Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2022)3(3) : 14-30
doi:10.48185/jmam.v3i3.523

Positivity results on the solutions for nonlinear two-term boundary value problem involving the ψ-Caputo fractional derivative

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• *Received: 03 July 2021* • *Accepted: 02 December 2022* • *Published Online: 30 December 2022*

Abstract

In this work, we consider a nonlinear two-term boundary value problem involving the ψ-Caputo fractional derivative with integral boundary conditions. By the construction of its associated Green function and application of the upper and lower solutions method together with some fixed point theorems due to Banach and Schauder, we establish the existence and uniqueness of positive solutions to our considered main problem. In the end some illustrative examples are provided to validate our theoretical results.

Keywords: Positive solution, ψ-Caputo fractional derivative, Green function, fixed point theorem.
2010 MSC: 34A08, 34A12, 34B15, 34B18, 47H10.

1. Introduction

Recently, many researchers have considered fractional calculus to be a generalization of integer order calculus. So, several definitions of derivatives and integrals of non-integer order have appeared. Currently the use of fractional calculus has become very important through many fields such as physics, mechanics, chemistry, biology, engineering, and electrochemistry, etc. see for example [1, 2, 3, 4, 5, 6]. In the literature, several works deal with boundary value and initial value problems using different definitions of differential operators, such as Caputo [7, 8], Generalized Caputo [9, 10], Atangana-Baleanu [11], Caputo-Fabrizio [12], Generalized Hadamard [13], Erdelyi-Kober [14, 15], Hilfer [16], Katugampola [17, 18], Riemann-Liouville [19, 20], Hadamard [21, 22], Generalized Katugampola [23], Generalized Hilfer [24], etc.

Several research works have been published recently dealing with the problems of existence of positive solutions of fractional differential equations on a cone, let us quote as examples [25, 26, 27, 28, 29, 30] and some references therein.

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Recentlly, in [31] the Researchers considered the following configuration

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0^+}\mathbf{y}(\mathfrak{z}) = \ell(\mathfrak{z},\mathbf{y}(\mathfrak{z})) + {}^{c}\mathcal{D}^{\alpha-1}_{0^+}\hbar(\mathfrak{z},\mathbf{y}(\mathfrak{z})), \quad 0<\mathfrak{z}<T, \\ \mathbf{y}(0)=\lambda_1>0, \quad \mathbf{y}'(0)=\lambda_2>0, \end{cases} \tag{1.1}$$

in the help of upper an lower solution method together with some classical fixed point theorems.

Also, Xu et al. [32] investigated the positivity of solution for the nonlinear fractional differential equation

$$\begin{cases} \mathcal{D}^{\alpha}_{0^+}\mathbf{y}(\mathfrak{z}) + \ell(\mathfrak{z},\mathbf{y}(\mathfrak{z})) + \mathcal{D}^{\beta}_{0^+}\hbar(\mathfrak{z},\mathbf{y}(\mathfrak{z})), \quad 0<\mathfrak{z}<1, \\ \mathbf{y}(0)=0, \\ \mathbf{y}(1) = \dfrac{1}{\Gamma(\alpha-\beta)}\displaystyle\int_0^1 \psi'(\mathfrak{m})\big(\psi(1)-\psi(\mathfrak{m})\big)^{\alpha-\beta-1}\hbar(\mathfrak{m},\mathbf{y}(\mathfrak{m}))\,d\mathfrak{m} \end{cases} \tag{1.2}$$

where fractional derivative is taken in the Riemann-Liouville sense.

In this work we are interested in the nonlinear two-term problem involving the generalized ψ-Caputo fractional derivative.

$$\begin{cases} {}^{c}\mathcal{D}^{\rho;\psi}_{0^+}\mathbf{y}(\mathfrak{z}) + \ell(\mathfrak{z},\mathbf{y}(\mathfrak{z})) = {}^{c}\mathcal{D}^{\sigma;\psi}_{0^+}\hbar(\mathfrak{z},\mathbf{y}(\mathfrak{z})), \quad 0<\mathfrak{z}<1, \\ \mathbf{y}(0)=0, \\ \mathbf{y}(1) = \eta\mathcal{J}^{\rho;\psi}_{0^+}\ell(1,\mathbf{y}(1)) + (1-\eta)\mathcal{J}^{\rho-\sigma;\psi}_{0^+}\hbar(1,\mathbf{y}(1)), \end{cases} \tag{1.3}$$

where $1<\rho\leqslant 2$, $0<\sigma\leqslant\rho-1$, $0<\eta\leqslant 1$, ${}^{c}\mathcal{D}^{\rho;\psi}_{0^+}$, ${}^{c}\mathcal{D}^{\sigma;\psi}_{0^+}$ are the ψ-Caputo derivatives depending on an increasing function ψ of orders ρ,σ respectively and $\ell,\hbar:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ are two known continuous functions where $\hbar(0,\mathbf{y}(0))=0$.
In order to study the existence and uniqueness of its positive solutions, we search the associated Green's function and apply the method of upper and lower solutions as well as the classical fixed point theorems due to Schauder and Banach.

2. Preliminary notions and basic tools

We denote by $\mathbf{E}=C([0,1],\mathbb{R})$ the Banach space of all continuous functions defined on $[0,1]$ with values in $\mathbb{R}$ provided with the norm $\|\mathbf{y}\| = \max_{\mathfrak{z}\in[0,1]}|\mathbf{y}(\mathfrak{z})|$, and $\mathbf{E}^1 = C^1([0,1],\mathbb{R})$.
Let us consider the space

$$\Xi = \{\mathbf{y}\in\mathbf{E} : \quad \mathbf{y}(\mathfrak{z})\geqslant 0, \quad \text{for all} \quad 0\leqslant\mathfrak{z}\leqslant 1\}.$$

It is obvious that Ξ is a subset of $\mathbf{E}$. We mean by a positive solution, each function $\mathbf{y}$ satisfies $\mathbf{y}\in\Xi$, $\mathbf{y}(0)=0$ and $\mathbf{y}(\mathfrak{z})>0$ for $0<\mathfrak{z}\leqslant 1$, i.e., $\mathbf{y}\in\Xi$.

Definition 2.1. The upper and lower functions are defined for each $\mathbf{y} \in [a, b]$ where $a, b \in \mathbb{R}^+$, by

$$U_\ell(\mathfrak{z}, \mathbf{y}) = \sup_{a \leqslant y \leqslant \mathbf{y}} \ell(\mathfrak{z}, y), \quad \text{and} \quad L_\ell(\mathfrak{z}, \mathbf{y}) = \inf_{\mathbf{y} \leqslant y \leqslant b} \ell(\mathfrak{z}, y)$$

$$U_\hbar(\mathfrak{z}, \mathbf{y}) = \sup_{a \leqslant y \leqslant \mathbf{y}} \hbar(\mathfrak{z}, y), \quad \text{and} \quad L_\hbar(\mathfrak{z}, \mathbf{y}) = \inf_{\mathbf{y} \leqslant y \leqslant b} \hbar(\mathfrak{z}, y).$$

It is obvious that the functions $U_\ell(\mathfrak{z}, \mathbf{y})$, $L_\ell(\mathfrak{z}, \mathbf{y})$, $U_\hbar(\mathfrak{z}, \mathbf{y})$, $L_\hbar(\mathfrak{z}, \mathbf{y})$ are non-decreasing with respect to the second variable and we have

$$L_\ell(\mathfrak{z}, \mathbf{y}) \leqslant \ell(\mathfrak{z}, \mathbf{y}) \leqslant U_\ell(\mathfrak{z}, \mathbf{y}) \tag{2.1}$$

$$L_\hbar(\mathfrak{z}, \mathbf{y}) \leqslant \hbar(\mathfrak{z}, \mathbf{y}) \leqslant U_\hbar(\mathfrak{z}, \mathbf{y}). \tag{2.2}$$

So that we can continue our work smoothly, we will present some useful definitions and lemmas throughout the proof of our main results.

Definition 2.2. [19] For $\rho \geqslant 0$, $\psi \in \mathbf{C}^n[a, b]$ is an increasing function which satisfies $\psi'(\mathfrak{z}) \neq 0$, for all $\mathfrak{z} \in [a, b]$ and $\varphi : [a, b] \to \mathbb{R}$ is integrable, the left-sided $\psi-$Riemann-Liouville integral and derivative of order ρ of the function φ are defined respectively, as

$$\mathcal{I}_{a^+}^{\rho;\psi} \varphi(\mathfrak{z}) = \frac{1}{\Gamma(\rho)} \int_a^{\mathfrak{z}} \psi'(m) \left[\psi(\mathfrak{z}) - \psi(m)\right]^{\rho - 1} \varphi(m) dm,$$

and

$$\mathcal{D}_{0^+}^{\rho;\psi} \varphi(\mathfrak{z}) = \left(\frac{1}{\psi'(\mathfrak{z})} \frac{d}{d\mathfrak{z}}\right)^n \mathcal{I}_{a^+}^{n-\rho;\psi} \varphi(\mathfrak{z}),$$

where $n = [\rho] + 1$ and Γ is the Euler gamma function.

Definition 2.3. [33] For $\rho \geqslant 0$, $\varphi, \psi \in \mathbf{C}^n[a, b]$ are two function such that ψ is increasing function which satisfies $\psi'(\mathfrak{z}) \neq 0$, for all $\mathfrak{z} \in [a, b]$. The left-sided $\psi-$Caputo fractional derivative of order ρ of the function φ is given as

$${}^c\mathcal{D}_{0^+}^{\rho;\psi} \varphi(\mathfrak{z}) = \mathcal{I}_{a^+}^{n-\rho;\psi} \mathcal{D}_{0^+}^{n;\psi} \varphi(\mathfrak{z}),$$

here, we have $n = [\rho] + 1$ if $\rho \notin \mathbb{N}$ and $n = \rho$ if $\rho \in \mathbb{N}$.

Proposition 2.4. *[33] For any $\rho > 0$, we have the following properties:*
- *For $\varphi \in \mathbf{C}^1[a, b]$, we have*

$${}^c\mathcal{D}_{0^+}^{\rho;\psi} \mathcal{I}_{a^+}^{\rho;\psi} \varphi(\mathfrak{z}) = \varphi(\mathfrak{z}).$$

- *For $\psi, \varphi \in \mathbf{C}^n[a, b]$, we have*

$$\mathcal{I}_{a^+}^{\rho;\psi} {}^c\mathcal{D}_{0^+}^{\rho;\psi} \varphi(\mathfrak{z}) = \varphi(\mathfrak{z}) - \sum_{j=0}^{n-1} \frac{\varphi_\psi^{[j]}}{j!} \left[\psi(\mathfrak{z}) - \psi(a)\right]^j,$$

here, we have $n = [\rho] + 1$ if $\rho \notin \mathbb{N}$ and $\varphi_\psi^{[j]}$ is defined expressed by $\varphi_\psi^{[j]}(\mathfrak{z}) = \left[\frac{1}{\psi'(\mathfrak{z})} \frac{d}{d\mathfrak{z}}\right]^j \varphi(\mathfrak{z})$.

Particularly, we have $\varphi_\psi^{'}(\mathfrak{z}) = \dfrac{\varphi'(\mathfrak{z})}{\psi'(\mathfrak{z})}$ and for $1 < \rho < 2$, we have

$$\mathcal{I}_{a^+}^{\rho;\psi} {}^c\mathcal{D}_{0^+}^{\rho;\psi} \varphi(\mathfrak{z}) = \varphi(\mathfrak{z}) - \varphi(a) - \varphi_\psi^{'}(a) \left[\psi(\mathfrak{z}) - \psi(a)\right],$$

Proposition 2.5. *[34] For $\rho > 0$, $\phi \in \mathbf{C}[a,b]$ and $\psi \in \mathbf{C}^1[a,b]$, we have for any $\mathfrak{z} \in [a,b]$*

- $\mathcal{J}_{a^+}^{\rho;\psi} : \mathbf{C}[a,b] \to \mathbf{C}[a,b]$ *is bounded.*
- $\lim\limits_{\mathfrak{z}\to a^+} \mathcal{J}_{a^+}^{\rho;\psi}\phi(\mathfrak{z}) = \mathcal{J}_{a^+}^{\rho;\psi}\phi(a) = 0.$

Proposition 2.6. *[9, 19] For $\rho, \sigma > 0$ and $\phi : [a,b] \to \mathbb{R}$, we have the following properties*

- $\mathcal{J}_{a^+}^{\rho;\psi}\mathcal{J}_{a^+}^{\sigma;\psi}\phi(\mathfrak{z}) = \mathcal{J}_{a^+}^{\rho+\sigma;\psi}\phi(\mathfrak{z}).$
- $\mathcal{J}_{a^+}^{\rho;\psi}\left[\psi(\mathfrak{z})-\psi(a)\right]^{\sigma-1} = \dfrac{\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\left[\psi(\mathfrak{z})-\psi(a)\right]^{\rho+\sigma-1}.$
- ${}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\left[\psi(\mathfrak{z})-\psi(a)\right]^{j} = 0, \quad$ *for all* $j = 0,1,\dots,n-1$ *and* $n \in \mathbb{N}$.

Our main results are based on the following fixed point theorems that we can find them in [35, 36]

Theorem 2.7. *(Banach) Let $\mathcal{T} : \Omega \to \Omega$ be a contraction operator, where Ω is a nonempty closed convex subset of a Banach space $\mathcal{B}$. Then, there exists a unique point $\mathbf{z} \in \Omega$ such that $\mathcal{T}\mathbf{z} = \mathbf{z}$.*

Theorem 2.8. *(Schauder) Let $\mathcal{T} : \Omega \to \Omega$ be a completely continuous operator, where Ω is nonempty bounded, closed and convex subset of a Banach space $\mathcal{B}$. Then, there exists at least $\mathbf{z} \in \Omega$ such that $\mathcal{T}\mathbf{z} = \mathbf{z}$.*

3. Existence and uniqueness results

Before starting the proof of our main results on the existence and uniqueness of our problem (1.3) thanks to the fixed point theorems of Schauder and Banach, we will present some lemmas which help us in the proof of our theorems.

Lemma 3.1. *Assume that $1 < \rho \leqslant 2$, $0 < \sigma \leqslant \rho - 1$, $0 < \eta \leqslant 1$, $\mathbf{y} \in \mathbf{E}$, $\psi, \mathbf{y}_\psi \in \mathbf{E}^1$ and $\ell, \hbar : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous functions such that $\hbar(0, \mathbf{y}(0)) = 0$. Then $\mathbf{y}$ is a solution of the problem (1.3) if and only if*

$$\mathbf{y}(\mathbf{t}) = \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)\hbar(m,\mathbf{y}(m))dm + \int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)\ell(m,\mathbf{y}(m))dm, \tag{3.1}$$

with

$$\mathcal{G}_\psi(\mathbf{t},m) = \frac{\eta\Theta(\mathbf{t})}{\Gamma(\rho-\sigma)}\begin{cases} \frac{1}{\eta\Theta(\mathbf{t})}\left[\psi(\mathfrak{z})-\psi(m)\right]^{\rho-\sigma-1} - \left[\psi(1)-\psi(m)\right]^{\rho-\sigma-1}, 0 \leqslant m \leqslant \mathfrak{z} \leqslant 1, \\ -\left[\psi(1)-\psi(m)\right]^{\rho-\sigma-1}, \qquad 0 \leqslant \mathfrak{z} \leqslant m \leqslant 1, \end{cases} \tag{3.2}$$

and

$$\mathcal{H}_\psi(\mathbf{t},m) = \frac{(\eta+1)\Theta(\mathbf{t})}{\Gamma(\rho)}\begin{cases} \left[\psi(1)-\psi(m)\right]^{\rho-1} - \frac{1}{(\eta+1)\Theta(\mathbf{t})}\left[\psi(\mathbf{t})-\psi(m)\right]^{\rho-1}, 0 \leqslant m \leqslant \mathfrak{z} \leqslant 1, \\ \left[\psi(1)-\psi(m)\right]^{\rho-1}, \qquad 0 \leqslant \mathfrak{z} \leqslant m \leqslant 1, \end{cases} \tag{3.3}$$

where $\Theta(\mathbf{t})=\dfrac{\psi(\mathbf{t})-\psi(0)}{\psi(1)-\psi(0)}$.

Proof. By applying the $\psi-$RL fractional integral $\mathcal{I}_{0^+}^{\rho;\psi}$ on both sides of aquation in (1.3), we get

$$\mathcal{I}_{0^+}^{\rho;\psi}{}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\mathbf{y}(\mathfrak{z})=-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)+\mathcal{I}_{0^+}^{\rho;\psi}{}^{c}\mathcal{D}_{0^+}^{\sigma;\psi}\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big).$$

By exploiting Proposition 2.6, it comes that

$$\begin{aligned}
&\mathbf{y}(\mathfrak{z})-\mathbf{y}(0)-\mathbf{y}'_{\psi}(0)\left[\psi(\mathfrak{z})-\psi(0)\right]\\
=\;&-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)+\mathcal{I}_{0^+}^{\rho;\psi}{}^{c}\mathcal{D}_{0^+}^{\sigma;\psi}\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)\\
=\;&-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)+\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\left(\mathcal{I}_{0^+}^{\sigma;\psi}{}^{c}\mathcal{D}_{0^+}^{\sigma;\psi}\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)\right)\\
=\;&-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)+\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\left(\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)-\hbar\big(0,\mathbf{y}(0)\big)\right)\\
=\;&-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)+\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)-\frac{\hbar\big(0,\mathbf{y}(0)\big)}{\Gamma(\rho-\sigma+1)}\left[\psi(\mathfrak{z})-\psi(0)\right]^{\rho-\sigma}.
\end{aligned}$$

Hence, from the hypothesis $\hbar\big(0,\mathbf{y}(0)\big)=0$ and the first initial condition $\mathbf{y}(0)=0$, it follows that

$$\mathbf{y}(\mathfrak{z})=\mathbf{y}'_{\psi}(0)\left[\psi(\mathfrak{z})-\psi(0)\right]-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)+\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big). \tag{3.4}$$

Then, by using the second boundary condition, we find

$$\mathbf{y}'_{\psi}(0)=\frac{-\eta}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar\big(1,\mathbf{y}(1)\big)+\frac{\eta+1}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(1,\mathbf{y}(1)\big). \tag{3.5}$$

By replacing (3.5) in (3.4) we obtain

$$\begin{aligned}
\mathbf{y}(\mathfrak{z})\;=\;&-\frac{\eta\big(\psi(\mathfrak{z})-\psi(0)\big)}{\big(\psi(1)-\psi(0)\big)\Gamma(\rho-\sigma)}\int_0^1\psi'(\mathfrak{m})\left[\psi(1)-\psi(\mathfrak{m})\right]^{\rho-\sigma-1}\hbar\big(\mathfrak{m},\mathbf{y}(\mathfrak{m})\big)d\mathfrak{m}\\
&+\frac{1}{\Gamma(\rho-\sigma)}\int_0^{\mathfrak{z}}\psi'(\mathfrak{m})\left[\psi(\mathfrak{z})-\psi(\mathfrak{m})\right]^{\rho-\sigma-1}\hbar\big(\mathfrak{m},\mathbf{y}(\mathfrak{m})\big)d\mathfrak{m}\\
&+\frac{(\eta+1)\big(\psi(\mathfrak{z})-\psi(0)\big)}{\big(\psi(1)-\psi(0)\big)\Gamma(\rho)}\int_0^1\psi'(\mathfrak{m})\left[\psi(1)-\psi(\mathfrak{m})\right]^{\rho-1}\ell\big(\mathfrak{m},\mathbf{y}(\mathfrak{m})\big)d\mathfrak{m}\\
&-\frac{1}{\Gamma(\rho)}\int_0^{\mathfrak{z}}\psi'(\mathfrak{m})\left[\psi(\mathfrak{z})-\psi(\mathfrak{m})\right]^{\rho-1}\ell\big(\mathfrak{m},\mathbf{y}(\mathfrak{m})\big)d\mathfrak{m},
\end{aligned} \tag{3.6}$$

otherwise written

$$\mathbf{y}(\mathbf{t})=\int_0^1\mathcal{G}_{\psi}(\mathbf{t},\mathfrak{m})\psi'(\mathfrak{m})\hbar(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}+\int_0^1\mathcal{H}_{\psi}(\mathbf{t},\mathfrak{m})\psi'(\mathfrak{m})\ell(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}$$

where $\mathcal{G}_{\psi}(\mathbf{t},\mathfrak{m})$ and $\mathcal{H}_{\psi}(\mathbf{t},\mathfrak{m})$ are the Green's functions defined respectively by (3.2) and (3.3) .
For the inverse case, just we express $\mathbf{y}(\mathbf{t})$ as

$$\begin{aligned}
\mathbf{y}(\mathfrak{z})\;=\;&-\eta\Theta(\mathfrak{z})\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar\big(1,\mathbf{y}(1)\big)+\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big)\\
&+(\eta+1)\Theta(\mathfrak{z})\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(1,\mathbf{y}(1)\big)-\mathcal{I}_{0^+}^{\rho;\psi}\ell\big(\mathfrak{z},\mathbf{y}(\mathfrak{z})\big).
\end{aligned} \tag{3.7}$$

Taking the operator ${}^{c}\mathcal{D}_{0^+}^{\rho;\psi}$ to both sides of (3.7), we obtain

$$\begin{aligned}{}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\mathbf{y}(\mathfrak{z}) &= -\eta\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar(1,\mathbf{y}(1))\,{}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\Theta(\mathfrak{z}) + {}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar(\mathfrak{z},\mathbf{y}(\mathfrak{z}))\\ &\quad +(\eta+1)\mathcal{I}_{0^+}^{\rho;\psi}\ell(1,\mathbf{y}(1))\,{}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\Theta(\mathfrak{z}) - {}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\mathcal{I}_{0^+}^{\rho;\psi}\ell(\mathfrak{z},\mathbf{y}(\mathfrak{z})).\end{aligned}$$

From Proposition 2.6 , we conclude that

$${}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\mathbf{y}(\mathfrak{z}) = \frac{1}{\psi(1)-\psi(0)}\,{}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\left[\psi(\mathfrak{z})-\psi(0)\right] = 0,$$

then, this leads to

$${}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\mathbf{y}(\mathfrak{z}) + \ell(\mathfrak{z},\mathbf{y}(\mathfrak{z})) = {}^{c}\mathcal{D}_{0^+}^{\sigma;\psi}\hbar(\mathfrak{z},\mathbf{y}(\mathfrak{z})).$$

By crossing the limits when $\mathfrak{z}$ tends to 0 and tends to 1 in equation (3.6) , we find respectively

$\mathbf{y}(0) = 0$ and $\mathbf{y}(1) = \eta\mathcal{I}_{0^+}^{\rho;\psi}\ell(1,\mathbf{y}(1)) + (1-\eta)\mathcal{I}_{0^+}^{\rho-\sigma;\psi}\hbar(1,\mathbf{y}(1))$.

Consequently, the problem (1.3) and the integral equation (3.1) are equivalent. □

Lemma 3.2. *The functions $\mathcal{G}_\psi$ and $\mathcal{H}_\psi$ satisfy the following assertions:*

(i) $\displaystyle\max_{0\leqslant\mathfrak{z}\leqslant1}\mathcal{G}_\psi(\mathfrak{z},\mathfrak{m}) = \frac{(\psi(1)-\psi(\mathfrak{m}))^{\rho-\sigma-1}}{\Gamma(\rho-\sigma)}, \quad 0<\mathfrak{m}<1,$

(ii) $\displaystyle\max_{0\leqslant\mathfrak{z}\leqslant1}\mathcal{H}_\psi(\mathfrak{z},\mathfrak{m}) = \frac{2(\psi(1)-\psi(\mathfrak{m}))^{\rho-1}}{\Gamma(\rho)}, \quad 0<\mathfrak{m}<1.$

Proof. To prove (i) and (ii), we distinguish the two cases: $0\leqslant\mathfrak{m}\leqslant\mathfrak{z}\leqslant1$ and $0\leqslant\mathfrak{z}\leqslant\mathfrak{m}\leqslant1$.

(i) For $0\leqslant\mathfrak{m}\leqslant\mathfrak{z}\leqslant1$, we have

$$\begin{aligned}\mathcal{G}_\psi(\mathfrak{z},\mathfrak{m}) &= \frac{1}{\Gamma(\rho-\sigma)}\left[(\psi(\mathfrak{z})-\psi(\mathfrak{m}))^{\rho-\sigma-1} - \eta\Theta(\mathbf{t})\left[\psi(1)-\psi(\mathfrak{m})\right]^{\rho-\sigma-1}\right]\\ &= \frac{(\psi(1)-\psi(\mathfrak{m}))^{\rho-\sigma-1}}{\Gamma(\rho-\sigma)}\left[\left(\frac{\psi(\mathfrak{z})-\psi(\mathfrak{m})}{\psi(1)-\psi(\mathfrak{m})}\right)^{\rho-\sigma-1} - \eta\Theta(\mathbf{t})\right].\end{aligned}$$

Since $0\leqslant\eta\Theta(\mathbf{t})\leqslant1$ and $0\leqslant\left(\dfrac{\psi(\mathfrak{z})-\psi(\mathfrak{m})}{\psi(1)-\psi(\mathfrak{m})}\right)^{\rho-\sigma-1}\leqslant1$, then

$\left(\dfrac{\psi(\mathfrak{z})-\psi(\mathfrak{m})}{\psi(1)-\psi(\mathfrak{m})}\right)^{\rho-\sigma-1} - \eta\Theta(\mathbf{t})\leqslant1$, hence $\mathcal{G}_\psi(\mathfrak{z},\mathfrak{m})\leqslant\dfrac{(\psi(1)-\psi(\mathfrak{m}))^{\rho-\sigma-1}}{\Gamma(\rho-\sigma)}$,

and for $0\leqslant\mathfrak{z}\leqslant\mathfrak{m}\leqslant1$, we have

$$\mathcal{G}_\psi(\mathfrak{z},\mathfrak{m}) = \frac{-\eta\Theta(\mathbf{t})}{\Gamma(\rho-\sigma)}(\psi(1)-\psi(\mathfrak{m}))^{\rho-\sigma-1}\leqslant0.$$

Consequently,

$$\max_{0\leqslant\mathfrak{z}\leqslant1}\mathcal{G}_\psi(\mathfrak{z},\mathfrak{m}) = \frac{(\psi(1)-\psi(\mathfrak{m}))^{\rho-\sigma-1}}{\Gamma(\rho-\sigma)}, \quad 0<\mathfrak{m}<1.$$

(ii) For $0 \leqslant \mathfrak{m} \leqslant \mathfrak{z} \leqslant 1$, we get

$$\begin{aligned}\mathcal{H}_{\psi}(\mathbf{t},\mathfrak{m}) &= \frac{1}{\Gamma(\rho)}\Big[(\eta+1)\Theta(\mathbf{t})\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}-\big(\psi(\mathfrak{z})-\psi(\mathfrak{m})\big)^{\rho-1}\Big]\\ &= \frac{\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}}{\Gamma(\rho)}\left[(\eta+1)\Theta(\mathbf{t})-\left(\frac{\psi(\mathfrak{z})-\psi(\mathfrak{m})}{\psi(1)-\psi(\mathfrak{m})}\right)^{\rho-1}\right],\end{aligned}$$

we know that $0 \leqslant \left(\frac{\psi(\mathfrak{z})-\psi(\mathfrak{m})}{\psi(1)-\psi(\mathfrak{m})}\right)^{\rho-1} \leqslant 1$ and $0 \leqslant (\eta+1)\Theta(\mathbf{t}) \leqslant 2$, then

$$-1 \leqslant (\eta+1)\Theta(\mathbf{t})-\left(\frac{\psi(\mathfrak{z})-\psi(\mathfrak{m})}{\psi(1)-\psi(\mathfrak{m})}\right)^{\rho-1} \leqslant 2,$$

therefore,

$$\mathcal{H}_{\psi}(\mathbf{t},\mathfrak{m}) \leqslant \frac{2\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}}{\Gamma(\rho)}.$$

On the other hand, for $0 \leqslant \mathfrak{z} \leqslant \mathfrak{m} \leqslant 1$, we get

$$\begin{aligned}\mathcal{H}_{\psi}(\mathbf{t},\mathfrak{m}) &= \frac{(\eta+1)\Theta(\mathbf{t})}{\Gamma(\rho)}\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}\\ &\leqslant \frac{2\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}}{\Gamma(\rho)}.\end{aligned}$$

Finally, we find

$$\max_{0\leqslant\mathfrak{z}\leqslant 1}\mathcal{H}_{\psi}(\mathfrak{z},\mathfrak{m}) = \frac{2\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}}{\Gamma(\rho)}, \quad 0<\mathfrak{m}<1.$$

□

In order to establish some existence and uniqueness results of solutions to our main problem (1.3), we define an operator $\mathcal{F}:\mathbf{E}\to\mathbf{E}$ as:

$$(\mathcal{F}\mathbf{y})(\mathfrak{z}) = \int_0^1 \mathcal{G}_{\psi}(\mathbf{t},\mathfrak{m})\psi'(\mathfrak{m})\hbar(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m} + \int_0^1 \mathcal{H}_{\psi}(\mathbf{t},\mathfrak{m})\psi'(\mathfrak{m})\ell(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}, \tag{3.8}$$

and we assume that the following assertion holds
$(\mathcal{ASS}1)$ For $\overline{\mathbf{y}},\underline{\mathbf{y}} \in \mathbf{E}$ such that $a \leqslant \underline{\mathbf{y}} \leqslant \overline{\mathbf{y}} \leqslant b$ we have

$$\begin{aligned}{}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\overline{\mathbf{y}}(\mathfrak{z})+U_{\ell}(\mathfrak{z},\overline{\mathbf{y}}(\mathfrak{z})) &\geqslant {}^{c}\mathcal{D}_{0^+}^{\rho;\psi}U_{\hbar}\big(\mathfrak{z},\overline{\mathbf{y}}(\mathfrak{z}))\big), \quad 0\leqslant\mathfrak{z}\leqslant 1,\\ {}^{c}\mathcal{D}_{0^+}^{\rho;\psi}\underline{\mathbf{y}}(\mathfrak{z})+L_{\ell}(\mathfrak{z},\underline{\mathbf{y}}(\mathfrak{z})) &\leqslant {}^{c}\mathcal{D}_{0^+}^{\rho;\psi}L_{\hbar}\big(\mathfrak{z},\underline{\mathbf{y}}(\mathfrak{z}))\big), \quad 0\leqslant\mathfrak{z}\leqslant 1,\end{aligned} \tag{3.9}$$

where $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ denote respectively the upper and the lower solutions of the fractional boundary value problem (1.3).

Theorem 3.3. *If the assertion $(\mathcal{ASS}1)$ holds, then at least one positive solution $\mathbf{y}\in\mathbf{E}$ exists for the FBVP problem (1.3) such that $\underline{\mathbf{y}}(\mathfrak{z}) \leqslant \mathbf{y}(\mathfrak{z}) \leqslant \overline{\mathbf{y}}(\mathfrak{z})$, for $0\leqslant\mathfrak{z}\leqslant 1$.*

Proof. Let us consider the cone

$$\mathbf{K} = \left\{\mathbf{y} \in \mathbf{E} : \quad \underline{\mathbf{y}}(\mathfrak{z}) \leqslant \mathbf{y}(\mathfrak{z}) \leqslant \overline{\mathbf{y}}(\mathfrak{z}), \quad 0 \leqslant \mathfrak{z} \leqslant 1\right\}$$

with $\|\mathbf{y}\| = \max\limits_{0\leqslant \mathfrak{z}\leqslant 1} |\mathbf{y}(\mathfrak{z})|$. Clearly, we have $\|\mathbf{y}\| \leqslant b$. Then, $\mathbf{K}$ is bounded, convex and closed subset of the Banach space $\mathbf{E}$. In addition, the continuity of the operator $\mathcal{F}$ on $\mathbf{K}$ is deduced from that of ℓ and $\hbar$. It is obvious that for any $\mathbf{y} \in \mathbf{K}$, there exist two positive real constants ω_ℓ and $\omega_\hbar$ satisfying

$$\max_{0\leqslant \mathfrak{z}\leqslant 1, \mathbf{y}(\mathfrak{z})\leqslant b} \ell\big(\mathfrak{z}, \mathbf{y}(\mathfrak{z})\big) \leqslant \omega_\ell,$$

and

$$\max_{0\leqslant \mathfrak{z}\leqslant 1, \mathbf{y}(\mathfrak{z})\leqslant b} \hbar\big(\mathfrak{z}, \mathbf{y}(\mathfrak{z})\big) \leqslant \omega_\hbar.$$

Therefore,

$$\begin{aligned}
(\mathcal{F}\mathbf{y})(\mathfrak{z}) &= \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)\hbar(m, \mathbf{y}(m))dm + \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)\ell(m, \mathbf{y}(m))dm \\
&\leqslant \int_0^1 \max_{0\leqslant \mathfrak{z}\leqslant 1} \mathcal{G}_\psi(\mathfrak{z}, m)\psi'(m)\hbar(m, \mathbf{y}(m))dm + \int_0^1 \max_{0\leqslant \mathfrak{z}\leqslant 1} \mathcal{H}_\psi(\mathfrak{z}, m)\psi'(m)\ell(m, \mathbf{y}(m))dm \\
&\leqslant \frac{\omega_\hbar}{\Gamma(\rho-\sigma)} \int_0^1 \psi'(m)\big(\psi(1)-\psi(m)\big)^{\rho-\sigma-1} dm + \frac{2\omega_\ell}{\Gamma(\rho)} \int_0^1 \psi'(m)\big(\psi(1)-\psi(m)\big)^{\rho-1} dm \\
&\leqslant \frac{\omega_\hbar}{\Gamma(\rho-\sigma+1)}\big(\psi(1)-\psi(0)\big)^{\rho-\sigma} + \frac{2\omega_\ell}{\Gamma(\rho+1)}\big(\psi(1)-\psi(0)\big)^{\rho},
\end{aligned}$$

this means that

$$\|\mathcal{F}\mathbf{y}\| \leqslant \left[\frac{\omega_\hbar}{\Gamma(\rho-\sigma+1)} + \frac{2\omega_\ell}{\Gamma(\rho+1)}\right]\big(\psi(1)-\psi(0)\big)^{\rho},$$

which implies the uniformly boundedness of $\mathcal{F}(\mathbf{K})$.
Now, we show that $\mathcal{F}(\mathbf{K})$ is equicontinuous. Let $\mathbf{y} \in \mathbf{K}$, then for all $\mathfrak{z}_1, \mathfrak{z}_2 \in [0, 1]$ such that

$\mathfrak{z}_1 < \mathfrak{z}_2$, we have

$$\begin{aligned}\left|(\mathcal{F}\mathbf{y})(\mathfrak{z}_2)-(\mathcal{F}\mathbf{y})(\mathfrak{z}_1)\right| &= \Bigg|\int_0^1\Big(\mathcal{G}_\psi(\mathbf{t}_2,\mathfrak{m})-\mathcal{G}_\psi(\mathbf{t}_1,\mathfrak{m})\Big)\psi'(\mathfrak{m})\hbar(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}\\ &\quad+\int_0^1\Big(\mathcal{H}_\psi(\mathbf{t}_2,\mathfrak{m})-\mathcal{H}_\psi(\mathbf{t}_1,\mathfrak{m})\Big)\psi'(\mathfrak{m})\ell(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}\Bigg|\\ &\leqslant \int_0^1\left|\mathcal{G}_\psi(\mathbf{t}_2,\mathfrak{m})-\mathcal{G}_\psi(\mathbf{t}_1,\mathfrak{m})\right|\psi'(\mathfrak{m})\hbar(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}\\ &\quad+\int_0^1\left|\mathcal{H}_\psi(\mathbf{t}_2,\mathfrak{m})-\mathcal{H}_\psi(\mathbf{t}_1,\mathfrak{m})\right|\psi'(\mathfrak{m})\ell(\mathfrak{m},\mathbf{y}(\mathfrak{m}))d\mathfrak{m}. \end{aligned} \tag{3.10}$$

Since

$$\begin{aligned}\left|\mathcal{G}_\psi(\mathbf{t}_2,\mathfrak{m})-\mathcal{G}_\psi(\mathbf{t}_1,\mathfrak{m})\right| &= \Bigg|\frac{1}{\Gamma(\rho-\sigma)}\Big[\big(\psi(\mathfrak{z}_2)-\psi(\mathfrak{m})\big)^{\rho-\sigma-1}-\big(\psi(\mathfrak{z}_1)-\psi(\mathfrak{m})\big)^{\rho-\sigma-1}\\ &\quad-\eta\big(\Theta(\mathbf{t}_2)-\Theta(\mathbf{t}_1)\big)\left[\psi(1)-\psi(\mathfrak{m})\right]^{\rho-\sigma-1}\Big]\Bigg|\\ &\leqslant \frac{1}{\Gamma(\rho-\sigma)}\Big[\big(\psi(\mathfrak{z}_2)-\psi(\mathfrak{m})\big)^{\rho-\sigma-1}-\big(\psi(\mathfrak{z}_1)-\psi(\mathfrak{m})\big)^{\rho-\sigma-1}\\ &\quad+\eta\big(\Theta(\mathbf{t}_2)-\Theta(\mathbf{t}_1)\big)\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-\sigma-1}\Big]\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{H}_\psi(\mathbf{t}_2,\mathfrak{m})-\mathcal{H}_\psi(\mathbf{t}_1,\mathfrak{m})\right| &= \Bigg|\frac{1}{\Gamma(\rho)}\Big[(\eta+1)\big(\Theta(\mathbf{t}_2)-\Theta(\mathbf{t}_1)\big)\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}\\ &\quad-\big(\psi(\mathfrak{z}_2)-\psi(\mathfrak{m})\big)^{\rho-1}+\big(\psi(\mathfrak{z}_1)-\psi(\mathfrak{m})\big)^{\rho-1}\Big]\Bigg|\\ &\leqslant \frac{1}{\Gamma(\rho)}\Big[(\eta+1)\big(\Theta(\mathbf{t}_2)-\Theta(\mathbf{t}_1)\big)\big(\psi(1)-\psi(\mathfrak{m})\big)^{\rho-1}\\ &\quad+\big(\psi(\mathfrak{z}_2)-\psi(\mathfrak{m})\big)^{\rho-1}-\big(\psi(\mathfrak{z}_1)-\psi(\mathfrak{m})\big)^{\rho-1}\Big],\end{aligned}$$

Hence, from (3.10), it follows that

$$\begin{aligned}\left|(\mathcal{F}\mathbf{y})(\mathfrak{z}_2)-(\mathcal{F}\mathbf{y})(\mathfrak{z}_1)\right| &\leqslant \frac{\omega_\hbar}{\Gamma(\rho-\sigma+1)}\Big[-\big(\psi(\mathfrak{z}_2)-\psi(1)\big)^{\rho-\sigma}+\big(\psi(\mathfrak{z}_2)-\psi(0)\big)^{\rho-\sigma} \\ &\quad+\big(\psi(\mathfrak{z}_1)-\psi(1)\big)^{\rho-\sigma}-\big(\psi(\mathfrak{z}_1)-\psi(0)\big)^{\rho-\sigma} \\ &\quad+\eta\big(\Theta(\mathbf{t}_2)-\Theta(\mathbf{t}_1)\big)\big(\psi(1)-\psi(0)\big)^{\rho-\sigma-1}\Big] \\ &\quad+\frac{\omega_\ell}{\Gamma(\rho+1)}\Big[(\eta+1)\big(\Theta(\mathbf{t}_2)-\Theta(\mathbf{t}_1)\big)\big(\psi(1)-\psi(0)\big)^{\rho}+\big(\psi(\mathfrak{z}_2)-\psi(1)\big)^{\rho} \\ &\quad-\big(\psi(\mathfrak{z}_2)-\psi(0)\big)^{\rho}-\big(\psi(\mathfrak{z}_2)-\psi(1)\big)^{\rho}+\big(\psi(\mathfrak{z}_1)-\psi(0)\big)^{\rho}\Big], \qquad (3.11)\end{aligned}$$

The right-hand side of inequality (3.11) goes to zero independently of $\mathbf{y}$ when $\mathbf{t}_2-\mathbf{t}_1\to 0$, Thus, $(\mathcal{F}\mathbf{K})$ is equicontinuous. Consequently, by Arzelà-Ascoli theorem it follows that the operator $\mathcal{F}:\mathbf{E}\to\mathbf{E}$ is compact.

Now, in order to apply Theorem 2.8, we have to show that $\mathcal{F}(\mathbf{K})\subset\mathbf{K}$. Let $\mathbf{y}\in\mathbf{K}$. Then in view of the assertion $(\mathcal{ASS}1)$ together with the definitions of upper and lower solutions, we obtain

$$\begin{aligned}(\mathcal{F}\mathbf{y})(\mathfrak{z}) &= \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)\hbar(m,\mathbf{y}(m))dm+\int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)\ell(m,\mathbf{y}(m))dm \\ &\leqslant \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)U_\hbar(m,\mathbf{y}(m))dm+\int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)U_\ell(m,\mathbf{y}(m))dm \\ &\leqslant \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)U_\hbar(m,\overline{\mathbf{y}}(m))dm+\int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)U_\ell(m,\overline{\mathbf{y}}(m))dm \\ &\leqslant \overline{\mathbf{y}}(\mathfrak{z}),\quad \mathfrak{z}\in[0,1], \qquad (3.12)\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F}\mathbf{y})(\mathfrak{z}) &= \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)\hbar(m,\mathbf{y}(m))dm+\int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)\ell(m,\mathbf{y}(m))dm \\ &\geqslant \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)L_\hbar(m,\mathbf{y}(m))dm+\int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)L_\ell(m,\mathbf{y}(m))dm \\ &\geqslant \int_0^1 \mathcal{G}_\psi(\mathbf{t},m)\psi'(m)L_\hbar(m,\underline{\mathbf{y}}(m))dm+\int_0^1 \mathcal{H}_\psi(\mathbf{t},m)\psi'(m)L_\ell(m,\underline{\mathbf{y}}(m))dm \\ &\geqslant \underline{\mathbf{y}}(\mathfrak{z}),\quad \mathfrak{z}\in[0,1]. \qquad (3.13)\end{aligned}$$

Thus, from the inequalities (3.12) and (3.13) we get $\underline{\mathbf{y}}(\mathfrak{z})\leqslant(\mathcal{F}\mathbf{y})(\mathfrak{z})\leqslant\overline{\mathbf{y}}(\mathfrak{z})$, for any $\mathfrak{z}\in[0,1]$. Therefore, the operator $\mathcal{F}$ fulfills all assumptions of Theorem 2.8, which ipmlies that $\mathcal{F}$ admits at least one solution $\mathbf{y} \in \mathbf{K}$. Consequently, our problem (1.3) has at least one solution $\mathbf{y} \in \mathbf{E}$ satisfying $\underline{\mathbf{y}}(\mathfrak{z}) \leqslant \mathbf{y}(\mathfrak{z}) \leqslant \overline{\mathbf{y}}(\mathfrak{z})$, $\mathfrak{z} \in [0,1]$. □

Corollary 3.4. *Assume that there exist four strictly real positive constants ℓ_1, ℓ_2, ℓ_3 and ℓ_4 satisfying the following inequalities*

$$\ell_1 \leqslant \ell(\mathfrak{z}, \mathbf{y}) \leqslant \ell_2, \quad (\mathfrak{z}, \mathbf{y}) \in [0,1] \times \mathbb{R}^+, \tag{3.14}$$

and

$$\ell_3 \leqslant \hbar(\mathfrak{z}, \mathbf{y}) \leqslant \ell_4, \quad (\mathfrak{z}, \mathbf{y}) \in [0,1] \times \mathbb{R}^+. \tag{3.15}$$

Then the problem (1.3) *admits at least one positive solution $\mathbf{y} \in \mathbf{K}$. In addition*

$$\mathbf{y}(\mathfrak{z}) \geqslant \ell_3 \int_0^1 \mathcal{G}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m} + \ell_1 \int_0^1 \mathcal{H}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m}, \tag{3.16}$$

and

$$\mathbf{y}(\mathfrak{z}) \leqslant \ell_4 \int_0^1 \mathcal{G}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m} + \ell_2 \int_0^1 \mathcal{H}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m}. \tag{3.17}$$

Proof. Let us define the following problems

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\rho;\psi} \overline{\mathbf{y}}(\mathfrak{z}) + \ell_2 = {}^c\ \mathcal{D}_{0^+}^{\sigma;\psi} \ell_4, \quad 0 < \mathfrak{z} < 1, \\ \overline{\mathbf{y}}(0) = 0, \\ \overline{\mathbf{y}}(1) = \eta \big(\mathcal{I}_{0^+}^{\rho;\psi} \ell_2 \big)(1) + (1-\eta) \big(\mathcal{I}_{0^+}^{\rho-\sigma;\psi} \ell_4 \big)(1), \end{cases} \tag{3.18}$$

and

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\rho;\psi} \underline{\mathbf{y}}(\mathfrak{z}) + \ell_1 = {}^c\ \mathcal{D}_{0^+}^{\sigma;\psi} \ell_3, \quad 0 < \mathfrak{z} < 1, \\ \underline{\mathbf{y}}(0) = 0, \\ \underline{\mathbf{y}}(1) = \eta \big(\mathcal{I}_{0^+}^{\rho;\psi} \ell_1 \big)(1) + (1-\eta) \big(\mathcal{I}_{0^+}^{\rho-\sigma;\psi} \ell_3 \big)(1). \end{cases} \tag{3.19}$$

By using Lemma 3.1, it easy to deduce that the problems (3.18) and (3.19) are respectively equivalent to the folowing integral equations

$$\overline{\mathbf{y}}(\mathfrak{z}) = \ell_4 \int_0^1 \mathcal{G}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m} + \ell_2 \int_0^1 \mathcal{H}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m}, \tag{3.20}$$

and

$$\underline{\mathbf{y}}(\mathfrak{z}) = \ell_3 \int_0^1 \mathcal{G}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m} + \ell_1 \int_0^1 \mathcal{H}_\psi(\mathbf{t}, \mathfrak{m}) \psi'(\mathfrak{m}) d\mathfrak{m}. \tag{3.21}$$

By exploiting assumptions (3.14) and (3.15) and inequalities (2.1) and (2.2), we obtain

$$\ell_1 \leqslant L_\ell(\mathfrak{z}, \mathbf{y}) \leqslant U_\ell(\mathfrak{z}, \mathbf{y}) \leqslant \ell_2, \quad (\mathfrak{z}, \mathbf{y}) \in [0,1] \times [a,b], \tag{3.22}$$

and

$$\ell_3 \leqslant L_\hbar(\mathfrak{z}, \mathbf{y}) \leqslant U_\hbar(\mathfrak{z}, \mathbf{y}) \leqslant \ell_4, \quad (\mathfrak{z}, \mathbf{y}) \in [0,1] \times [a,b], \tag{3.23}$$

where $a = \min\limits_{\mathfrak{z}\in[0,1]} \mathbf{y}(\mathfrak{z}), \quad b = \max\limits_{\mathfrak{z}\in[0,1]} \mathbf{y}(\mathfrak{z})$. Therefore,

$$\begin{aligned}
\mathbf{y}(\mathbf{t}) &= \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)\hbar(m, \mathbf{y}(m))dm + \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)\ell(m, \mathbf{y}(m))dm \\
&\geqslant \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)L_\hbar(m, \mathbf{y})dm + \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)L_\ell(m, \mathbf{y})dm \\
&\geqslant \ell_3 \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)dm + \ell_1 \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)dm \\
&= \underline{\mathbf{y}}(\mathbf{t}),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{y}(\mathbf{t}) &= \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)\hbar(m, \mathbf{y}(m))dm + \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)\ell(m, \mathbf{y}(m))dm \\
&\leqslant \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)U_\hbar(m, \mathbf{y})dm + \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)U_\ell(m, \mathbf{y})dm \\
&\leqslant \ell_4 \int_0^1 \mathcal{G}_\psi(\mathbf{t}, m)\psi'(m)dm + \ell_2 \int_0^1 \mathcal{H}_\psi(\mathbf{t}, m)\psi'(m)dm \\
&= \overline{\mathbf{y}}(\mathbf{t}).
\end{aligned}$$

□

Example 3.5. With regard to the fractional boundary value problem (1.3) , we consider in this example

$$\ell(m, \mathbf{y}) = \ell_1 + (\ell_2 - \ell_1)m, \quad 0 \leqslant m \leqslant 1,$$

and

$$\hbar(m, \mathbf{y}) = \ell_3 + (\ell_4 - \ell_3)m, \quad 0 \leqslant m \leqslant 1.$$

We have clearly,

$$\ell_1 \leqslant \ell(m, \mathbf{y}) \leqslant \ell_2, \quad \text{and} \quad \ell_3 \leqslant \hbar(m, \mathbf{y}) \leqslant \ell_4.$$

So that we can plot $\underline{\mathbf{y}}(\mathbf{t}), \overline{\mathbf{y}}(\mathbf{t})$ and $\mathbf{y}(\mathbf{t})$, we take the following numerical values.

$$\rho = 1.5, \quad \sigma = 0.5, \quad \eta = 0.25, \quad \ell_1 = 1, \quad \ell_2 = \ell_3 = 2, \quad \ell_4 = 3,$$

with the increasing function $\psi(t) = t$. After computing, we obtain

$$\begin{aligned}\overline{y}(t) &= 2.4167t + 1.1543t(1-t)^{1.5} - 1.5045t^{1.5},\\ \underline{y}(t) &= 1.5000t + 0.5771t(1-t)^{1.5} - 0.7523t^{1.5},\\ y(t) &= 1.9583t - 0.0560t^2 - 0.1250t^3 - 1.1284t^{1.5} + 0.0571t(1+t)(1-t)^{1.5}\\ &\quad +0.8657t(1-t)^{1.5} - 0.1250t(1-t)^{2.5} - 0.0071(1+t)(1-t)^{1.5}.\end{aligned}$$

Theorem 3.6. *Assume that $\ell, \hbar : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous functions such that there exist two real constants $\mu_1, \mu_2 > 0$ which satisfy*

$$|\ell(t,y) - \ell(t,y)| \leqslant \mu_1|y-y|, \quad t \in [0,1], \quad (y,y) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

and

$$|\hbar(t,y) - \hbar(t,y)| \leqslant \mu_2|y-y|, \quad t \in [0,1], \quad (y,y) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

If

$$\omega = \frac{\mu_1\big(\psi(1)-\psi(0)\big)^{\rho-\sigma}}{\Gamma(\rho-\sigma+1)} + \frac{2\mu_2\big(\psi(1)-\psi(0)\big)^{\rho}}{\Gamma(\rho+1)} < 1, \tag{3.25}$$

then the FBVP (1.3) admits a unique solution $y \in K$.

Proof. From Theorem 3.3 , it follows that the problem (1.3) has at least one positive solution in K. So, it suffices to show that the operator $\mathcal{F}$ expressed by (3.8) is a contraction mapping in K. It is clear that if $y \in K$, then $\mathcal{F}y \in K$. In addition, for any $t \in [0,1]$ and $y, y \in \mathbb{R}^+$, we have

$$\begin{aligned}\|\mathcal{F}y - \mathcal{F}y\| &= \max_{t\in[0,1]} |\mathcal{F}y(t) - \mathcal{F}y(t)|\\ &\leqslant \max_{t\in[0,1]} \left(\int_0^1 \mathcal{G}_\psi(t,m)\psi'(m)|\hbar(m,y(m)) - \hbar(m,y(m))|dm\right.\\ &\quad \left.+\int_0^1 \mathcal{H}_\psi(t,m)\psi'(m)|\ell(m,y(m)) - \ell(m,y(m))|dm\right)\\ &\leqslant \frac{1}{\Gamma(\rho-\sigma)}\int_0^1 \big(\psi(1)-\psi(m)\big)^{\rho-\sigma-1}\mu_1\|y-y\|dm\\ &\quad +\frac{2}{\Gamma(\rho)}\int_0^1 \big(\psi(1)-\psi(m)\big)^{\rho-1}\mu_2\|y-y\|dm\\ &\leqslant \left(\frac{\mu_1\big(\psi(1)-\psi(0)\big)^{\rho-\sigma}}{\Gamma(\rho-\sigma+1)} + \frac{2\mu_2\big(\psi(1)-\psi(0)\big)^{\rho}}{\Gamma(\rho+1)}\right)\|y-y\|\\ &= \omega\|y-y\|.\end{aligned}$$

Since from the condtion (3.25) we have $\omega < 1$, hence the operator $\mathcal{F}$ is a conraction. Thus, The Theorem 2.7 ensures that the FBVP (1.3) has a unique solution $y \in K$. □

Example 3.7. With regard to the FBVP (1.3), we consider in this situation the following problem

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\frac{3}{2};e^{\mathfrak{z}}}\mathbf{y}(\mathfrak{z}) + \dfrac{1}{(2+\mathfrak{z})e^2}\left(2+\dfrac{\mathfrak{z}\mathbf{y}(\mathfrak{z})}{1+\mathbf{y}(\mathfrak{z})}\right) = {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2};e^{\mathfrak{z}}}\dfrac{1}{\pi^2}\left(\dfrac{1+\mathbf{y}(\mathfrak{z})}{3+\mathbf{y}(\mathfrak{z})}-\dfrac{1}{3}\right), \quad 0<\mathfrak{z}<1, \\ \mathbf{y}(0)=0, \\ \mathbf{y}(1) = \dfrac{1}{12e^2}\mathcal{J}_{0^+}^{\frac{3}{2};e^{\mathfrak{z}}}\left(2+\dfrac{\mathbf{y}(1)}{1+\mathbf{y}(1)}\right) + \dfrac{3}{4\pi^2}\mathcal{J}_{0^+}^{1;e^{\mathfrak{z}}}\left(\dfrac{1+\mathbf{y}(1)}{3+\mathbf{y}(1)}-\dfrac{1}{3}\right), \end{cases} \tag{3.26}$$

In this construction, we have

$$\rho = \frac{3}{2}, \quad \sigma = \frac{1}{2}, \quad \eta = \frac{1}{4}, \quad \psi(\mathfrak{z}) = e^{\mathfrak{z}},$$

$$\ell(\mathfrak{z},\mathbf{y}) = \frac{1}{(2+\mathfrak{z})e^2}\left(2+\frac{\mathfrak{z}\mathbf{y}}{1+\mathbf{y}}\right),$$

and

$$\hbar(\mathfrak{z},\mathbf{y}) = \frac{1}{\pi^2}\left(\frac{1+\mathbf{y}}{3+\mathbf{y}}-\frac{1}{3}\right).$$

Hence, the condition $\hbar(0,\mathbf{y}(0)) = 0$ holds and for any $\mathbf{y},\mathbf{y} \in \mathbb{R}^+$ and each $\mathfrak{z} \in [0,1]$, we have

$$\begin{aligned} |\ell(\mathfrak{z},\mathbf{y}) - \ell(\mathfrak{z},\mathbf{y})| &= \frac{1}{(2+\mathfrak{z})e^2}\left|\frac{\mathfrak{z}\mathbf{y}}{1+\mathbf{y}} - \frac{\mathfrak{z}\mathbf{y}}{1+\mathbf{y}}\right| \\ &\leqslant \frac{1}{2e^2}|\mathbf{y}-\mathbf{y}|, \end{aligned}$$

and

$$\begin{aligned} |\hbar(\mathfrak{z},\mathbf{y}) - \hbar(\mathfrak{z},\mathbf{y})| &= \frac{1}{\pi^2}\left|\frac{1+\mathbf{y}}{3+\mathbf{y}} - \frac{1+\mathbf{y}}{3+\mathbf{y}}\right| \\ &\leqslant \frac{2}{9\pi^2}|\mathbf{y}-\mathbf{y}|, \end{aligned}$$

then, we have

$$\mu_1 = \frac{1}{2e^2}, \quad \mu_2 = \frac{2}{9\pi^2}.$$

Therefore, we get

$$\omega = \frac{e-1}{2e^2} + \frac{4(e-1)^{\frac{3}{2}}}{9\pi^2\Gamma\left(\frac{5}{2}\right)} \approx 0.1926 < 1.$$

Consequently, by using Theorem 3.6 we conclude that the FBVP (3.26) has a unique solution.

If we go further, we note that the functions ℓ and $\hbar$ are nondecreasing on the second varible $\mathbf{y}$ with

$$\lim_{\mathbf{y}\to+\infty}\ell(\mathfrak{z},\mathbf{y}) = \frac{1}{e^2}, \quad \lim_{\mathbf{y}\to+\infty}\hbar(\mathfrak{z},\mathbf{y}) = \frac{2}{3\pi^2},$$

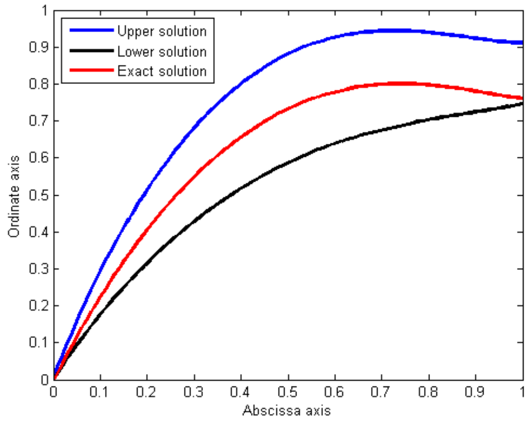

Figure 1: Graph of the exact solution $\mathbf{y}$, compared with the upper and lower solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$.

and

$$\frac{2}{3e^2} \leqslant \ell(\mathfrak{z}, \mathbf{y}) \leqslant \frac{1}{e^2}, \quad \frac{1}{3\pi^2} \leqslant \hbar(\mathfrak{z}, \mathbf{y}) \leqslant \frac{2}{3\pi^2},$$

for any $\mathfrak{z} \in [0, 1]$ and $\mathbf{y} \in \mathbb{R}^+$. Hence, by taking into account $\ell_1 = \frac{2}{3e^2}$, $\ell_2 = \frac{1}{e^2}$, $\ell_3 = \frac{1}{3\pi^2}$, $\ell_4 = \frac{2}{3\pi^2}$, and applying Corollary 3.4, we deduce that the FBVP (3.26) admits at least one solution which satisfies $\underline{\mathbf{y}}(\mathfrak{z}) \leqslant \mathbf{y}(\mathfrak{z}) \leqslant \overline{\mathbf{y}}(\mathfrak{z})$ where $\overline{\mathbf{y}}(\mathfrak{z})$ and $\underline{\mathbf{y}}(\mathfrak{z})$ can be easily calculated by using the expressions (3.20) and (3.21).

Acknowledgement

The author would like to thank the referees and the handling editor for their careful reading and for relevant remarks/suggestions to improve the paper.

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