of oil thefts.

Keywords: Government Oil Revenue, Consumers Purchasing Power, Commodity Markets, Deterministic Model, Time Delay.

1. Introduction

Nigeria is one of the leading crude oil importing countries in the world and the second importer of crude oil in Sub-Saharan Africa [1]. Despite the status as the largest producer of crude oil, the Nigerian crude oil refineries that could have catered for local

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consumption have been in a moribund state for years [2]. This has resulted in the Nigerian government's reliance on European fuel importation to cater to local consumption [3]. With the alarming rate of poverty, high unemployment, and uneven equality in income in Nigeria, the local consumption of imported crude oil products is subsidized to ensure the products are avoidable to the Nigerian populace [4, 5]. The determination of local price for imported crude oil products was done by the Nigerian Petroleum Products Pricing Regulatory Agency (PPPRA) through estimation of possible gaps between the expected price of imported petroleum products and domestic prices indicators [6].

Fuel is an important factor in Nigerian economic growth as its significant roles are prominent in good and service productions, including livelihood and the poor populace participation in the economic activities [7]. The support arguments for subsidy in most developing countries, including Nigeria, hinge on the need for the state to fulfill the implicit social contract [8, 9]. However, the fluctuations in crude oil price globally in the last decades and the subsequent fluctuations in Nigerian government oil revenue led to an increase in calls for subsidy reform as arguments are advanced against the sustenance of fuel subsidy [1, 10]. There are several arguments for the fuel subsidy reform based on the need for efficiency and equity considerations [11], and most importantly, its overbearing fiscal implications [2]. Fuel subsidy has been argued to be characterized by Nigerian income leakage as it benefits minority groups rather than the target poor population [12, 13, 14].

However, the nexus between fuel subsidies and the social contract has been a major challenge facing the removal of fuel subsidies. The removal of fuel subsidy by the Nigerian government without clear social welfare palliatives is viewed by the labor unions and social activists as a unilateral change in the social contract, which might degenerate into labor unrest and protest against the government [6, 15, 16]. The impact of the removal of fuel subsidy on both the consumer price index, local markets, and the livelihood of the citizens has been so problematic [17, 18, 19]. The successful implementation of subsidy removal must hinge on the government's capacity to engage stakeholders and factor their concerns into the subsidy removal strategy policy [20, 21].

The dwindling Nigerian oil income has also been attributed to the activities of pirates and pipeline vandalism in the Niger Delta region of Nigeria. Oil theft accounts for the loss of 470,000 barrels of crude oil daily, amounting to 700 million U.S. dollars monthly [22, 23]. The growing rate of oil theft is leaked to security compromise and criminal tendencies of oil traders, both local and international, which has led to loss of human and material resources, fall in oil revenue, and increase in state insecurity, among others [24]. There have been number of studies recently on the dynamics of Nigerian fuel subsidy see [25, 26, 27].

Mathematics model has been exploited in unraveling the dynamics of biological phenomena including coronavirus see [28, 29], cancers [30, 31, 32], Measles [33] among others. Also, the uses of Mathematics modeling in studying the periodic heat transfer as in [34], fractional electric circuits as in [35]. Recently, Mathematical models have also been used in studying the impact of subsidy dynamics in the Nigerian economy: Odior [36] assess the likelihood of 142.8 percent (₦200 Per liter) subsidy removal on both the Nigerian welfare and macroeconomic using a Structuralist Computable General-Equilibrium (CGE) model to simulate the effect of fuel subsidy removal between 2015 – 2020. Their findings indicate that, though government income will grow positively due to the increase in PMS price to ₦200 Per liter, the income and consumption losses and worsening inflation will characterize the economy. Omotosho et al.[37] proposed a New-Keynesian DSGE model to study the impacts of oil price shock and fuel subsidy regime on macroeconomics in Nigeria. Their results highlight that removing fuel subsidies might lead to "higher macroeconomic instabilities and generates non-trivial implications for the response of monetary policy to an oil price shock". They recommend that for a successful fuel subsidy removal, there should be well-factored safety measures and mechanisms.

In the wake of the unrest generated by the fuel subsidy removal, this paper (unlike others cited above whose analyses are stochastic) proposes a deterministic model to study the impact of fuel subsidy on the dynamics of Nigerian government income, the consumer purchasing power, the oil-pirating groups, and the commodity markets. The choice of a deterministic model is to avoid arbitrary selection of performances or variables and ensure that the findings are determined by the model's parameters and initial values. Section 2 of this paper includes formulating and analyzing the deterministic model for fuel-subsidy impacts on the dynamics of Nigerian government income, consumer purchasing power, the oil-pirating groups, and the commodity markets. Also, in Section 2, a time delay is used to depict Nigerian securities' control of oil pirates. The analysis of the proposed model is done in Section 3, which includes the existence of positive solutions and the existence and stability analysis of steady-states. The analytical results obtained in Section 3 are numerically verified in Section 4 using MATLAB DDE23 solver. The discussion is provided to sum up the work in Section 5. Findings, conclusions, and recommendations are made in Section 6.

2. Model formulation and Analysis

Nigeria media in recent months have been saturated by the overview of the fuel subsidy removal, the intended subsidy palliative measures by the government, the outcries by trade unions and civil activists, and public analysts' expert opinions. The appraisal of the divergent views of these players highlights the need to search for an amicable modality that will integrate all players' opinions into the government policy on subsidy removal, such that the result will be a win-win outcome. In the wake of the unrest generated by the fuel subsidy removal, this paper proposes a deterministic model to study the dynamics of the fuel subsidy, the consumer purchasing power, the oil-pirating groups, and the commodity markets based on the assumptions given below:

- That the Nigerian government's oil revenue is determined by its crude oil production rate.
- That there exists crude oil pirates' encroachment in Nigeria, whose operation undermines the crude oil revenue of the Nigerian government.
- There is subsidy intervention on oil products to make it avoidable to the teaming population.
- That the effect of this subsidy reduces the cost of production of local goods and transportation.

- There is a relative stagnation in consumer purchasing income due to economic challenges arising from COVID and the Central Bank of Nigeria's cashless policy.
- That the commercial market growth is based on production and sales.
- That the pirates' group proliferates to become a cartel if their activities are unchecked.

Let R represents the Nigeria government oil revenue, C represents the consumer purchasing power, P depicts the pirate groups and the M represents commodity market. Since the oil thefts by pirate groups do not go unchecked by the Nigerian security forces, the time delay is used to model the control of oil theft. This will help incorporate security control measures in curbing oil-theft rather than the mere depletion of oil theft on Nigerian income. Therefore, the rate of changes of $R,\ C,\ P$ and M are defined by the interactive and deterministic delay model as given below:

$$\begin{aligned}
\frac{dR}{dt} &= \alpha R(t) - \beta_1 R(t)C(t) - \upsilon_1 R(t-\tau)P(t-\tau) - \theta R(t),\\
\frac{dC}{dt} &= \sigma C(t) + \beta_2 R(t)C(t) - \upsilon_2 C(t)M(t) - \vartheta C(t),\\
\frac{dP}{dt} &= \phi P(t) + \upsilon_3 R(t-\tau)P(t-\tau) - \rho P(t)\\
\frac{dM}{dt} &= \omega M(t) - \Omega M(t).
\end{aligned} \tag{2.1}$$

with initial conditions:

$$R_0 = \phi_1(\xi),\ \ C_0 = \phi_2(\xi),\ \ P_0 = \phi_3(\xi),\ M_0 = \phi_4(\xi)\ \forall\ \xi \in [-\tau, 0].$$

The first equation in (1) is the rate of changes in government oil revenue with the first term depicting the growth rate of oil income by the Nigerian government (α), the second term is the rate of oil income depletion due to subsidy (β_1), the third is the rate of depletion of oil-income due to oil-theft by pirates (υ_1) which is subject to control τ and the last term describes other oil-income depletion rates due to sabotage among others (θ).

The second equation of (1) is the rate of changes in the consumer purchasing power with the first term representing the average consumer's income rate (σ), the second term referring to the expenses' recovery rate due to government subsidy intervention (β_2), the third is the depletion rate of their income due to purchase of needed commodities (υ_2). The last term refers to other depletion in the consumer income due to savings or other standing commitments (ϑ).

The third equation in (1) represents the rate of changes in the oil pirates' incomes with the first term describing the income proliferation rate of pirates (ϕ), the second is the income rate of proceeds from oil thefts(υ_3) which is subject to security control τ, the third term is the depletion rate of pirates incomes (ρ) due to other factors.

The fourth equation in (1) depicts the commercial market dynamics with the first term representing the production rate (ω) and the last term representing the sales rate (Ω).

3. Model Analysis

3.1. Existence of Non-negative Solution

Here, the existence of non-negativity of the model solutions is verified so as to guarantee that the results obtained are dynamically meaningful and conform with the model assumptions. Therefore, the Lemma 3.1 below captures the existence of non-negativity of the solution for System (2.1).

Lemma 3.1. *Given that the initial conditions for System (2.1) are non-negative* $\forall\ \xi \in [-\tau, 0]$. *Then, there exist a non-negative solutions for* $R(t),\ C(t),\ p(t)$ *and* $M(t)$ *of System (2.1)* $\forall\ \xi \in [-\tau, 0]$.

Proof: Solving for $R(t),\ C(t),\ p(t)$ and $M(t)$ in (1), firstly, the first equation in System (2.1) is re-arranged to solve $R(t)$ as follows:

$$\frac{dR}{dt} - [\alpha - \beta_1 C(t) - \theta]R(t) = -\upsilon_1 R(t-\tau)P(t-\tau). \tag{3.1}$$

To solve Equation (3.1), we make use of its integrating factor defined as:

$$I_R(t) = e^{-[\alpha - \beta_1 C(t) - \theta]t}. \tag{3.2}$$

Multiplying equation (3.1) through by $I_R(t)$ yields

$$\frac{dR}{dt}e^{-[\alpha-\beta_1 C(t)-\theta]t} - [\alpha - \beta C(t) - \theta]R(t)e^{-[\alpha-\beta_1 C(t)-\theta]t} = -\upsilon_1 R(t-\tau)P(t-\tau)e^{-[\alpha-\beta_1 C(t)-\theta]t},$$
$$\equiv \frac{d}{dt}\left[e^{-[\alpha-\beta_1 C(t)-\theta]t}R(t)\right] = -\upsilon_1 R(t-\tau)P(t-\tau)e^{-[\alpha-\beta_1 C(t)-\theta]t}. \tag{3.3}$$

Integrating Equation (3.3) with respect to "t" in the interval $[0, t]$ yields

$$R(t) = \phi_1(\xi)e^{[\alpha+\beta_1 y(t)+\upsilon_1 z(t)-\theta]t} - \upsilon_1 \int_{-\tau}^{0} R(t-\tau)P(t-\tau)dt, \tag{3.4}$$

Since

$$\int_{-\tau}^{0} R(t-\tau)P(t-\tau)dt = \int_{t-\tau}^{t} R(\xi)P(\xi)d\xi, \tag{3.5}$$

then (3.4) takes the form:

$$R(t) = \phi_1(\xi)\exp[\alpha + \beta_1 y(t) + \upsilon_1 z(t) - \theta]t - \upsilon_1 \int_{t-\tau}^{t} R(\xi)P(\xi)d\xi. \tag{3.6}$$

Obviously, $R(t)$ is non-negative if $\phi_1(\xi)$ is non-negative. The non-negativity of the second and the third equations in (2.1) is determined using the same process and the obtained solution of $C(t),\ P(t)$ and $M(t)$ are as given below:

$$\begin{aligned} C(t) &= [C_1 + \phi_2(\xi)]\exp -[\sigma + \beta_2 x(t) + \upsilon_2 z(t) - \vartheta]t, \\ P(t) &= \phi_3(\xi)\exp\,[\phi P(t) - \rho]t + \upsilon_3 \int_{t-\tau}^{\tau} R(\xi)P(\xi)d\xi, \\ M(t) &= [C_2 + \phi_4(\xi)]\exp\,[\omega - \Omega]t. \end{aligned} \tag{3.7}$$

Therefore, System (2.1) has non-negative solution for $R(t),\ C(t),\ P(t)$ and $M(t)$ $\forall\ \xi \in [-\tau, 0]$, if all the assumptions in Lemma (3.1) hold. □

3.2. Existence of Steady-state

The controversies surrounding fuel subsidy removal are hitched on finding a state of balance where all parties' arguments hold such that the argument for and against subsidy removal is harmonized in win-win outcomes. These balanced states in a dynamical system are referred to as steady states, and obtaining the conditions for arriving at such states is sacrosanct to finding harmonized win-win outcomes. Therefore, the conditions for the existence of steady-states for System (2.1) are weighed in three possible circumstances, namely oil-theft-free circumstance, fuel subsidy-free circumstance, and the critical circumstance where all variables are present as captured in Theorem (3.2) below:

Theorem 3.2. *Assume, there exist positive invariant states* $R^*,\ C^*,\ P^*$ *and* M^* *for System (2.1), then there exist oil-theft free steady-state, fuel-subsidy free steady-state and critical steady-state for System (2.1).*

Proof: Suppose the assumptions in Theorem (3.2) hold, then $R^*,\ C^*,\ P^*$ and M^* of System (2.1) satisfy the followings:

$$\begin{aligned}\alpha R^* - \beta_1 R^* C^* - \upsilon_1 R^{*+} P^{*+} - \theta R^* &= 0,\\ \sigma C^* + \beta_2 R^* C^* - \upsilon_2 C^* M^* - \vartheta C(t) &= 0,\\ \varphi P^* + \upsilon_3 R^{*+} P^{*+} - \rho P^* &= 0\\ \omega M^* - \Omega M^* &= 0.\end{aligned} \tag{3.8}$$

1. For oil-theft free steady-state, we set $P^* = 0$ in (3.8) and solve for $R^*,\ C^*$ and M^* to have:

$$(R^*, C^*, 0, M^*) = \left(\frac{[\vartheta + \upsilon_2 M^* - \sigma]}{\beta_2},\ \frac{[\alpha - \theta]}{\beta_1},\ 0,\ M^*\right). \tag{3.9}$$

   where M^* implies that if $M^* \neq 0$ then $\omega = \Omega$.

2. For the subsidy-free steady-state, we set β_n where $n = 1, 2.$ in Equation (3.8), and solve for $R^*,\ C^*,\ P^*$ and M^* to have:

$$\left(R^*,\ C^*,\ P^*,\ M^*\right) = \left(\frac{\rho - \varphi}{\upsilon_3},\ C^*,\ \frac{\alpha - \theta}{\upsilon_1},\ M^*\right) \tag{3.10}$$

   Where $C^* > 0$ and $M^* = 0$ implies $\omega = \Omega$.
3. For the critical steady-state, we solve for $R^*,\ C^*,\ P^*$ and M^* in Equation (3.8) to have:

$$\left(R^*,\ C^*,\ P^*,\ M^*\right) = \left(\frac{\vartheta + \upsilon_2 M^* - \sigma}{\beta_2},\ \frac{\theta + \upsilon_1 P^* - \alpha}{\beta_1},\ P^*,\ \frac{\beta_2[\rho - \varphi] - \upsilon_3[\vartheta - \sigma]}{\upsilon_2 \upsilon_3}\right), \tag{3.11}$$

   where $P^* \neq 0$ implies $R^* = \frac{\rho - \varphi}{\upsilon_3}$.
   Hence, there exist positive oil-theft-free, subsidy-free and critical steady-states for System (2.1) for all positive invariant $R^*,\ C^*,\ P^*,\ \&\ M^*$.

□

3.3. Local Stability Analysis

The arguments for and against the removal of fuel subsidy by all parties centered on the need to harmonize the concerns of all stakeholders and the possibility of blocking leakages occasioned by oil thefts and vandals. In the wake of these furors, we seek in this section to determine the status of the oil-theft-free, the subsidy-free, and the critical steady-states for System (2.1). To provide insight into the behaviors of System (2.1) around these steady-states. Linearizing System (2.1) around the steady-state, we obtain a characteristic equation of the form:

$$\begin{vmatrix} \lambda-(\alpha+\beta_1C^*-\theta-\upsilon_1P^*e^{-\lambda\tau}) & \beta_1R^* & -\upsilon_1R^*e^{-\lambda\tau} & 0 \\ \beta_2C^* & \lambda-[\sigma-\vartheta+\beta_2R^*-\upsilon_2M^*] & 0 & \upsilon_2C^* \\ \upsilon_3P^*e^{-\lambda\tau} & 0 & \lambda-[\phi+\upsilon_3R^*-\rho] & 0 \\ 0 & 0 & 0 & \lambda-[\omega-\Omega] \end{vmatrix}=0. \tag{3.12}$$

The simplified form of Equation (3.12) is

$$\begin{aligned} D(\lambda,\tau)=&[\lambda+[\Omega-\omega]]\Big[\lambda^3-\big([\sigma-\vartheta+\beta_2R^*-\upsilon_2M^*]+[\phi+\upsilon_3R^*-\rho]+(\alpha+\beta_1C^*-\theta \\ &-\upsilon_1P^*e^{-\lambda\tau})\big)\lambda^2+\Big([\sigma-\vartheta+\beta_2R^*-\upsilon_2M^*][\phi+\upsilon_3R^*-\rho]+(\alpha+\beta_1C^*-\theta \\ &-\upsilon_1P^*e^{-\lambda\tau})\big([\sigma-\vartheta+\beta_2R^*-\upsilon_2M^*]+[\phi+\upsilon_3R^*-\rho]\big)-\beta_1R^*\big[\beta_2C^*\big]\Big)\lambda- \\ &-(\alpha+\beta_1C^*-\theta-\upsilon_1P^*e^{-\lambda\tau})([\sigma-\vartheta+\beta_2R^*-\upsilon_2M^*][\phi+\upsilon_3R^*-\rho]) \\ &+\beta_1R^*\big[[\beta_2C^*][\phi+\upsilon_3R^*-\rho]\big]\Big]=0. \end{aligned} \tag{3.13}$$

Clearly, $\lambda_1=-[\Omega-\omega]$ remains a negative root of Equation (3.13) if $\Omega>\omega$, otherwise it unstable. This implies that commodity markets will fall if the sales' rate is less than productions' rate, and it will flourish if the sales' rate is greater than productions' rate. To ascertain the longtime behaviors of the other model's variables, we consider the cubic polynomial of (3.13) to obtain the other roots. Therefore, the longtime status of steady-states are defined by the following theorems:

Theorem 3.3. *The oil-theft free steady-state of System (2.1) is always unstable for a positive invariant.*

Proof: Substituting $(R^*,C^*,0,M^*)=\left(\frac{[\vartheta+\upsilon_2M^*-\sigma]}{\beta_2},\ \frac{[\alpha-\theta]}{\beta_1},\ 0,\ M^*\right)$ into Equation (3.13), the resulting cubic polynomial of (3.13) becomes:

$$\begin{aligned} D(\lambda,\tau)=\lambda^3-\Big([\phi+\upsilon_3\frac{[\vartheta+\upsilon_2M^*-\sigma]}{\beta_2}-\rho]\Big)\lambda^2+\Big([\phi+\upsilon_3\frac{[\vartheta+\upsilon_2M^*-\sigma]}{\beta_2}-\rho]- \\ [\vartheta+\upsilon_2M^*-\sigma][\alpha-\theta]\Big)\lambda+[\vartheta+\upsilon_2M^*-\sigma][\alpha-\theta][\phi+\upsilon_3\frac{[\vartheta+\upsilon_2M^*-\sigma]}{\beta_2}-\rho]=0. \end{aligned} \tag{3.14}$$

The simplified version of Equation (3.14) is

$$\lambda^3-h_1\lambda^2+h_2\lambda+h_3=0, \tag{3.15}$$

where

$$\begin{aligned}h_1 =&\phi + \upsilon_3 \frac{[\vartheta + \upsilon_2 M^* - \sigma]}{\beta_2} - \rho,\\ h_2 =&[\phi + \upsilon_3 \frac{[\vartheta + \upsilon_2 M^* - \sigma]}{\beta_2} - \rho] - [\vartheta + \upsilon_2 M^* - \sigma][\alpha - \theta],\\ h_3 =&[\vartheta + \upsilon_2 M^* - \sigma][\alpha - \theta][\phi + \upsilon_3 \frac{[\vartheta + \upsilon_2 M^* - \sigma]}{\beta_2} - \rho].\end{aligned} \tag{3.16}$$

By the Routh-Hurwitz criterion, Equation (3.14) has all negative roots if and only if

$$h_1 < 0, \;\; h_3 > 0, \; \& \; h_1 h_2 - h_3 > 0. \tag{3.17}$$

The conditions in (3.17) hold, if Equation (3.14) has all positive coefficients. This is only assured if and only if:

$$\begin{aligned}\phi &< \rho,\\ \vartheta + \upsilon_2 M^* &< \sigma,\\ \phi + \upsilon_3 \frac{[\vartheta + \upsilon_2 M^* - \sigma]}{\beta_2} - \rho &< [\vartheta + \upsilon_2 M^* - \sigma][\alpha - \theta].\end{aligned} \tag{3.18}$$

Since $\vartheta + \upsilon_2 M^* < \sigma$ negates existence of positive invariant R^* as initially assumed in Theorem (3.3), then the oil-theft free steady-state is always unstable. The economic implication of (3.18) is that: In an oil-theft-free dynamics, the consumer purchasing power is always sacrosanct to commodity markets, such that their income is greater than the cost of their needs. □

Theorem 3.4. *The subsidy-free steady-state of System (2.1) is always unstable for $\tau = 0$, and experiences a transcritical bifurcation as τ varies.*

Proof: Substituting $\left(R^*, \; C^*, \; P^*, \; M^*\right) = \left(\frac{\rho-\phi}{\upsilon_3}, \; C^*, \; \frac{\alpha-\theta}{\upsilon_1}, \; M^*\right)$ into Equation (3.13), the resulting cubic polynomial of (3.13) becomes:

$$\begin{aligned}D(\lambda, \tau) =& \lambda^3 - \big([\sigma - \vartheta + \beta_2 \frac{\rho - \phi}{\upsilon_3} - \upsilon_2 M^*] + [\alpha - \theta + \beta_1 C^* - [\alpha - \theta] e^{-\lambda\tau}]\big)\lambda^2 +\\ &\Big((\alpha - \theta + \beta_1 C^* - [\alpha - \theta] e^{-\lambda\tau})\big([\sigma - \vartheta + \beta_2 \frac{\rho - \phi}{\upsilon_3} - \upsilon_2 M^*]\big) - \beta_1 \frac{\rho - \phi}{\upsilon_3}[\beta_2 C^*]\Big)\lambda = 0.\end{aligned} \tag{3.19}$$

The simplified version of Equation (3.19) is

$$\lambda\Big(\lambda^2 + [f_{11} - f_{12}]\lambda + [f_{21} - f_{22}]\Big) = 0, \tag{3.20}$$

where

$$\begin{aligned}f_{11} =& [\sigma - \vartheta + \beta_2 \frac{\rho - \phi}{\upsilon_3} - \upsilon_2 M^*] + \alpha - \theta + \beta_1 C^*\\ f_{12} =& \alpha - \theta,\\ f_{21} =& (\alpha - \theta + \beta_1 C^*)\big([\sigma - \vartheta + \beta_2 \frac{\rho - \phi}{\upsilon_3} - \upsilon_2 M^*]\big) - \beta_1 \frac{\rho - \phi}{\upsilon_3}[\beta_2 C^*],\\ f_{22} =& [\alpha - \theta]\big([\sigma - \vartheta + \beta_2 \frac{\rho - \phi}{\upsilon_3} - \upsilon_2 M^*]\big).\end{aligned} \tag{3.21}$$

Obviously, Equation (3.20) has a root equal to zero on the imaginary axis. There is a need to determine whether or not the remaining roots of (3.20) are either positive or negative. This will suggest whether or not, the model variables regress or progress near the subsidy-free steady-state. Considering only the quadratic part of (3.20), we have

$$\lambda^2 + [f_{11} - f_{12}e^{-\lambda\tau}]\lambda + [f_{21} - f_{22}e^{-\lambda\tau}] = 0. \tag{3.22}$$

In order to assess the impact of oil theft on the dynamics of the model, we consider, the case without oil-theft control $\tau = 0$ and the case with control $\tau \geqslant 0$ in (3.22).

For $\tau = 0$*:* we set $\tau = 0$ in (3.22) to have

$$\lambda^2 + [f_{11} - f_{12}]\lambda + [f_{21} - f_{22}] = 0, \tag{3.23}$$

and solving using quadratic formula yield

$$\lambda_{1,2} = \frac{-[f_{11} - f_{12}] \pm \sqrt{[f_{11} - f_{12}]^2 - 4[f_{21} - f_{22}]}}{2}. \tag{3.24}$$

Where

$$\begin{aligned} f_{11} - f_{12} =& [\sigma - \vartheta + \beta_2 \frac{\rho - \phi}{\upsilon_3} - \upsilon_2 M^*] + \beta_1 C^* \\ f_{21} - f_{22} =& \beta_1 C^* [\sigma - \vartheta - \upsilon_2 M^*]. \end{aligned} \tag{3.25}$$

Obviously, $[f_1 - f_{12}]^2 > 4[f_{21} - f_{22}]$, and Equation (3.25) has two distinct real roots. Hence, the subsidy-free steady-state is also unstable for $\tau = 0$ that is when oil-theft is unchecked.

For $\tau > 0$*:* we set $\lambda = a + ik$ where $a = 0$ and k is a real number, and re-write the exponential in terms of trigonometric functions in Equation (3.22) to have:

$$-k^2 + [f_{11} - f_{12}[\cos(k\tau) - i\sin(k\tau)]]ik + [f_{21} - f_{22}[\cos(k\tau) - i\sin(k\tau)]] = 0. \tag{3.26}$$

Separating real and imaginary parts in (3.26) yields

$$\begin{aligned} -k^2 + f_{21} =& kf_{12}\sin(k\tau) + f_{22}\cos(k\tau) \\ kf_{11} =& -kf_{12}\cos(k\tau) - f_{22}\sin(k\tau). \end{aligned} \tag{3.27}$$

Squaring and adding up the equations in Equation (3.27) gives

$$k^4 + [2f_{21} + f_{11}^2 - f_{12}^2 - f_{22}^2]k^2 + f_{21}^2 = 0 \tag{3.28}$$

Letting $G = k^2$, then Equation (3.28) becomes

$$G^2 + [2f_{21} + f_{11}^2 - f_{12}^2 - f_{22}^2]G + f_{21}^2 = 0 \tag{3.29}$$

Computing (3.28) based on the definitions in (3.16) reveals that $f_{21} + f_{11}^2 - f_{12}^2 - f_{22}^2 > 0$ and $f_{21}^2 > 0$. Since $f_{21}^2 > 0$, Obviously, the roots of (3.16) are positive. This suggests that there exists an oil theft's control-induced transcritical bifurcation at τ^* where the subsidy-free steady state starts to be unstable and the model variables grow healthily. In the following, we derive the corresponding value of τ^* at which transcritical bifurcation occurs and the direction of the switching. Solving for τ^* in Equation (3.27) yields

$$\tau* = \frac{1}{k}\arcsin\Big(\frac{k^3 f_{12} - kf_{12}f_{21} - f_{22}kf_{11}}{f_{22}^2 - k^2 f_{12}^2}\Big). \tag{3.30}$$

To obtain the transversality condition for the occurrence of the transcritical bifurcation at $\tau = \tau^*$, we differentiate Equation (3.22) with respect to τ and obtain

$$2\lambda + f_{11} - f_{12}e^{-\lambda\tau} + \tau[f_{12}\lambda + f_{22}]e^{-\lambda\tau}\frac{\partial\lambda}{\partial\tau_1} = \lambda[f_{12}\lambda + f_{22}]e^{-\lambda\tau}. \tag{3.31}$$

Re-setting Equation (3.31) to reflect the changes in the form of τ gives

$$\left(\frac{\partial\lambda}{\partial\tau_1}\right)^{-1} = \frac{[2\lambda + f_{11}]e^{\lambda\tau}}{\lambda[f_{12}\lambda + f_{22}]} - \frac{f_{12}}{\lambda[f_{12}\lambda + f_{22}]} + \frac{\tau}{\lambda}. \tag{3.32}$$

From Equation (3.22),

$$e^{-\lambda\tau} = \frac{\lambda^2 + f_{11}\lambda + f_{21}}{f_{12}\lambda + f_{22}}. \tag{3.33}$$

Substituting Equation (3.33) into (3.31), we have

$$\left(\frac{\partial\lambda}{\partial\tau_1}\right)^{-1} = \frac{[2\lambda + f_{11}]}{\lambda[\lambda^2 + f_{11}\lambda + f_{21}]} - \frac{f_{12}}{\lambda[f_{12}\lambda + f_{22}]} + \frac{\tau}{\lambda}. \tag{3.34}$$

Hence,

$$\text{Sign}\left\{\text{Re}\frac{d\lambda}{d\tau_1}\right\}^{-1}_{\lambda=ui} = \text{Sign}\left\{\frac{[2\lambda + f_{11}]}{\lambda[\lambda^2 + f_{11}\lambda + f_{21}]} - \frac{f_{12}}{\lambda[f_{12}\lambda + f_{22}]} + \frac{\tau}{\lambda}\right\}_{\lambda=ik}. \tag{3.35}$$

Inserting $\lambda = ik$ into Equation (3.35) and putting $i^2 = -1$ yields

$$\text{Sign}\left\{\text{Re}\frac{d\lambda}{d\tau_1}\right\}^{-1}_{\lambda=ui} = \text{Sign}\left\{-\left[\frac{[2ik + f_{11}]}{[ik^3 + f_{11}k^2 - f_{21}ik]} - \frac{f_{12}}{[f_{12}k^2 - f_{22}ki]}\right]\right\}_{\lambda=ik}. < 0. \tag{3.36}$$

Since the sign for the real part of (3.34) at $\lambda = ik$ is negative, then, the transversality condition holds. Therefore, transcritical bifurcation occurs at $\tau = \tau^*$ where the model variables become unstable. □

Theorem 3.5. *The critical steady-state of System (2.1) is always unstable for $\tau \geqslant 0$.*

Proof: Substituting $\left(R^*,\ C^*,\ P^*,\ M^*\right) = \left(\frac{\vartheta+\upsilon_2 M^*-\sigma}{\beta_2}, \frac{\theta+\upsilon_1 P^*-\alpha}{\beta_1},\ P^*,\ \frac{\beta_2[\rho-\phi]-\upsilon_3[\vartheta-\sigma]}{\upsilon_2\upsilon_3}\right)$ into Equation (3.13), the resulting cubic polynomial of (3.13) becomes:

$$\begin{aligned} D(\lambda,\tau) =& \lambda^3 - \left(\upsilon_1 P^* - \upsilon_1 P^* e^{-\lambda\tau}\right)\lambda^2 + \left([\upsilon_1 P^* - \upsilon_1 P^* e^{-\lambda\tau}] - [\frac{\beta_2[\rho-\phi]}{\upsilon_3}][\theta + \upsilon_1 P^* - \alpha]\right)\lambda \\ &+ \frac{\beta_1\beta_2}{\upsilon_3}[\rho - \phi][\theta + \upsilon_1 P^* - \alpha]\rho = 0. \end{aligned} \tag{3.37}$$

Simplifying Equation (3.37) yields

$$\lambda^3 + [D_{11} - D_{12}e^{-\lambda\tau}]\lambda^2 + [D_{21} - D_{22}e^{-\lambda\tau}]\lambda + D_3 = 0 \tag{3.38}$$

where

$$\begin{aligned}
D_{11} &= \upsilon_1 P^*,\\
D_{12} &= \upsilon_1 P^*,\\
D_{21} &= \upsilon_1 P^* - [\frac{\beta_2[\rho-\phi]}{\upsilon_3}][\theta+\upsilon_1 P^* - \alpha],\\
D_{22} &= \upsilon_1 P^*,\\
D_3 &= \frac{\beta_1\beta_2}{\upsilon_3}[\rho-\phi][\theta+\upsilon_1 P^*-\alpha]\rho. \qquad (3.39)
\end{aligned}$$

To assess the behavior of System (2.1) around the critical steady-state, we seek to know what becomes the state of government oil income, the consumer purchasing power, and commodity market dynamics when oil-thefts is unchecked ($\tau = 0$) and when it is checked ($\tau \geqslant 0$).

for $\tau = 0$, Equation (3.38) becomes:

$$D(\lambda,\tau) = \lambda^3 - [D_{21} - D_{22}]\lambda + D_3 = 0. \qquad (3.40)$$

By the Routh-Hurwitz criterion, Equation (3.40) has all negative roots, if and only if

$$D_{11} - D_{12} > 0, \;\; D_3 > 0, \;\&\; [D_{11} - D_{12}][D_{21} - D_{22}] - D_3 > 0. \qquad (3.41)$$

Obviously, the conditions in (3.41) are not hold in (3.40) as $D_{11} - D_{12} = 0$. Hence, the critical steady-state is unstable for $\tau = 0$.

For $\tau > 0$*:* we set $\lambda = a + iw$, where $a = 0$ and w is a real number, and re-write the exponential in terms of trigonometric functions in Equation (3.38) to have:

$$-iw^3 + \left(D_{11} - D_{12}[\cos(w\tau) - i\sin(w\tau)]\right)w^2 + \left(D_{21} - D_{22}[\cos(w\tau) - i\sin(w\tau)]\right)iw + D_3 = 0. \qquad (3.42)$$

Separating real and imaginary parts in (3.42) yields

$$\begin{aligned}
-w^2 D_{11} - D_3 &= -w^2 D_{12}\cos(w\tau) - D_{22} w\sin(w\tau)\\
w^3 - D_{21} w &= w^2 D_{12}\sin(w\tau) - D_{22} w\cos(w\tau). \qquad (3.43)
\end{aligned}$$

Squaring and adding up the equations in Equation (3.43) gives

$$w^6 + [D_{11}^2 - 2D_{21} - D_{12}^2]w^4 + [D_{21}^2 + 2D_{11}D_3 - D_{22}^2]w^2 + D_3^2 = 0. \qquad (3.44)$$

Letting $m = w^2$, then Equation (3.44) becomes

$$m^3 + [D_{11}^2 - 2D_{21} - D_{12}^2]m^2 + [D_{21}^2 + 2D_{11}D_3 - D_{22}^2]m + D_3^2 = 0. \qquad (3.45)$$

Since D_3^2 is positive and factorized, then the roots of Eqn. (3.45) becomes:

$$\begin{aligned}
m_1 &= -\beta_1\beta_2\rho,\\
m_2 &= -\frac{\rho-\phi}{\upsilon_3},\\
m_3 &= -(\theta + \upsilon_1 P^* - \alpha).
\end{aligned}$$

Thus, the critical steady-state is unstable for $\tau = 0$, stable for $\tau > 0$ if and only if $\rho > \phi \;\; \& \; \theta + \upsilon_1 P^* > \alpha$ otherwise it is unstable. This condition suggests that

1. oil-theft depletion rate should be greater than its proliferation rate.
2. The sum of the Nigerian government's oil income depletion due to production capacity and oil theft should be greater than its oil income growth rate.

The suggestion (2) is indeed un-admirable for oil-income growths, as the contrary will guarantee the desirable growth of government oil income. Hence, the critical steady-state is unstable for $\tau \geqslant 0$.

4. Numerical Simulation

Here, we verify analytical results obtained in Section 3 for System (2.1) using Matlab DDE223 solver with the estimated parameters' values in Table 1, based on the prevailing dynamics in Nigeria, recently. This is done by implementing the conditions of the theorems on the simulation of System (2.1) as given below:

Table 1: Model Estimated Parameter Values

Parameter	Estimated Value	Source
α	1	Estimated from [22]
β_1	0.051	" " [16]
θ	0.126	" " [14]
σ	0.	" " [10]
β_2	1.5	" " [16]
ϑ	0.5	Assumed
υ_1	0.1	Estimated from [23]
υ_2	0.5	Assumed
ϕ	0.4	"
υ_3	0.2	Estimated from[23]
υ_4	0.35	Assumed
ρ	0.2	"

Figure 1 depicts the simulation of the oil-theft-free steady-state of System (2.1) based on the conditions in Theorem 2. That is when the consumer income is greater than its living expenses, the pirates' decimation is greater than its proliferation and the commodity market is stable in an oil-theft-free state, the interacting model variables' dynamics are characterized by the continuous growth of government income and commodity markets. However, the consumer purchasing power grows to reach its peak before regressing to its initial state with future briefed upsurges.

Figure 2 accounts for the numerical simulations of the subsidy-free steady-state of System (2.1) based on the stated conditions in Theorem 3: that even with sufficient income for consumers, the fuel-subsidy-free steady-state is unstable when the pirates' activities are unchecked but undergone a transcritical bifurcation as pirates activities are checked in a subsidy-free state. (a) is when the pirates' activities are unchecked, i.e., when $\tau = 0$. The time evolution curves exhibit a quick and steady growth of consumers' income. However, this scenario characterized the inter-twisting oscillatory growth of government income and pirate groups. The commodity markets' time evolution remains unperturbed. (b) is when pirates' activities are checked, i.e., when $\tau > 0$. The scenario is characterized by delayed-induced pirate checks, which paved the way for an upsurge in immediate government incomes at a transcritical bifurcation point. Consumer income growth also starts around the transcritical bifurcation point before growing progressively. The commodity markets' time evolution remains as seen in (a).

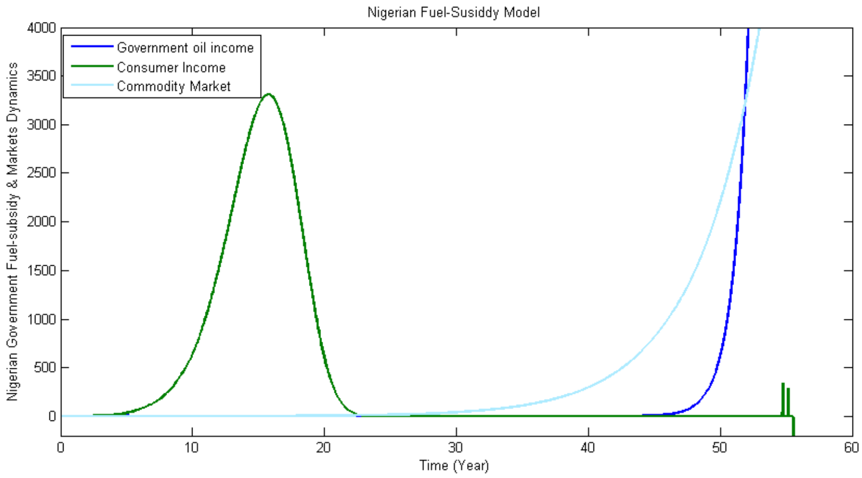


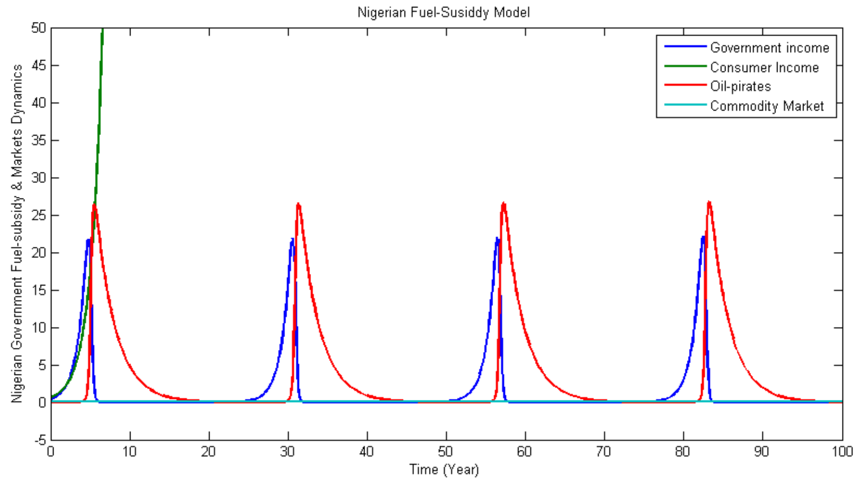
Figure 1: Numerical simulation of oil-theft-free steady-state of System (2.1) based on Theorem 2 conditions.

Figure 3 describes the numerical simulations of subsidy-free steady-state of System (2.1) based on the stated conditions in Theorem 3: that the accommodation of fuel subsidy in the Nigerian economy will lead to unstable model's variables dynamics both in the presence of oil-pirates checks or un-check. (a) depicts when the oil-pirates groups are not checked i.e. when $\tau = 0$. The time evolution dynamics are characterized by consumers' income curve flattening to zero, the commodity markets' slight growths, and un-pronounced oil-pirates growths. However, The chaotic Nigerian oil income growth is also featured in this scenario. (b) captures the dynamics when the oil-pirates groups are checked i.e. when $\tau > 0$. The dynamics exhibit the dormancy of oil-pirates groups and the increased growth of Nigerian government oil income and commodity markets. The consumer income curve is also seen to grow to a peak before returning to its initial state with future upsurges.

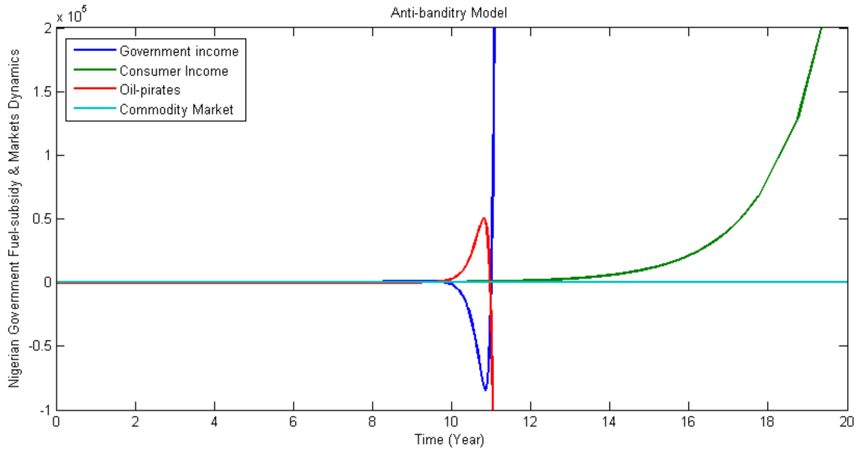
5. Discussion

In this work, a deterministic model is proposed and analyzed to determine the appropriate strategy to achieve win-win outcomes from the present oil subsidy uproars, such that the government oil incomes increase desirably without adversity on the general well-being of the citizenry. To ensure the economic reality of our findings and are true to the model's assumptions, the existence of positive solutions is determined. Mindful of the need for common ground among stakeholders in fuel-subsidy dynamics in Nigeria, the condition for the existence of steady states based on the imminent opinions of the stakeholders is determined for the proposed model. These efforts yield three steady-states: the oil-theft-free steady-state, the subsidy-free steady-state, and the critical steady-state.

To determine the condition for regression or progression of the proposed model variables from these steady states, the stability analysis is performed on the steady states. The Analysis of System (2.1) around the oil-theft-free steady-state predicted an unstable scenario characterized by the need for a living wage that will ensure consumers' income is greater than their living expenses and, subsequently, ensure commodity markets' growth. For the subsidy-free steady-state, the analysis predicted an unstable models' variables dynamics when there is no oil-theft policing, i.e., when $\tau = 0$, and a transcritical bifurcation when there is oil-theft policing i.e. when $\tau > 0$. The analysis of the critical steady-state where there are oil theft and fuel subsidies in the dynamics of Nigeria's



(a)

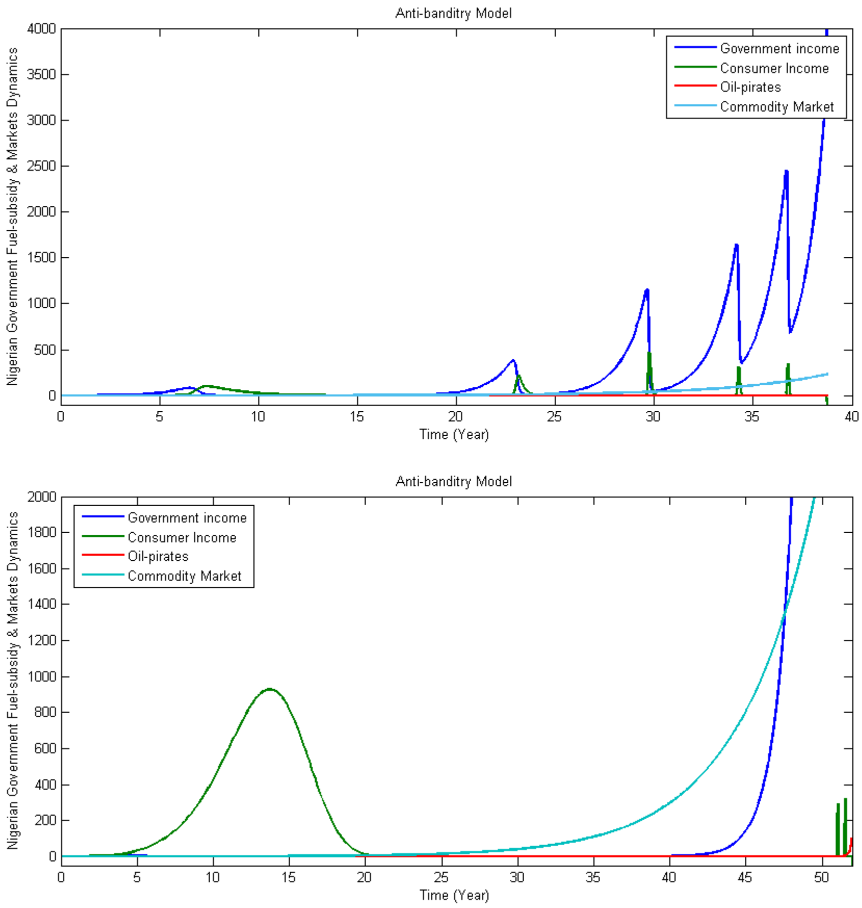


(b)

Figure 2: Numerical simulation of Subsidy-free steady-state of System (2.1) based on Theorem 3 conditions. (a) When $\tau = 0$. (b)When $\tau \geqslant 0$

oil income and the commodity markets predicts an unstable model variable dynamics for the case of oil theft policing and non-oil theft policing.

The numerical solutions verify the analytical results in Section 3 and reveal the graphical dynamics of the model's variables. The simulation of System (2.1) around the oil-theft-free steady-state based on the assumptions in Theorem 2 exhibited the predicted unstable scenario characterized by consumers' income progression to the peak, the Nigerian government oil-income and the commodity markets' continuous progression (see Fig. 1). Also, the simulation of System (2.1) around the subsidy-free steady-state verified the analytical result prediction of unstable stability for $\tau = 0$, and the occurrence of a transcritical bifurcation as $\tau > 0$s (see Fig. 2[a & b]). The numerical simulation of System (2.1) around the critical steady-state based on Theorem 4 verified the unstable dynamics as predicted analytically (see Fig. 3). The impact of delay τ is presented graphically as for $\tau = 0$, the consumer income is seen paralyzed, the commodity markets is practically dormant, and the government oil-income exhibited chaotic growth (see Fig. 3a). Also, for $\tau > 0$, the delay-induced consumers' income undulated growth coupled with the normal and continuous growth of government oil incomes and the commodity markets (see Fig. 3b).



(a)

(b)

Figure 3: Numerical simulation of critical steady-state of System (2.1) based on Theorem 4 conditions. (a) When $\tau = 0$. (b)When $\tau \geqslant 0$

6. Conclusion

This work has considered the sentiments and arguments for and against fuel subsidy removal in Nigeria to propose a deterministic model of Fuel subsidy in Nigeria's commodity market dynamics and its implication on Nigerian government oil income, the consumer, and the commodity markets. Also incorporated is the depleting impact of oil pirates on Nigerian oil income, and time delay is used to model the security efforts to curb the pirates' activity(s). Steady states for win-win resolutions were determined and analyzed. Our findings highlight that:

1. If oil theft can be eradicated, the government oil incomes and the commodity markets will witness normal growth, and consumers' income will experience undulated growth for some time if their incomes are higher than their required living expenses.
2. If the fuel subsidy is removed and the oil-pirate activities are unchecked, the government oil incomes and the oil-pirate activity(s) inter-twisting oscillation coupled with consumer income growth if their incomes exceed their required living expenses.
3. If the fuel subsidy is removed and the oil-pirate activities are checked, the government income will transcritically upsurge instantly, and the pirates' activities will vanish. The consumers' income will progress more normally under this scenario if their incomes exceed their required living expenses.

Based on these findings, this paper recommends the need for the Nigerian government to ensure improvement in the livelihood of her citizenry such that an average Nigerian can cater for their required living expenses and ensure the blockage of oil theft as a strategy for sustainable removal of fuel subsidy without causing social unrest and economy hardship on their citizenry.

7. Authorship contribution statement

Abdulkareem Afolabi Ibrahim: Conceptualization, Methodology, Coding, Writing - Review and Editing. **Jibril Mbaya**: Conceptualization, Writing - Review and Editing. **Dahiru Alhaji-Bala Birnintsaba:** Social-economic Conceptualization and Validation. **Baba Gimba Alhassan:** Software Formal Analysis and Reviewing.

8. Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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