**Journal of Mathematical Analysis and Modeling**
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2022)3(3) : 1-13
doi:10.48185/jmam.v3i3.629

# Numerical solution of systems of fractional order integro-differential equations with a Tau method based on monic Laguerre polynomials

Mohamed Saad Akel[α], Hussein Shafei Hussein[α], Marwa Masoud Mohammed [α], Ismail Gad Ameen [α]

[α]Mathematics department, Faculty of Science, South Valley University, Qena, Egypt.

• *Received: 11 November 2022* • *Accepted: 17 December 2022* • *Published Online: 30 December 2022*

## Abstract

In this paper, numerical technique based on monic Laguerre polynomials is proposed to obtain approximate solutions of initial value problems for systems of fractional order integro-differential equations (FIDEs). Operational fractional integral matrix is constructed. This operational matrix is applied together with the monic Laguerre Tau method to solve systems of FIDEs. This systems of FIDEs will be transformed into a system of algebraic equations which can be solved easily. Numerical results and comparisons with other methods are also presented to show the efficiency and applicability of the proposed method.

Keywords: Fractional integro-differential equations, Monic Laguerre polynomials, Operational matrix, Tau method.
*2010 MSC:* 34A08 , 33F05, 65M70.

## 1. Introduction

Fractional calculus (FC) is the branch of mathematical analysis which concerned the integrals and derivatives of fractional order of the function. FC have many different applications in various phenomena in the fields of science, engineering, physical, chemistry, electric circuits and mechanical systems [1]. In fact, there are various definitions of fractional derivative which do not coincide in general, such as Caputo [2], Atangana-Baleanu [3], Riemann-Liouville [4], Hadamard [5], Caputo-Fabrizio [6] and Riesz derivative [7], The Caputo derivative commonly used in various applications of fractional differential equation (FDEs). There is new definition of fractional differential derivative called Caputo-Fabrizio (CF), which based on the exponential function. The FIDEs play an important role in different physical phenomena from science and engineering [8].

*Corresponding author: marwa.masoud@sci.svu.edu.eg

Khader and Sweilam [9] presented numerical approximation based on Chebyshev pseudo-spectral method to solve system of linear and non-linear FIDEs of Volterra type. Wang et al. [10] presented Bernoulli wavelets and operational matrix to solve coupled systems of non-linear fractional order integro-differential equations. Loh and Phang [11] introduced Genocchi polynomials to solve system of Volterra integro-differential equations numerically. Ghasemi et al.[12] introduced a reproducing kernel method to solve system of integro-differential equations with nonlocal boundary conditions. Haar Collocation Scheme (HCS) is devoted to obtain the solution of non-linear Volterra-Fredholm (FIDEs) and fractional linear integro-differential equations with variable order [13, 14]. Rida and Hussien [15] presented a new approximation method for solving fractional differential equations based on Mittag-Leffler function. There are many numerical methods have been developed to solve numerical solutions for system of fractional integro-differential equations , such as variational iteration method [16, 17], homotopy perturbation method [18, 19], Adomian decomposition method [20], homotopy analysis method [21] and collocation method [22, 23]. Haar wavelet collocation scheme (HWCS) devoted for solving delay fractional order differential equations (FODEs) and non-linear fractional integral equations (NFIEs) [24, 25]. Shah et al.[26] presented Haar Wavelet technique (HWT) for solving non-linear variable order integro-differential equations (VO-IDEs). Abbasbandy and Taati [27] employed the operational Tau method for solving a system of non-linear Volterra integro-differential equations with non-linear differential part.

Other than previous methods, we present Tau monic Laguerre method based on operational matrix for solving systems of FIDEs. let's consider the following system of FIDEs [28]:

$$D^{\mu}u_{\ell}(t)+\sum_{j=1}^{m}\left(u_j(t)+\int_0^t k_{\ell j}(t,x)u_j(x)dx\right)=f_{\ell}(t), \ell=1,2,...,m, \tag{1.1}$$

with initial conditions:

$$u_{\ell}^{(r)}(0)=\sigma_{r\ell} \quad r=0,1,...n-1, \quad \ell=1,2,...,m, \quad m\in\mathbb{N}, \tag{1.2}$$

$n-1<\mu\leqslant n$, $n\in\mathbb{N}$, where $u_{\ell}$ are the unknown functions, $k_{\ell j}$ and $f_{\ell}$ are the continuous functions.
The main target of this work is to extend monic Lagurre Tau method based on operational matrix to solve systems of FIDEs..........................

The rest of the paper is organized as follows: Section 2 contains properties of the fractional calculus and monic Laguerre polynomials. Section 3 present monic Laguerre operational matrix of fractional integrals. Section 4 present Tau monic Laguerre method based on operational matrices for systems of FIDEs. Section 5 display the numerical results for some examples. Finally, in Section 6 the conclusion is drawn.

## 2. Preliminaries

In this section, we present some definitions and notions of FC and monic Laguerre polynomials which are utilized in this paper.

### 2.1. Caputo fractional derivative and the Riemann-Liouville fractional integral

**Definition.** The Riemann-Liouville fractional integral $I^{\mu}$ of order $\mu$, is defined as [4]:

$$I^{\mu}u(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}u(s)ds, t>0, & \mu>0, \\ u(t), & \mu=0. \end{cases} \tag{2.1}$$

For the Riemann-Liouville fractional integral, we have the properties (see [4, 2]):

$$I^{\mu}\Big(\lambda_1 u_1(t) + \lambda_2 u_2(t) + ... + \lambda_n u_n(t)\Big) = \Big(\lambda_1 I^{\mu} u_1(t) + \lambda_2 I^{\mu} u_2(t) + ... + \lambda_n I^{\mu} u_n(t)\Big), \tag{2.2}$$

where $\lambda_1, \lambda_2, ...., \lambda_n$ are constants.

$$I^{\mu}t^{w} = \frac{\Gamma(w+1) \quad t^{w+\mu}}{\Gamma(w+1+\mu)}, \quad w > -1. \tag{2.3}$$

**Definition.** Caputo's fractional derivative of order $\mu$ is defined as [2]:

$$D^{\mu}u(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-s)^{n-\mu-1}u^{(n)}(s)ds, t>0, \mu>0, \tag{2.4}$$

where $n-1<\mu\leqslant n, n \in \mathbb{N}$.
For the Caputo derivative, we have the following important properties (see [2]):

**(i)**

$$D^{\mu}t^{w} = \begin{cases} \frac{\Gamma(w+1)}{\Gamma(w+1-\mu)}t^{w-\mu}, & \text{for } w \in \mathbb{N}_0, w \geqslant \lceil\mu\rceil, \\ 0, & \text{for } w \in \mathbb{N}_0, w < \lceil\mu\rceil, \end{cases} \tag{2.5}$$

where $\lceil\mu\rceil$ denote to the smallest integer greater than or equal to $\mu$.

**(ii)**

$$D^{\mu}I^{\mu}u(t) = u(t). \tag{2.6}$$

**(iii)**

$$I^{\mu}D^{\mu}u(t) = u(t) - \sum_{k=0}^{n-1} u^{(k)}(0^{+})\frac{t^k}{k!}; t>0. \tag{2.7}$$

### 2.2. Some properties of Monic Laguerre polynomials

The monic Laguerre polynomials $\check{L}_{\alpha,n}(t)$, for $n \in \mathbb{N}$ and $\alpha > -1$, can be defined on the interval $[0,\infty[$ ( see e.g. [32]):

$$\check{L}_{\alpha,n}(t) = \sum_{r=0}^{n} \frac{(-1)^{r+n}\Gamma(\alpha+n+1)\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(\alpha+r+1)\Gamma(r+1)}t^r, n \geqslant 1, \tag{2.8}$$

where $\check{L}_{\alpha,0}(t) = 1$ and $\check{L}_{\alpha,1}(t) = t-(1+\alpha)$.

The Orthogonal property of monic Laguerre Polynomials is given by:

$$< \check{L}_{\alpha,n}(t), \check{L}_{\alpha,m}(t) >= \int_0^\infty \check{L}_{\alpha,n}(t)\check{L}_{\alpha,m}(t)t^{k+\alpha}e^{-t}dt = \zeta_{\alpha,n}\delta_{nm},$$

where $\zeta_{\alpha,n} = n!\Gamma(n+\alpha+1)$ and $\delta_{nm} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$

Let $u(t)$ have a monic Laguerre expansion series as:

$$u(t) = \sum_{k=0}^{\infty} a_k \check{L}_{\alpha,k}(t), \tag{2.9}$$

where the coefficients $a_k$ are given by:

$$a_k = \frac{1}{\zeta_{\alpha,k}} \int_0^\infty t^{\alpha+k}e^{-t}u(t)\check{L}_{\alpha,k}(t)dt, k = 0, 1, 2, .... \tag{2.10}$$

In practice, we truncate the infinite series up to $(N+1)$ terms monic Laguerre polynomials are considered. Then we have

$$u_N(t) \simeq \sum_{k=0}^{N} a_k \check{L}_{\alpha,k}(t) = A^T\psi(t), \tag{2.11}$$

where $A^T = [a_0, a_1, a_2, ..., a_N]$ and $\psi(t) = [\check{L}_{\alpha,0}(t), \check{L}_{\alpha,1}(t), \check{L}_{\alpha,2}(t), ..., \check{L}_{\alpha,N}(t)]$.

## 3. Monic Laguerre operational matrices of fractional integration

Here, we derive the monic Laguerre polynomials of fractional integral.

**Lemma:** The fractional integral of monic Laguerre polynomials is given by:

$$I^\mu \check{L}_{\alpha,n}(t) = \sum_{i=0}^{N} \vartheta_\mu(n,i)\check{L}_{\alpha,i}(t), \tag{3.1}$$

where $\vartheta_\mu(n,i)$ is the operational matrix of integration of $\check{L}_{\alpha,n}$.

**Proof**: By appling Eq. (2.3) to the monic Laguerre polynomials defined by Eq.(2.8), we get

$$I^\mu \check{L}_{\alpha,n}(t) = \sum_{r=\mu}^{n} \frac{(-1)^{r+n}\Gamma(\alpha+n+1)\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(\alpha+r+1)\Gamma(r+\mu+1)} t^{r+\mu} \quad n \geqslant 1. \tag{3.2}$$

Approximating $t^{r+\mu}$ in terms of the monic Laguerre polynomials as:

$$t^{r+\mu} = \sum_{i=0}^{N} E_i \check{L}_{\alpha,i}(t), \tag{3.3}$$

where

$$\begin{aligned} E_i &= \frac{1}{i!\Gamma(\alpha+i+1)} \int_0^\infty e^{-t}t^{\alpha+r+\mu}\check{L}_{\alpha,i}(t)dt \\ &= \sum_{m=0}^{i} \frac{(-1)^{m+i}}{\Gamma(i-m+1)\Gamma(\alpha+m+1)\Gamma(m+1)} \int_0^\infty e^{-t}t^{\alpha+r+\mu+m}dt \\ &= \sum_{m=0}^{i} \frac{(-1)^{m+i}\Gamma(\alpha+r+\mu+m+1)}{\Gamma(i-m+1)\Gamma(\alpha+m+1)\Gamma(m+1)}. \end{aligned} \tag{3.4}$$

Substituting from Eq. (3.4) into Eq. (3.3), then we have

$$t^{r+\mu} = \sum_{i=0}^{N} \left[ \sum_{m=0}^{i} \frac{(-1)^{m+i}\Gamma(\alpha+r+\mu+m+1)}{\Gamma(i-m+1)\Gamma(\alpha+m+1)\Gamma(m+1)} \right] \check{L}_{\alpha,i}(t). \tag{3.5}$$

Thus loading Eq.(3.5) in Eq. (3.2), yields:

$$I^{\mu}\check{L}_{\alpha,n}(t) = \sum_{i=0}^{N} \vartheta_{\mu}(n,i)\check{L}_{\alpha,i}(t), \tag{3.6}$$

where

$$\vartheta_{\mu}(n,i) = \sum_{r=[\mu]}^{n} \sum_{m=0}^{i} \left( \frac{(-1)^{r+n+m+i}\Gamma(\alpha+n+1)\Gamma(n+1)\Gamma(\alpha+r+\mu+m+1)}{\Gamma(n-r+1)\Gamma(\alpha+r+1)\Gamma(r+\mu+1)\Gamma(i-m+1)\Gamma(\alpha+m+1)\Gamma(m+1)} \right). \tag{3.7}$$

## 4. Implementation of the proposed method

Applying the Riemann-Liouville fractional integral of order $\mu$ on Eq. (1.1) and using Eq.(2.7), we get the integrated formula of Eq. (1.1), as follows:

$$u_{\ell}(t) - \sum_{j=0}^{n-1} u_{\ell}^{(j)}(0^{+})\frac{t^{j}}{j!} + I^{\mu}\left[ \sum_{j=1}^{m} \left( u_j(t) + \int_0^t k_{\ell j}(t,x)u_j(x)dx \right) = f_{\ell}(t) \right], \tag{4.1}$$

$$u_{\ell}(t) - z_{\ell}(t) + I^{\mu}\left[ \sum_{j=1}^{m} \left( u_j(t) + \int_0^t k_{\ell j}(t,x)u_j(x)dx \right) = f_{\ell}(t) \right], \ell = 1,2,...,m, \tag{4.2}$$

where

$$z_{\ell}(t) = \sum_{j=0}^{n-1} u_{\ell}^{(j)}(0^{+})\frac{t^{j}}{j!}. \tag{4.3}$$

The function terms in Eq. (4.2) can be approximated as:

$$\begin{aligned} u_{\ell}(t) &\simeq U_{\ell}^{T}\psi(t) = \psi^{T}(t)U_{\ell}, \\ k_{\ell j}(t,x) &\simeq \psi^{T}(t)K_{\ell j}\psi(x), \\ f_{\ell}(t) &\simeq F_{\ell}^{T}\psi(t) = \psi^{T}(t)F_{\ell}, \\ z_{\ell}(t) &\simeq Z_{\ell}^{T}\psi(t) = \psi^{T}(t)Z_{\ell}, \end{aligned} \tag{4.4}$$

where $U_{\ell}$, $Z_{\ell}$ and $F_{\ell}$ are $(N+1)$ vector and $K_{\ell j}$ is $(N+1)\times(N+1)$ matrix $\ell, j = 1,2,...,n$. Hence, integral term in Eq.(4.2) is approximated as:

$$\begin{aligned} &\int_0^t k_{\ell j}(t,x)u_j(x)dx \simeq \psi^{T}(t)K_{\ell j}\int_0^t \psi(x)\psi^{T}(x)U_j\, dx = \\ &\psi^{T}(t)K_{\ell j}\, U_j \int_0^t \psi(x)\psi^{T}(x)dx = \psi^{T}(t)K_{\ell j}\, U_j\, M^{(0,t)} \\ &I^{\mu}\left( \int_0^t k_{\ell j}(t,x)u_j(x)dx \right) \simeq I^{\mu}\left( \psi^{T}(t)K_{\ell j}\, U_j\, M^{(0,t)} \right), \end{aligned} \tag{4.5}$$

where

$$M^{(0,t)} = \int_0^t \psi(x)\psi^T(x)dx.$$

Substituting from Eqs.(4.4) and Eq.(4.5) into Eq.(4.2), we obtain

$$R_N(t_i) = \left(\psi^T(t_i)U_\ell - \psi^T(t_i)Z_\ell + I^\mu\psi^T(t_i)\left[\sum_{j=1}^{m}\left(U_j + K_{\ell j}\ U_j\ M^{(0,t)}\right) - F_\ell\right]\right). \quad (4.6)$$

As in a typical Tau method (see [30]), we generate $(N+1)$ linear algebraic equations by applying

$$< R_N(t), \check{L}_{\alpha,k}(t) >= \int_0^\infty R_N(t)\check{L}_{\alpha,k}(t)w(t)dt = 0, \quad k = 0,1,2,...,N-n. \quad (4.7)$$

To approximate the integration in Eq. (4.7), as follow in [31], we use

$$\int_0^\infty w(t)f(t)dt = \sum_{i=0}^{n} w_i\ f(t_i) + \frac{(\Gamma(n+1))^2}{(\Gamma[2(n+1)])^2} f^{(2n)}(\zeta), \zeta \in (0,\infty). \quad (4.8)$$

Then Eq.(4.7) becomes:

$$\bar{R}(U_\ell(t_i)) = \sum_{i=0}^{n}\left(R_N(t_i)\check{L}_{\alpha,k}(t_i)\ w_i\right). \quad (4.9)$$

Eqs.(4.9) generate $(N+1)$ system of algebraic equation. This system solved by using least squares approximation to be an unconstrained optimization problems with objective function:

$$S = \sum_{i=0}^{n}\left[\bar{R}(U_\ell(t_i))\right]^2. \quad (4.10)$$

The unknown values $U_\ell(t_i)$ can be solved by using Leap Frog Algorithm (LFOPC) method (see for more details [33]).

## 5. Numerical Results

In order to show the efficiency and applicability of present method, we introduce three examples for solving system of FIDEs.

**Example 1.** Consider the following system linear Volterra FIDEs [9]:

$$\begin{cases} D^\alpha u_1(t) = f_1(t) - u_2(t) - \int_0^t \left(u_1(s) + u_2(s)\right)ds, \\ D^\alpha u_2(t) = f_2(t) - u_1(t) - \int_0^t \left(u_1(s) - u_2(s)\right)ds, \end{cases} \quad (5.1)$$

where $f_1(t) = 1 + t + t^2$ and $f_2(t) = -(1+t)$, $t \in [0,1]$, $\alpha \in ]0,1]$, with the conditions $u_1(0) = 1, u_2(0) = -1$. The exact solution for $\alpha = 1$ is $u_1(t) = t + e^t$ and $u_2(t) = t - e^t$.

By applying the proposed method with $n = 7$, we plotted the approximate solution at some selected values of $\alpha$ in Fig 1. In Table 1, we present the absolute error of $u_1$ and $u_2$. It is clear from Fig 2, the absolute error of proposed method is better than absolute error for Chebyshev pseudo-spectral method [9].

Table 1: Absolute error of Example 1 for $n = 7$ and $\alpha = 1$.

| $t$ | $u_1$ | $u_2$ |
|---|---|---|
| 0.00000 | 4.7851e-14 | 2.2204e-14 |
| 1.4286e-01 | 8.5899e-10 | 5.4253e-10 |
| 2.8571e-01 | 2.2263e-09 | 8.9617e-10 |
| 4.2857e-01 | 2.1737e-09 | 1.0056e-09 |
| 5.7143e-01 | 3.5702e-09 | 8.9099e-10 |
| 7.1429e-01 | 6.3894e-09 | 1.6161e-09 |
| 8.5714e-01 | 1.1485e-09 | 8.0605e-10 |
| 1.0000e+00 | 4.1407e-09 | 1.3921e-10 |

Figure 1: Numerical solutions for different values of $\alpha$ for Example 1.

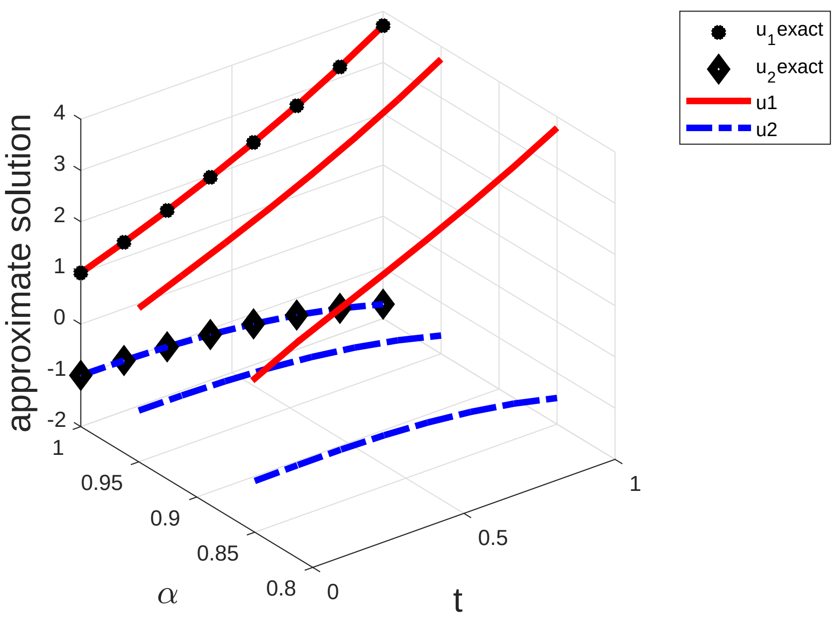
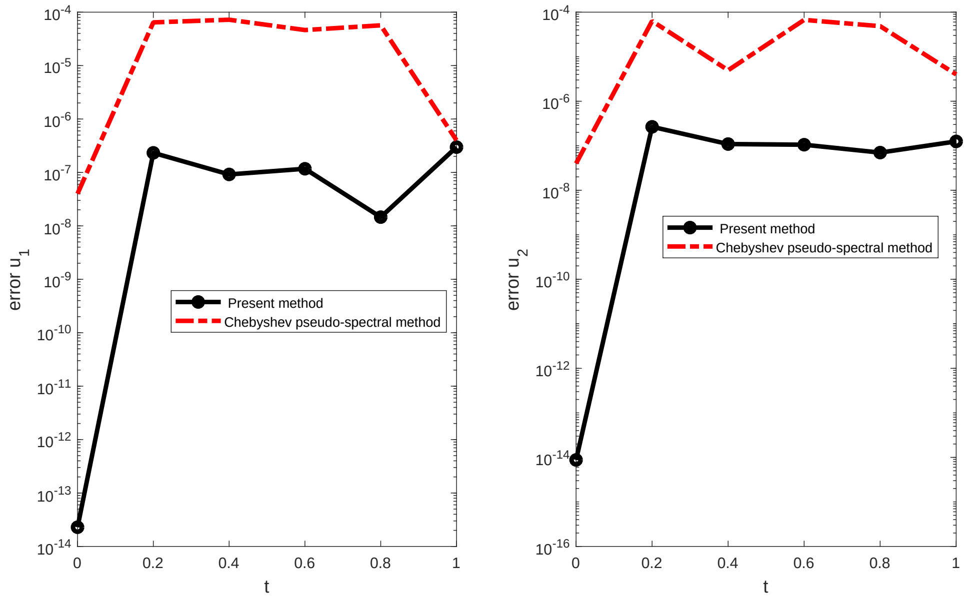


Figure 2: The absolute error comparison of present method with Chebyshev pseudo-spectral method [9] of Example 1.




**Example 2.** Now we consider the system of nonlinear Volterra FIDEs [10]:

$$\begin{cases} D^{\beta}u_1(t) = \frac{1}{3}u_1(t)u_2(t) + \frac{1}{2}u_2^2(t) + 2u_2(t) - \int_0^t \big(u_1(s)+u_2(s)\big)ds, \\ D^{\beta}u_2(t) = \frac{1}{3}u_1(t)u_2(t) - u_1(t) + 1 - \int_0^t \big(u_1(s)-2u_2(s)\big)ds, \end{cases} \tag{5.2}$$

where $t>0$, $\beta \in ]0,1]$, with the conditions $u_1(0)=0, u_2(0)=0$.
The exact solution for $\beta = 1$ is $u_1(t) = t^2$ and $u_2(t) = t$.The approximate solution is plotted at some selected values of and $\beta$ in Fig 3. In Table 2, we present the absolute error of $u_1$ and $u_2$. In Fig 4, the absolute error of proposed method is better than absolute error for wavelets method (BWM) [10].

Table 2: Absolute error of Example 2 for $n=7$ and $\beta=1$.

| $t$ | $u_1$ | $u_2$ |
|---|---|---|
| 0.00000 | 4.6313e-10 | 9.7930e-11 |
| 1.4286e-01 | 8.3883e-10 | 2.7338e-09 |
| 2.8571e-01 | 8.3878e-10 | 4.3003e-10 |
| 4.2857e-01 | 6.7606e-10 | 2.7042e-09 |
| 5.7143e-01 | 1.1265e-09 | 3.4787e-10 |
| 7.1429e-01 | 2.1987e-09 | 6.8821e-10 |
| 8.5714e-01 | 1.7535e-09 | 2.4683e-09 |
| 1.0000e+00 | 9.2903e-10 | 2.0162e-09 |

Figure 3: Numerical solutions for different values of β for Example 2.

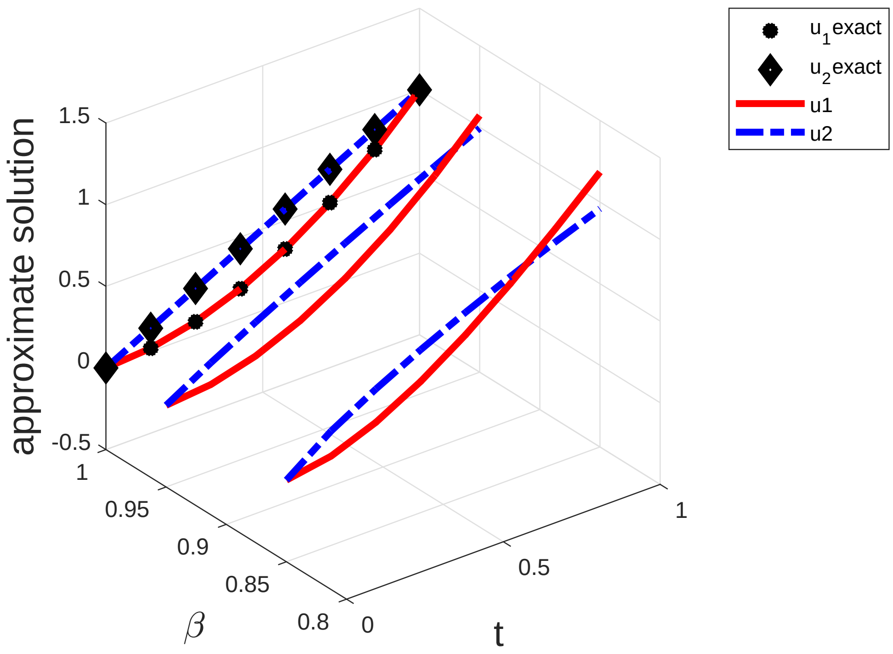
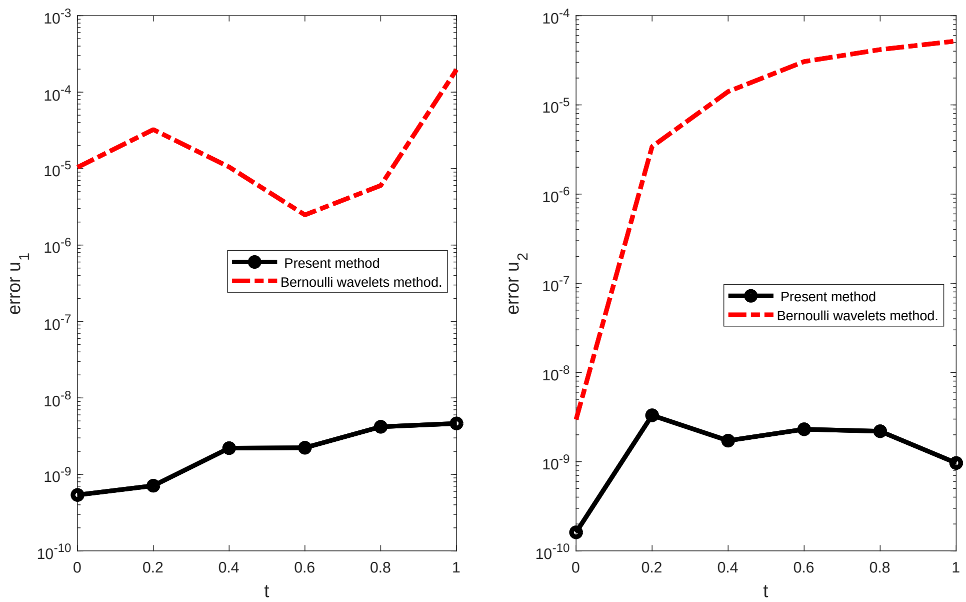



Figure 4: The absolute error comparison of present method with BWM [10] for Example 2.



**Example 3.** In this example, we consider the following system of nonlinear FIDEs [10]:

$$\begin{cases} D^{\gamma}u_1(t) = u_1^2(t) + u_2^2(t) - \int_0^t u_1(s)ds, \\ D^{\gamma}u_2(t) = \frac{-1}{2}\left(u_2^2(t) + 2u_1(t) - 1\right) - \int_0^t u_1(s)u_2(s)ds, \end{cases} \tag{5.3}$$

where $\gamma \in ]0,1], t > 0$, with the conditions $u_1(0) = 0, u_2(0) = 1$.
The exact solution for $\gamma = 1$ is $u_1(t) = \sin t$, $u_2(t) = \cos t$.
We have plotted the approximate solution at some selected values of $\gamma$ in Fig 5. In Table 3, we present the absolute error of $u_1$ and $u_2$. The absolute error of proposed method Less than absolute error for BWM [10] as it shown in Fig 6.

Table 3: Absolute error of Example 3 for $n = 7$ and $\gamma = 1$.

| $t$ | $u_1$ | $u_2$ |
|---|---|---|
| 0.00000 | 6.0054e-10 | 6.8090e-10 |
| 1.4286e-01 | 3.2167e-09 | 5.6199e-09 |
| 2.8571e-01 | 3.3274e-09 | 2.2910e-09 |
| 4.2857e-01 | 3.2031e-10 | 4.4477e-09 |
| 5.7143e-01 | 8.5872e-09 | 9.0616e-10 |
| 7.1429e-01 | 5.4139e-09 | 2.6741e-09 |
| 8.5714e-01 | 9.6374e-09 | 4.4517e-09 |
| 1.0000e+00 | 8.9055e-09 | 8.7073e-09 |

Figure 5: Numerical solutions for different values of $\gamma$ for Example 3.

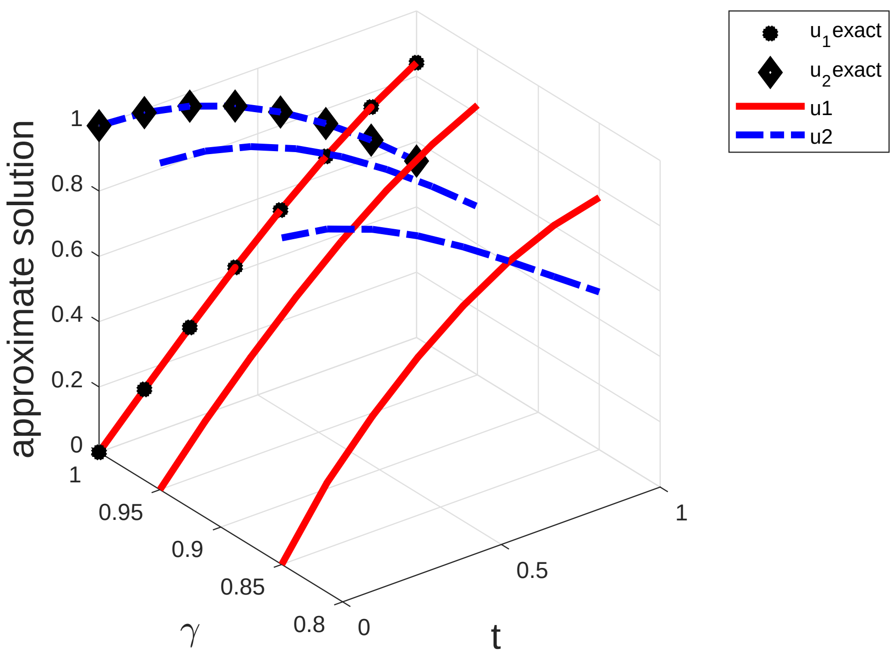
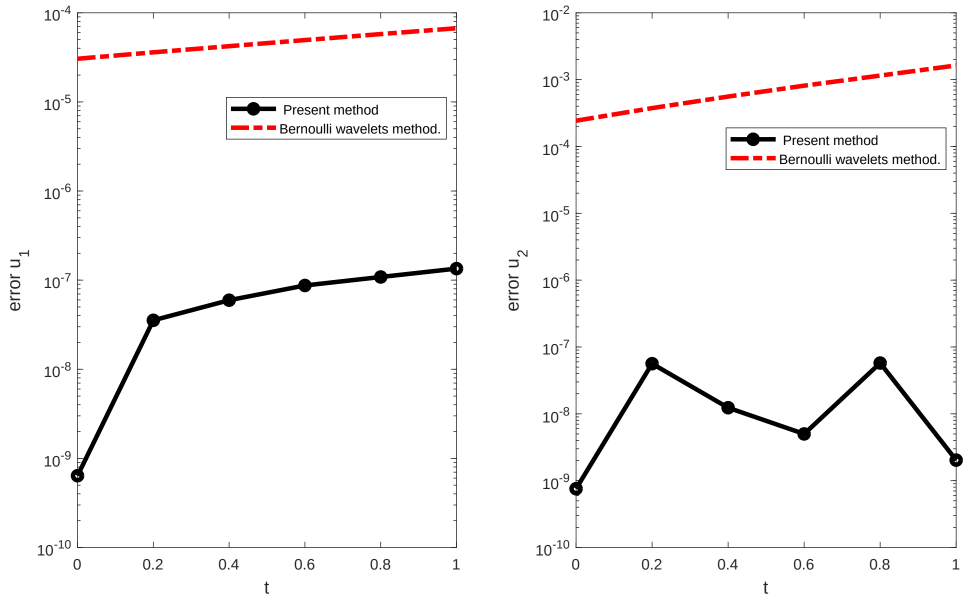


Figure 6: The absolute error comparison of present method with BWM [10] for Example 3.




In Table 4, we present CPU time and optimization error of the proposed method for all examples.

Table 4: CPU time in seconds and optimization error S.

| | | S | | |
|---|---|---|---|---|
| Example No. | CPU time | $\mu = 1$ | $\mu = 0.95$ | $\mu = 0.85$ |
| 1 | 15.20 | 8.2707e-27 | 3.3166e-27 | 4.4556e-27 |
| 2 | 23.38 | 2.5758e-17 | 3.0060e-17 | 4.1878e-17 |
| 3 | 32.30 | 9.0691e-17 | 9.8870e-17 | 1.1862e-16 |

## 6. Conclusion

In this paper, we have introduced monic Laguerre operational matrix of fractional integral. We have obtained Tau monic Laguerre method based on operational matrix. This operational matrix is applied to approximate the solution of initial value problems for systems of FIDEs. We have obtained accurate solutions for the systems of FIDEs by using Tau monic Laguerre method based on operational matrix of fractional integral. We have presented the effects of fractional order on the solution and simulations of approximate solutions graphically. The proposed method gives good accuracy with a small number of unknowns for all discussed examples and the results are more accurate compared with other treatments. We have presented the CPU time and optimization error of discussed examples. We have utilized MATLAB software of all computational work.

## Acknowledgement

The authors are thankful to the editorial board and the anonymous reviewers for their insightful comments on our paper.

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