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Parametric Poisson Bifurcated Autoregressive Process: Application to Worldwide, Regional, and Peculiar Countries' of Automobile Production

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Abstract

This article introduces Bifurcated Autoregressive (BAR) process with two apart marginal distribution error terms of ω_{2t} and ω_{2t+1} of Poisson white noises to make it Poisson Bifurcated Autoregressive (PBAR) in a parametric setting. The statistical definition of PBAR (1) process with parameters β_1 and β_2 that must be $|\beta_1|$ and $|\beta_2| < 1$ for stationary process was spelt-out. Weighted Least Squares (WLS) parameter estimation technique was adopted and the process limiting distribution was carried-out via the combination of martingale process and Lindeberg's condition. Monthly automobile production in Japan, Outside Japan, America, USA, Europe, Asia, and China that approximately tantamount to worldwide, regional, and peculiar countries' of automobile production was subjected to the PBAR process. In conclusion, Japan automobile production possessed the highest and largest error correlation $(\omega_{2t},\ \omega_{2t+1})$ of 0.6582 (65%) with first order PBAR, with $\beta_1 Y_{\left(\frac{t}{2}\right)}$, such that, $\beta_1 = 0.2228$ of degenerated two major divisions of automobile production of Registrations and Mini-Vehicles with descendant of different brands (models).

Keywords: Automobile Production, Lindeberg's Condition, Martingale Process, Poisson Bifurcated Autoregressive (PBAR), Weighted Least Squares (WLS).
2010 MSC: 34C23, 37G15, 17B63.

1. Introduction

Some linear time series models lacked the ability to capture conditional distributions with time-varying dependency and distributional properties that deviate from the Gaussian assumptions. However, some non-linear time series models relaxed these properties

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the linear time series models failed to treat but not all [1, 2]. Non-linear time series models, such as the Generalized Auto-Regressive Conditional Heteroscedasticity (GARCH) and its variants; Self-Exciting Threshold Autoregressive (SETAR), Gaussian Distribution Mixture Autoregressive (GDTM), Threshold Autoregressive (TAR) etc. had been correctly used to analyze uniform time-varying series in order to capture volatility and some stylized attributes linear time series models could not detect [3, 4, 5]. Empirical research indicated that some non-linear time series models are yet to fully capture varying and clustering volatility, more than two regime-shifts (ability to handle cycles), bifurcated series, and lineage data structure (structural tree series and inherited generated series [6, 7]. According to [8] and [9], non-linear time series models, because of their ability to capture complex stylized traits in financial market returns, such as, exchange rates, interest rates, crude oil prices, bank shares, stock indices, and so on, are best suitable for describing financial return series.

According to [10], the above-mentioned models (both linear and non-linear models) are all-pass models that are Autoregressive (AR) based process that its coefficient(s) must lies within the unit roots of the process for any order of any polynomial. AR model and its modifications are of regarded interest because of its flexibility to model any ergodic process. The uncorrelated error term in these all-pass models of time series made the observed series to be independent and possibly suitable for non-Gaussian noise [11, 12, 13]. These all-pass processes can be a case of commingling two white noises of a detached series from a parental time uniform series or subjected to different non-Gaussian distribution(s) error term in order to measure the detached series correlation. The ability to measure, access, and estimate the correlation of two detached time series via autoregressive process is what [14] and [15] referred to as Bifurcated Autoregressive (BAR) process.

Bifurcated Autoregressive (BAR) process is one of the rare and infamous time series models that have the versatile ability to be applied to two coded (binary) structural tree system for biometric related and genetic problems. It is usually applied to cell lineage of measurement data that contain features like growth rate, their duration, length, and time until division [16, 17]. BAR can also be regarded as dichotomous-splitting autoregressive or tree-indexed autoregressive processes of descendants or precursors of an individual or financial series, in which it gives to two or more offspring, subseries, subsequence or bi-regime productions or generations [18, 19, 20]. These traits are usually used to measure and estimate the inherited correlation (association) effects of evolution of the characteristics on interest. Ref.[15] propounded the BAR model for cell lineage data where everyone in one generation gives rise to two or more offspring in the next generation by adopting Autoregressive (AR) to the dyadic structural tree to give a BAR process.

Recently, bifurcated models do not full address the problem of characterizing lineage (parental series) variation via contextual dynamics, branching process, asymmetric distributions, accounting for discreteness (discrete variables), branching processes; comparable and recognizable correlation structures via autoregressive coefficients of two bifurcated series [21, 22, 7, 23]. It will be unjustifiable and inappropriate, if discrete variates that are count data time series (positive integer readings or observations) are approximated, subjected, shirked, or compressed to continuous distributional variates, because the error term of the model is in it conventional form or assumption of normality of mean zero and unity variance. Examples of financial series that falls in this category are number of stocks sold per week, number of withdraws in a particular Automated Teller Machine (ATM) per week, month or weekly number of depositors in a bank, number of stock market transactions monthly etc.

However, this trait of integer valued observations (count data) or series is not a waiver for BAR process that has been compressed for Gaussian error term for any type of empirical data. In connection to this, [24] and [25] adopted and investigated count series via integer value series of autoregressive model to make it Integer Valued Autoregressive (INAR) model. What this connotes is that the error term was subjected to a discrete distributional form of Binomial distribution to make it Binomial-Autoregressive (BINAR) for a finite and not lengthy series. Such adjustment, modification and accommodation was adopted to correct BAR process with parental or progenitor (whole series of the financial series) distributional form of count data or positive integer-valued observations. Unlike the BINAR process, Poisson distribution form of undetached (whole series) or parental series was traced, absolved, adopted, and transferred to the bi-series of autoregressive processes. By doing so, a befitting distribution of non-negativity and inclusion of non-response (inclusion of zero events or missing events) makes it signifiable and justifiable to propound a non-linear time series for integer-valued BAR called Poisson Bifurcated Autoregressive (PBAR) process. According to [25] and [26], count data models built with either Binomial or Poisson distribution are not only suitably and accountable for dependence among observations, but also conditioned on the mean of the immediate past and previous observations. PBAR will not be an exceptional because of the individual autoregressive process that make-up PBAR model will be described, incorporated, and structuralized the correlated traits of the detached series via the dependency of observations from the parental or mother series. PBAR process will take into consideration the asymmetric succession and environmental effects (correlated structure of the bi-autoregressive coefficients) to explain the quantitative traits of the powerful descendant.

2. Overview of Bifurcated Autoregressive Related Models and their Applications.

The profounder and introducer of Bifurcating Autoregressive model was traced back to [15], when they have a cell division lineage structural data at their disposal, such that, the individuality of the mother, progenitor or source gave birth to bi-offspring in the descendant generation. Each of the degenerating equation of the descendant was viewed, seen, and subjected to autoregressive process of order one. In their view, since two degenerated sister cells from the mother possessed a higher likelihood of being associated, the AR (1) process was then perceived in a two (bi) divided processes to form BAR process to estimate correlation of the bi-sister cells. Although, [17] extended the study to p^{th}-ordered bifurcated autoregressive, that is, BAR (p) as well as moving from observational bi-sister cells from a source to observational datasets of several correlations among cousins. In autoregressive settings, if the autoregressive process was successfully specified and fitted, what comes next is the Moving Average (MA) via the residuals of the remnants of subtraction of the actual and estimated observations to form a new actual series for MA with order “q”. In lieu of this, [27] elongated BAR model to Autoregressive Moving Average (ARMA) of order “p” and “q”, that is, BARMA (p, q). The rationale behind their stretched-out was to capture and conform to an extensive range in dependency of family tree-indexed structure. Ref.[28] also worked out the parameter estimation of the BAR (p) with normally distributed white noise via maximum likelihood estimation. They further ascertained the goodness of their estimators (estimated parameters) via the firm confirmation of consistency and asymptotic property of the normality of the estimators. Ref.[16, 29] affirmed the need to deviate from the Gaussian distributional form of the white noise of BAR (p) in order to incorporate geometrical series of the error term. By doing so, they emphasized on the need to subject the white noise to exponentiated form, which is exponential bifurcated autoregressive model. The same maximum likelihood estimation method was employed to parameterize the solutions needed for exponential bifurcated autoregressive of order one [30].

Ref.[31] proposed a combination of local meshless method based on radial basis function (RBF) with Laplace transform. The proposed method was for handling fractional order derivation in a heterogeneous and homogenous setting in order to overcome time independent effect in order to achieve accuracy and stability in a Laplace setting. Ref.[32] proposed a fractional-order derivative in Caputo context for modeling COVID-19 cases in eight different classes. Result regarding the uniqueness existence of the solution for the proposed model was fine-tuned via fractional –order Taylor's series. They affirmed that simulation results for the fractional differential order model displayed stability of the results. In a similar vein, [33] studied a nonlinear fractional order Lotka-Volterra prey-predator type dynamical system under a conformable fractional order derivative (CFOD). They concluded that Caputo fractional order derivative is the ideal algorithm for solving the uniqueness of the established fractional order Lotka-Volterra prey-predator. In addition, [34] proposed Tempered Fractional J-Transform (TFJT) technique for solving linear and nonlinear dynamical system in a Rienann-Liouville and Caputo settings. They came to term that the proposed technique will converge faster when guess values are first precise confirmations of TFJT in situations where tempered calculus are applied to physical sciences.

Ref.[35] elaborately explained the diversification method to employ when there is likelihood of different formation via various noticeable genealogical structural or indexed trees. They take into consideration not only missed observations, but also joint noticeable information from various dataset to maneuver the problem of uncertain parameterization via miniature single structural index. The two-type Galton Watson process, that is Galton Watson-BAR process was adopted instead of the BAR process and the Least Square (LS) technique of estimation to estimate involved parameters. Thereafter, the solutions were applied to Escherichia coli divisional data and simulation to ascertain the convergence error rate of estimations and significance level of missing observations via the power of a test. Ref.[36] worked on outliers contaminated series of BAR; a vigorous but evincing procedures of parameter estimation was adopted due to the contaminated source outliers or stimulant outliers. The first procedure was designed to attach weights to estimates with the help of Wilcoxon non-parametric test embedded in Monte Carlo. This procedure was referred to as Weighted L1, whereas, when the weights are constant in the individual split source, the penalized estimation procedure of Least Absolute Deviation (LAD) will be adopted. Real life data and simulation applications were used to validate that no model or autoregressive process was inferior to others when weighted and un-weighted procedure via the types of outliers were adopted. The notable deduction was that when LAD is adopted, the weights are usually constant and equal to one, while the residual robust estimates in factored and stimulus spaces (outliers) were not only high, but also failed to proof the gradients of linear form and the normally distributed asymptotic properties [3, 20, 37, 38].

2.1. Poisson Bifurcated Autoregressive Model.

In real life scenarios, bivariate Poisson distribution of p^{th}-order integer-valued autoregressive to a binary-tree indexed or non-binary-tree indexed structure might be the ideal and satisfying relationship and representation. This asymmetric classification and generalization are common to time unit division, cell lineage data, bivariate data, dual source of buying and selling of a currency rate, dual source of buying and selling of number of stocks, split temperature effects, number of withdrawers and depositors in a particular Automated Teller Machine or Bank etc. The sighted data examples are not only mimic or fitted in, in identifying the signal noise measurements required by a PBAR process, but also are applicable to the theory of the branching processes. Considering this, the conditional distribution of the PBAR error term will be of two apart marginal distributions of error terms, such that the present white noise depends on the immediate previous one. The research aim of this article is to workout Poisson Bifurcated Autoregressive (PBAR) process, its parameter estimation, and limiting distribution via the combination of martingale process and Lindeberg's condition. The process will now be applied to ancestral series of segmented worldwide automobile production data. The monthly automobile production of cars, trucks, jeeps, vans etc. worldwide as extracted by Japan Automobile Manufacturers Association (JAMA) Inc. being the headquarters of automobile production. The monthly estimated production of automobiles were categorized into seven (7) productive of automobiles' countries/continent in the world. These comprise of Japan, automobiles outside Japan, America, USA, Europe, Asia, and China that approximately make-up worldwide, regional, and peculiar countries' of automobile production. The monthly data to be considered will start from January 2019 to February 2023.

2.2. Binary Counted-Valued Autoregressive (BICVAR)

In generality, if there exist t^{th}-generation, such that, $t \geqslant 1$, then the general term of t^{th}-possible outcome, subseries, segmentation or division is

$$H_n = \left\{2^t, 2^t + 1, \cdots, 2^{t+1} - 1\right\} \tag{2.1}$$

Where;
At $n = 0$; H_0 connotes the initial generation, the source, parental series, or original ancestors.
At $n = 0$; H_1 is the first generation from the source.
At $n = r_t$; H_{r_t} is the generation of individual at which implies that $r_t = \left(\log_2(t)\right)$, since the two descendants of individual "t" are 2^t and 2^{t+1}, reversing it, the individual is "t', $\left(\frac{t}{2}\right)$, where $[y]$ is the largest integer less than or equal to y. So,

$$\prod_t = \bigcup_{i=o}^{t} H_i \tag{2.2}$$

will be the union, sub-division, sub-detachment, sub-tree indexed of all characterization emanated from the ancestor's traits or series up to t^{th}-succession. Then, in generality, the number of elements of $|H_t|$ of H_t is nothing but 2^n. The cardinality of $|\prod_t|$ of $\prod_t$ is $2^{t+1}-1$.

2.3. The Bifurcated Autoregressive Model

Ref.[16] defined a p^{th}-order Bifurcated Autoregressive process (BAR) to be

$$Y_t = \beta_0 + \beta_1 Y_{(\frac{t}{2})} + \beta_2 Y_{(\frac{t}{4})} + \cdots + \beta_p Y_{(\frac{t}{2^p})} + \omega_t = \sum_i \beta_i Y_{(\frac{t}{2^i})} \tag{2.3}$$

Where,
$\forall\ i = 0, 1, \cdots, p$; $(\omega_{2t}, \omega_{2t+1})$ is a series of Independent and Identical Distributed (IID) with $\omega_t \approx (0,\ \sigma^2)$, that is, $E(\omega_{2t}) = E(\omega_{2t+1}) = 0$; $Var(\omega_{2t}) = Var(\omega_{2t+1}) = \sigma^2$; such that, for BAR(1)

$$Y_t = \beta_1 Y_{(\frac{t}{2})} + \omega_t \quad \text{for } t \geqslant 2 \tag{2.4}$$

Alternatively, adopting the simplified BAR (1) by [15]. Let Y_t connotes the quantitative characteristics of a random variable with individual "t", then the first-order BAR process for $t \geqslant 1$ is given by

$$Y_t = \begin{cases} Y_{2t} = \alpha_1 + \beta_1 Y_t + \omega_{2t} \\ Y_{2t+1} = \alpha_2 + \beta_2 Y_t + \omega_{2t+1} \end{cases} \tag{2.5}$$

Where,
ω_{2t} and ω_{2t+1} implies the individual white noises with autoregressive parameters of β_1 and β_2 that must be $|\beta_1|$ and $|\beta_2| < 1$ for a stationary process where α_1 and α_2 are the constant term that is usually negligible in application.

2.4. The Probability Mass Function (PMF) of Poisson

Let Y_t be a random variable such that it entails or accommodates only number of events or happenings with time, then the Probability Mass Function (PMF) of Y_t is

$$P(Y_t/y_t) = \frac{\beta^{y_t}\exp(-\beta)}{y_t!}\ ; \qquad y_t = 0, 1, 2, 3, \cdots \tag{2.6}$$

Such that, $E(Y_t) = Var(Y_t) = \beta$

2.5. Statistical Definition of the Poisson Bifurcated Autoregressive Process (PBAR)

Considering the BAR (1) defined by [15], $\forall\ t \geqslant 1$, then PBAR of p^{th}-order PBAR (p) is given by

$$Y_t = \begin{cases} Y_{2t} = \beta_{1i} Y_t + \omega_{2t} \\ Y_{2t+1} = \beta_{2i} Y_t + \omega_{2t+1} \end{cases} \tag{2.7}$$

Where PBAR (1),

$$Y_t = \begin{cases} Y_{2t} = \beta_1 Y_t + \omega_{2t} \\ Y_{2t+1} = \beta_2 Y_t + \omega_{2t+1} \end{cases} \tag{2.8}$$

Where,

$$\beta_{1i}Y_t = \sum_{i=0}^{Y_t} \beta_{1i}X_{i-t};\ \beta_{2i}Y_t = \sum_{i=0}^{Y_t} \beta_{2i}W_{i-t};\ \beta_1 Y_t = \sum_{i=0}^{Y_t} \beta_1 X_i;\ \beta_2 Y_t = \sum_{i=0}^{Y_t} \beta_2 W_i \quad (2.9)$$

Where Y_t is the initial count data of the forebear series, progenitor, mother series process, ω_{2t} and ω_{2t+1} are the white noise of the non-negative integer values of the bi-autoregressive processes, such that, ω_{2t} & $\omega_{2t+1} \approx Y_t = (\beta,\ \beta)$ with $\left(\sum_{i=0}^{Y_t} \beta_{1i}X_{i-t}\ \&\ \sum_{i=0}^{Y_t} \beta_{2i}W_{i-t}\right)$ and $\left(\sum_{i=0}^{Y_t} \beta_1 X_i\ \&\ \sum_{i=0}^{Y_t} \beta_2 W_i\right)$ two sub-sequence, sub-series with independent and identically distributed integer-valued variates for PBAR(p) and PBAR(1) respectively. β_1, β_2, β_{1i} & β_{2i} must lie within the unit circle condition to satisfy the stationary condition.

3. Parameter Estimation of PBAR via Weighted Least Squares (WLS)

Let $\Im = (F_n)_{n\geqslant 0}$ be the corresponding filtration associated to the first-order PBAR process of equation (7), such that, F_n is the $\sigma-$algebra bring-forth by persons up to n^{th}-generation. That is, $\Im = \sigma\{Y_t,\ t \in \Pi_n\}$, for all $t \in H_i$.

$$\begin{cases} E\left[\omega_{2t}/\Im\right] = \beta_1 \\ E\left[\omega_{2t+1}/\Im\right] = \beta_2 \end{cases} \quad (3.1)$$

Consequently, this implies that

$$\begin{cases} Y_{2t} = \beta_{1i}Y_t + g + \varepsilon_{2t}, \\ Y_{2t+1} = \beta_{2i}Y_{t+1} + h + \varepsilon_{2t+1}, \end{cases} \quad (3.2)$$

Where $\varepsilon_{2t} = Y_{2t} - [\omega_{2t}/\Im]$ and $\varepsilon_{2t+1} = Y_{2t+1} - [\omega_{2t+1}/\Im]$. Therefore, the two relations can be re-written in a matrix form as:

$$\Sigma = \Theta^t\Gamma_n + \Psi_n \quad (3.3)$$

Where,
$\Sigma = \begin{pmatrix} Y_{2t} \\ Y_{2ti} \end{pmatrix}$; $\Gamma_n = \begin{pmatrix} Y_t \\ 1 \end{pmatrix}$; $\Psi_n = \begin{pmatrix} \varepsilon_{2t} \\ \varepsilon_{2t+1} \end{pmatrix}$; such that, $\Theta^t = \begin{pmatrix} a & b \\ g & h \end{pmatrix}$

The main objective is to estimate Θ from the observation of all persons up to Π_n. That is, WLS estimator $\hat{\Theta}$ to Θ which minimizes

$$\Omega_n(\Theta) = \frac{1}{2}\sum_{t\in\Pi_{n-1}} \frac{1}{g_t}\left\|\Sigma - \Theta^t\Gamma_n\right\|^2 \quad (3.4)$$

Such that the choice of the weighting sequence $(g_n)_{n\geqslant 1}$ can be chosen via $g_n = 1 + Y_n$ via

$$\Theta^t = S_{n-1}^{-1}\sum_{t\in\Pi_{n-1}} \frac{1}{g_t}\Gamma_t Y_t^t\ , \text{ where } S_n = \sum_{t\in\Pi_n} \frac{1}{g_t}\Gamma_t\Gamma_t^t \quad \forall\ n \geqslant 1$$

3.1. The Limiting Distribution of the PBAR

We shall be employing the Central Limit Theorem (CTL) via arrays of triangular vector of martingales. Assuming H_n is a pair-wise filtration to $\mathfrak{I} = (F_n)_{n \geqslant 0}$, such that, $H_n = \sigma[Y_1, (Y_{2t}, Y_{2t+1}),\ 1 \leqslant t \leqslant n]$. Let us assume further that $\Psi^{(n)} = \left(\Psi_t^{(n)}\right)$ is a square integrable vector of martingale of $Y_{2t},\ Y_{2t+1},\ \varepsilon_2,\ \varepsilon_{2t+1}$, such that,

$$I_i = \frac{1}{g_i}\begin{pmatrix} Y_i \varepsilon_{2i} \\ \varepsilon_{2i} \\ Y_i \varepsilon_{2i+1} \\ \varepsilon_{2i+1} \end{pmatrix} \tag{3.5}$$

Such that, $\Psi_t^{(n)} = \frac{1}{\sqrt{|\Pi_t|}} \sum_{i=1}^{t} I_i$

We have it that,

$$\Psi_{t_n}^{(n)} = \frac{1}{\sqrt{|\Pi_t|}} \sum_{i=1}^{t_n} I_i = \frac{1}{\sqrt{|\Pi_t|}} \Psi_{n+1} \tag{3.6}$$

Where,
$\pi_t = \prod_t$, the monotone increasing function $\left(\Psi_t^{(n)}\right)$ is given by

$$\left\langle \Psi^{(n)} \right\rangle_t = \frac{1}{|\Pi_n|} \sum_{i=1}^{t} E\left[H_i H_i^t | H_{i-1}\right] \tag{3.7}$$

$$= \frac{1}{|\Pi_n|} \sum_{i=1}^{t} E \frac{1}{g_i^2} \begin{pmatrix} \sigma_a^2 Y_i + \sigma_c^2 & \mu \\ \mu & \sigma_b^2 Y_i + \sigma_d^2 \end{pmatrix} \otimes \begin{pmatrix} Y_i^2 & Y_i \\ Y & 1 \end{pmatrix} \quad (a.s.) \tag{3.8}$$

For almost sure convergence, it is to be assumed that (ε_n) validates $\lim_{n \to \infty} \frac{\langle \Psi \rangle_n}{|\Pi_{n-1}|} = L$.
Where "$\otimes$ " is the Tensor or Kronecker product and $\langle \Psi^{(n)} \rangle$ is the langle of $\Psi^{(n)}$.
Consequently, $\lim_{n \to \infty} \langle \Psi^{(n)} \rangle_{t_n} = L \quad (a.s.)$
Verifying equation (16) by using the Lindeberg's condition, we have,

$$\eta_n = \sum_{t=1}^{t_n} E\left[\left\|\Psi_t^{(n)} - \Psi_{t-1}^{(n)}\right\|^4 | H_{t-1}\right] \tag{3.9}$$

$$= \frac{1}{|\Pi_n|^2} \sum_{t=1}^{t_n} E\left[\frac{(1+Y_t^2)}{g_t^4} (\varepsilon_{2t}^2 + \varepsilon_{2t+1}^2)^2 | H_{t-1}\right] \tag{3.10}$$

$$\leqslant \sum_{t=1}^{t_n} \left[E\left[\varepsilon_{2t}^4 | H_{t-1}\right] + E\left[\varepsilon_{2t+1}^4 | H_{t-1}\right]\right] \frac{1}{|\Pi_n|^2} \tag{3.11}$$

From the property of $\lim_{n \to \infty} \frac{1}{|\Pi_n|} \sum_{t \in \Pi_n} g(y_t) = E[(T)]$ and $\overline{\Psi}_{\Delta_n}(g) = \lim_{n \to \infty} \frac{1}{|\Delta_n|} \sum_{t \in \Delta_n} g(y_t)$
It implies that,

$$E\left[\varepsilon_{2t}^4|H_{t-1}\right] \leqslant \mu_{ag}^4 g_n^2; \quad E\left[\varepsilon_{2t+1}^4|H_{t-1}\right] \leqslant \mu_{bh}^4 g_n^2 \quad (a.s) \tag{3.12}$$

$$\eta_n \leqslant \frac{2\mu^4 \sum_{t=1}^{t_n} g_t^2}{|\Pi_n|^2} \tag{3.13}$$

This connotes that,

$$\lim_{n\to\infty} \eta_n = 0 \quad (a.s) \tag{3.14}$$

Using the Lindeberg's condition, we can deduce that,

$$\frac{\Psi_n}{\sqrt{|\Pi_{n-1}|}} \underset{\rightarrow}{L} (0,\ L) \tag{3.15}$$

Moreover, the increasing process associated to ”L” makes the convergence of equation (24) follows the distribution of Gaussian law. This implies that the Poisson Bifurcated Autoregressive process might converge to standard Gaussian noise Bifurcated Autoregressive in the long-run.

4. Numerical Analysis

The numerical data analysis to be subjected to the PBAR model is the monthly data production of automobile as recorded by Japan Automobile Manufacturers Association (JAMA) Inc. in collaboration with Honda Motor Co., Ltd. The monthly automobile production of cars, trucks, jeeps, vans etc. worldwide as extracted by JAMA, being the headquarters and centralized collation center of automobile production. The monthly recorded production of automobiles were categorized into seven (7) productive of automobiles' countries/continent in the world. These comprises of Japan, automobiles outside Japan, America, USA, Europe, Asia and China to make-up worldwide, regional, and peculiar countries' of automobile production. The automobiles are in terms of Registrations and Mini-Vehicles. The monthly data considered started from January 2019 to February 2023. The time plot as well as the structural tree-indexed of the Poisson Bifurcated Autoregressive (PBAR) process of each category will be constructed. The estimation of the bifurcating autoregressive models of any order, “p”, PBAR (p) will be considered.

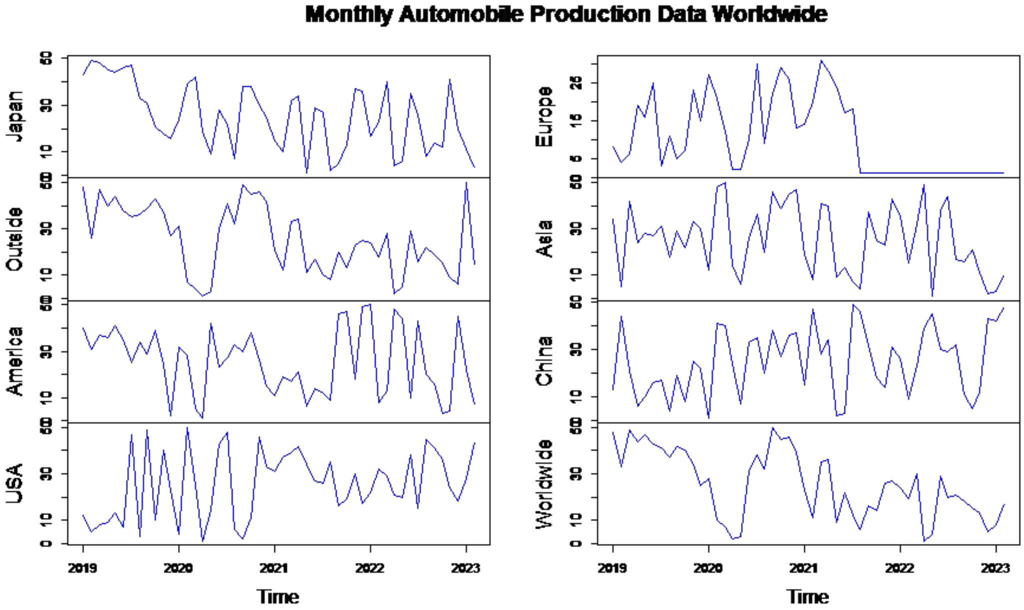

Figure 1: Time Plot of the Monthly Automobile Production Data Worldwide.

From Figure 1, it is obvious that there was a decline production of automobiles towards ending of 2019 till mid of 2022 in Japan, Outside Japan, America, USA, Europe and China which led to worldwide decline in the production of automobiles. United States experienced a pure zigzag trend manner in the production of automobile production from the beginning of 2019 till towards ending of 2020. The production of the automobiles experienced a long-term oscillation (cycles) about the average trend line of 30,000 for Japan, America, Asia, USA, Outside Japan, and China from the beginning of 2021 till very early of March of 2023. The cycles were not periodic (not regular) and the cyclical movement for each country/continent was greater than one-year business cycle, which embodied prosperity, recession, depression, and recovery. This can be classified as splitting trend that will lead to count tree-indexed autoregressive processes of descendants or precursors of automobile production of financial series, which will lead to subsequence production. The worldwide production of automobile from 2021 virtually experienced declining cyclical movements from 2021 to early 2023 that might produce a degenerating tree-indexed autoregressive process.

Table 1: Poisson Bifurcated Autoregressive Coefficients for Japan Automobile Production

<table>
<tr><th>Intercept</th><th>β_1</th><th>Err.</th><th>5% C.I of Coef.</th><th>95% C.I of Coef.</th><th>P-value</th><th>Cov. Matrix</th></tr>
<tr><td rowspan="2">11.1550</td><td rowspan="2">0.2228</td><td rowspan="2">0.6582</td><td>Intercept | β_1</td><td>Intercept | β_1</td><td>Intercept | β_1</td><td rowspan="2">$\begin{pmatrix} 342.75 & -23.63 \\ -23.63 & 1.64 \end{pmatrix}$</td></tr>
<tr><td>6.8047|-0.078</td><td>15.505 | 0.524</td><td>(0.017)|(0.224)</td></tr>
</table>

Keys: Err. = Error Correlation $(\omega_{2t}, \ \omega_{2t+1})$; C.I of Coef. = Confidence Interval of Model Coefficients; Cov. Matrix = Covariance Matrix

Table 2: Poisson Bifurcated Autoregressive Coefficients for Asia Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
12.5391	0.1496	0.6078	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 401.03 & -27.01 \\ -27.01 & 1.83 \end{pmatrix}$
			7.8335|-0.1684	17.2448 | 0.4675	(0.008)|(0.439)	

Table 3: Poisson Bifurcated Autoregressive Coefficients for Worldwide Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
15.6492	-0.1567	0.2872	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 582.37 & -42.84 \\ -42.84 & 3.16 \end{pmatrix}$
			9.9786|-0.5745	21.3199 | 0.2610	(0.014)|(0.537)	

Table 4: Poisson Bifurcated Autoregressive Coefficients for Outside Japan Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
20.928	0.7941	0.32	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 12247.57 & -121.54 \\ -121.54 & 1.21 \end{pmatrix}$
			-5.0766|0.5360	46.9331 | 1.0521	(0.186)|(0.004)	

Table 5: Poisson Bifurcated Autoregressive Coefficients for North America Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
18.1922	-0.3275	0.1421	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 386.74 & -28.16 \\ -28.16 & 2.06 \end{pmatrix}$
			13.5711|-0.6646	22.8132 | 0.0096	(0.0002)|(0.11)	

Table 6: Poisson Bifurcated Autoregressive Coefficients for USA Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
12.56	0.05	0.215	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 548.66 & -41.872 \\ -41.872 & 3.205 \end{pmatrix}$
			7.0578|-0.3706	18.0659 | 0.4707	(0.0001)|(0.85)	

Table 7: Poisson Bifurcated Autoregressive Coefficients for Europe Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
10.4796	0.076	0.462	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 382.99 & -33.69 \\ -33.69 & 2.98 \end{pmatrix}$
			5.8810|-0.3297	15.0782 | 0.4811	(0.0001)|(0.7589)	

Table 8: Poisson Bifurcated Autoregressive Coefficients for China Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
11.024	0.194	0.286	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 382.99 & -33.69 \\ -33.69 & 2.98 \end{pmatrix}$
			6.5521|-0.1385	15.4956 | 0.5266	(0.0337)|(0.3372)	

Table 9: Poisson Bifurcated Autoregressive Coefficients for Others Automobile Production

Intercept	β_1	Err.	**5% C.I of Coef.**	**95% C.I of Coef.**	**P-value**	**Cov. Matrix**
9.676	0.194	0.286	Intercept | β_1	Intercept | β_1	Intercept | β_1	$\begin{pmatrix} 382.99 & -33.69 \\ -33.69 & 2.98 \end{pmatrix}$
			6.5521|-0.1385	15.4956 | 0.5266	(0.0337)|(0.3372)	

4.1. Discussion:

It can be deduced from table one (1) to nine (9) above that the error correlation $(\omega_{2t},\ \omega_{2t+1})$ for Japan Automobile production possessed the highest and largest 0.6582 (65%) among others. This implies that a first order PBAR with $\beta_1 Y_{(\frac{t}{2})}$, such that, $\beta_1 =$ 0.2228 of degenerated two major divisions automobile production of Registrations and Mini-Vehicles with descendant of different brands (models) was attached to Japan production of automobile from 2019 to 2023 with 65% positive bifurcated correlation of the two degenerated series. In addition, the 5% and 95% confidence intervals of model coefficients of the first order PBAR gave $\beta_1 Y_{(\frac{t}{2})}$ of -0.078 and 0.5238 with intercepts 6.8047 and 15.5053 respectively. The p-values of the 5% and 95% confidence intervals of model coefficients were 0.0166 <0.05 and 0.2235>0.05. Next in line with the most significant bifurcated automobile production is Asia automobile production with error correlation $(\omega_{2t},\ \omega_{2t+1})$ of 60% positive correlation and PBAR with $\beta_1 Y_{(\frac{t}{2})}$, such that, $\beta_1 = 0.1496$ of degenerated two major divisions automobile production of Registrations and Mini-Vehicles with descendant of different brands (models) was attached to Japan production of automobile from 2019 to 2023. Concisely, there is low positive correlation among Japan, Outside Japan, America, USA, Europe, and China that led to worldwide automobile production with low positive error correlation 0.2872 (28%) with $\beta_1 = -0.1567$. Overall, it is to be noted that $|\beta_1| < 1$ for all the PBAR coefficients as well as at 5% confidence intervals of model coefficients, coupled with the deduction fact that PBAR(1) is the ideal realization process for the studied regional and countries' automobile production.

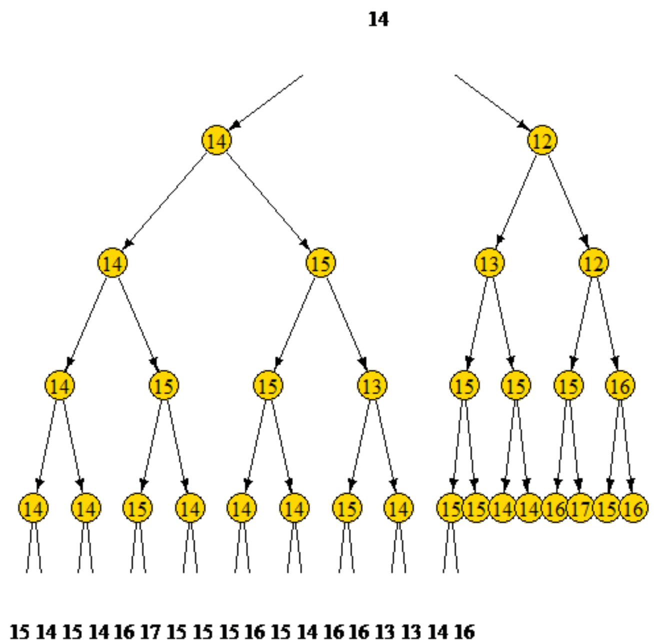

Figure 2: Time Plot of the Monthly Automobile Production Data Worldwide.

From Figure 2 above, it can be deduced that the production of Japan automobile production for sales from 2019 to 2023 is a labeling convention of two major divisions of automobile production of Registrations and Mini-Vehicles in their 14 and 12 million respectively. Each of Registrations and Mini-Vehicles has two brands each that degenerated to four types of automobile production. Each of the four types of automobile production for Registrations and Mini-Vehicles has eight different descendants (models) each in their 14, 14, 15, 14, 14, 14, 15, 14; and 15, 15, 14, 14,16 ,17, 15, 16 thousands produced from 2019 to 2023. Same pattern goes with the Poisson Bifurcated Autoregressive tree plots of automobile production Asia, Japan, and worldwide, but in their different thousands. In the same manner for figure 3 to figure 10.

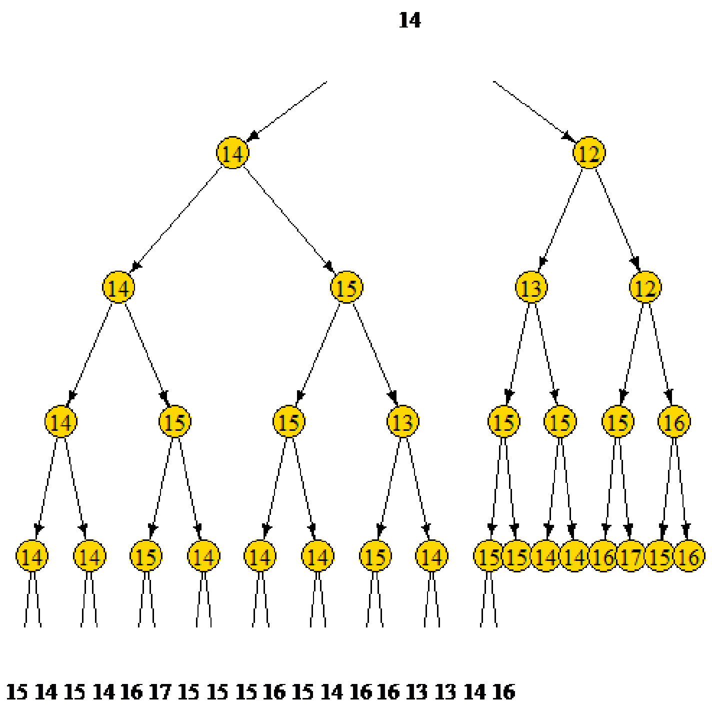

Figure 3: The Count Bifurcated Autoregressive Tree Plot for Japan Automobile Production from 2019 to 2023.

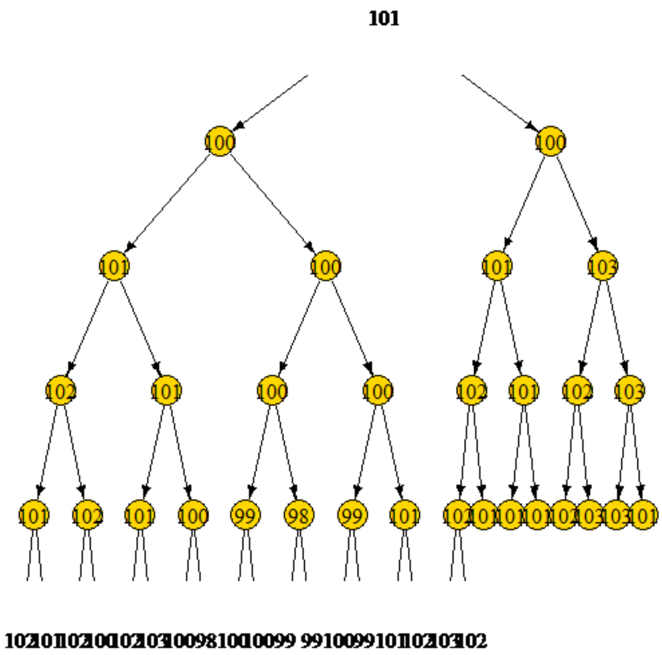

Figure 4: The Count Bifurcated Autoregressive Tree Plot for Outside Japan Automobile Production from 2019 to 2023

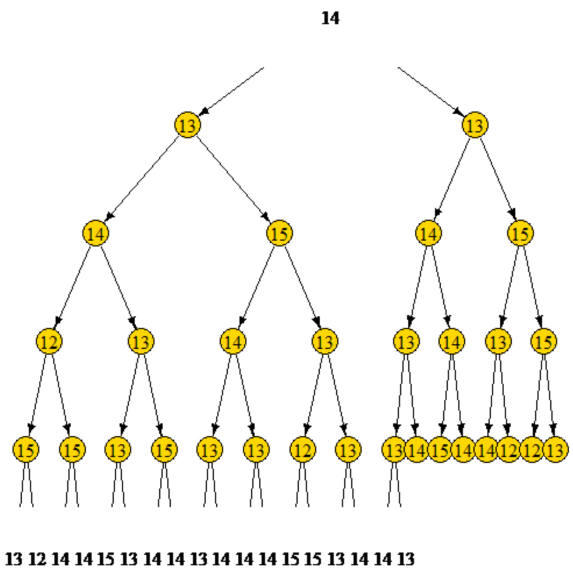

Figure 5: The Count Bifurcated Autoregressive Tree Plot for North America Automobile Production from 2019 to 2023

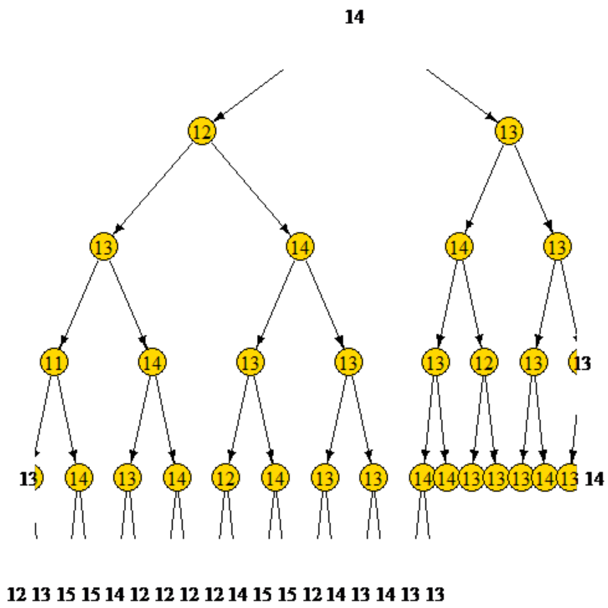

Figure 6: The Count Bifurcated Autoregressive Tree Plot for USA Automobile Production from 2019 to 2023

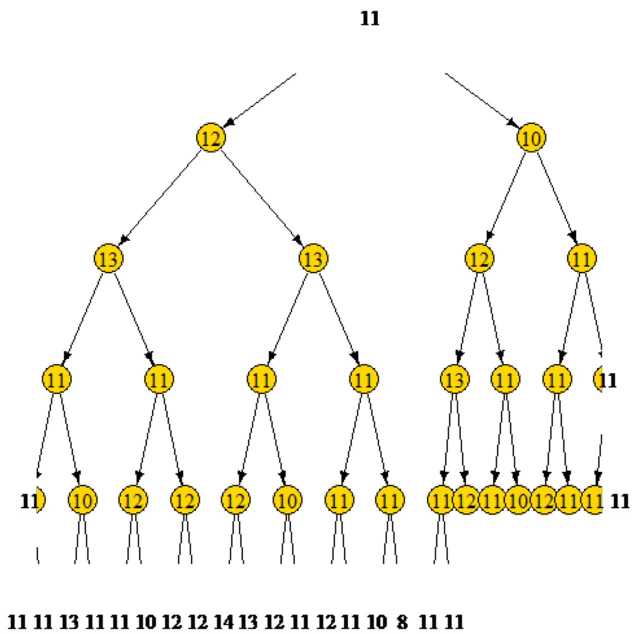

Figure 7: The Count Bifurcated Autoregressive Tree Plot for Europe Automobile Production from 2019 to 2023

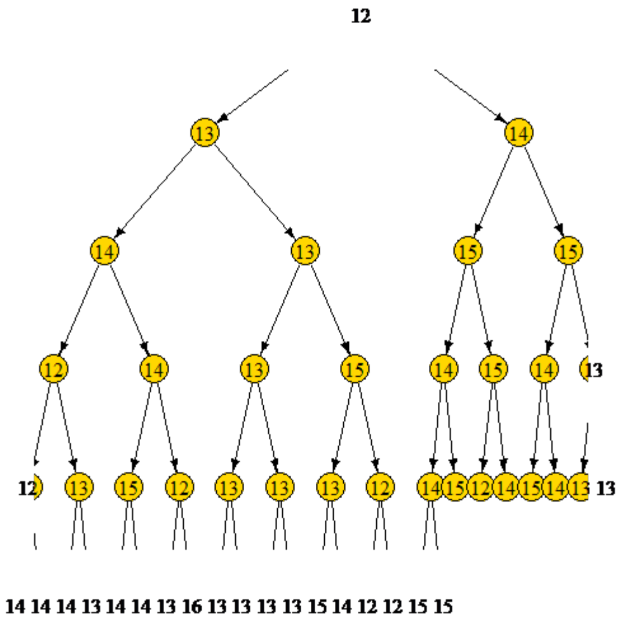

Figure 8: The Count Bifurcated Autoregressive Tree Plot for Asia Automobile Production from 2019 to 2023

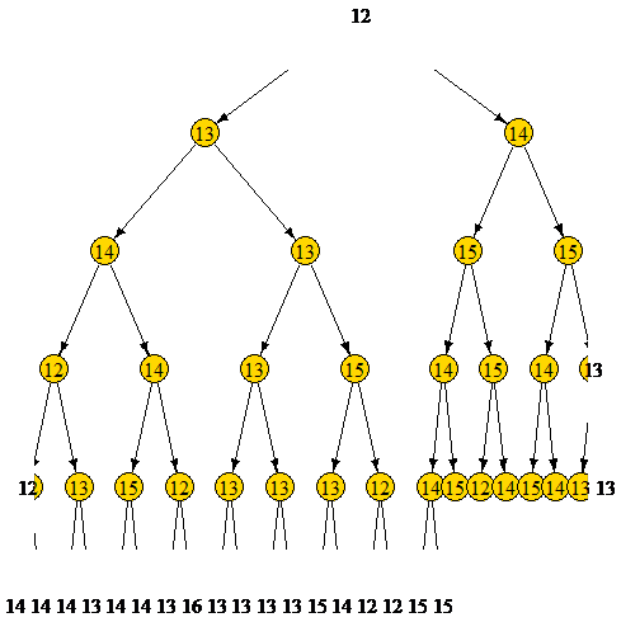

Figure 9: The Count Bifurcated Autoregressive Tree Plot for China Automobile Production from 2019 to 2023

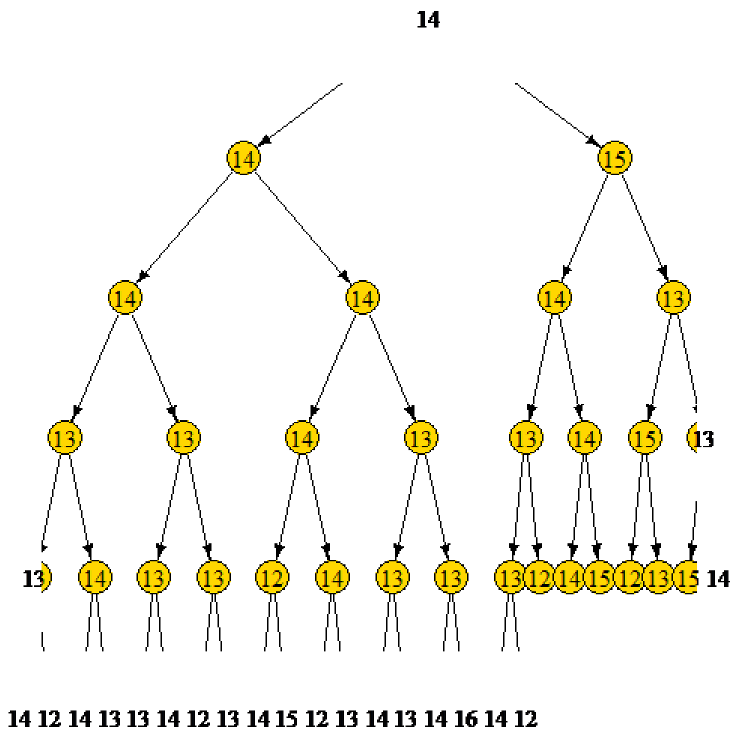

Figure 10: The Count Bifurcated Autoregressive Tree Plot for Worldwide Automobile Production from 2019 to 2023

5. Conclusion

A full-blown parametric Poisson Bifurcated Autoregressive (PBAR) process of p^{th}-order with two apart marginal distribution error terms of ω_{2t} and ω_{2t+1} of Poisson white noises. The PBAR process statistical definition was spelt-out with parameter β_i, such that, $|\beta_i| < 1$ for stationary process that must be ascertained. Weighted Least Squares (WLS) parameter estimation technique was adopted and the limiting distribution of the PBAR process was carried out via Central Limit Theorem (CLT) with the aid of martingale and Lindeberg's condition. In conclusion, Japan automobile production possessed the highest and largest error correlation $(\omega_{2t}, \omega_{2t+1})$ of 0.6582 (65%) with first order PBAR with $\beta_1 Y_{(\frac{t}{2})}$, such that, $\beta_1 = 0.2228$ of degenerated two major divisions automobile production of Registrations and Mini-Vehicles with descendant of different brands (models).

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Conflicts of Interest

We the authors certify that there is no conflict of interest whatsoever with any affiliation, or involvement with any organization, financial and non-financial entity.

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