On Orthogonality of Elementary Operators in Normed Spaces

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Abstract

In this paper, we give a detailed survey on characterization of orthogonality of elementary operators in normed spaces particularly in the space of norm-attainable operators. In particular, we consider these operators when they are finite and unveil new conditions which are necessary and sufficient for their orthogonality. Lastly, we characterize Birkhoff-James orthogonality for this class of operators. We have shown that finite elementary operators satisfy orthogonality in the sense of Birkhoff-James if they are bounded, isometric and normal.

Keywords: Elementary operator, Inner derivation, Generalized derivation, Finiteness, Orthogonality.
2010 MSC: put your Mathematics Subject Classification 2010 (MSC) 47B47, 47A30 .

1. Introduction

Studies involving operators and mathematical models (see [1],[2], [3] and the references therein) have been studied by many researchers with a lot of results obtained. In [4] the authors described Fractal-Fractional Differential Equations through giving approximations using numerical inverse Laplace transforms while the works of [5] and [6] studied and gave a mathematical model on Integer and Fractional Order for COVID-19. Most recently, the authors in [7], [8] and [9] gave results involving mathematical model of cutaneous leishmaniasis disease with inputs on how it can be treated. Orthogonality of operators in normed spaces has also attracted the attention of many mathematicians for a very long period of time and it still remains interesting (see [10], [11], [12], [13] and [14]). The usual definition of orthogonality of vectors of a metric space is that $q \perp h$ [15] if and only if the inner product $\langle q, h \rangle = 0$. Orthogonality in any normed linear space can not be defined in the same way of an inner product space because a normed space is not always an inner product space [16]. Hence, since 1934 various concepts of orthogonality in Hilbert spaces have been studied and introduced by [17], [18], [19], [20], [21], [22], [23], [24], [25], [26] and [27] among others. These studies lead to several versions of orthogonality such as:

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(i). Rorberts orthogonality(1934): $\|j - \gamma k\| = \|j + \gamma k\|$, for all $\gamma \in \mathbb{R}$.

(ii). Birkhoff orthogonality(1935): $\|j\| \leqslant \|j + \gamma k\|$, for all $\gamma \in \mathbb{R}$.

(iii). Isosceles orthogonality(1945): $\|j - k\| = \|j + k\|$.

(iv). Pythagorean orthogonality(1945): $\|j - k\|^2 = \|j\|^2 + \|k\|^2$.

(v). Singer orthogonality(1957): $j = 0$ or $k = 0$ or $\|\frac{j}{\|j\|} + \frac{k}{\|k\|}\| = \|\frac{j}{\|j\|} - \frac{k}{\|k\|}\|$.

(vi). a-isosceles orthogonality(1988): $\|j - ak\| = \|j + ak\|$.

(vii). a-pythagorean orthogonality(1988): $\|j - ak\|^2 = \|j\|^2 + a^2\|k\|^2$.

(viii). Carlsson(1961): $\sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0$ where $m \geqslant 2$ and $a_k, b_k, c_k \in \mathbb{R}$, $\sum_{k=1}^{m} a_k b_k c_k \neq 0$, $\sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0$.

(ix). ab(1978): $\|ap + bq\|^2 + \|p + q\|^2 = \|ap + q\|^2 + \|p + bq\|^2$.

(x). a(1983): $(1 + a^2)\|q + r\|^2 = \|aq + r\|^2 + \|q + ar\|^2$.

(xi). U-isosceles(1957): either $\|q\|\|r\| = 0$, or $\|q\|^{-1}q$ is isosceles-orthogonal to $\|r\|^{-1}r$.

(xii). U-pythagorean(1986): either $\|q\|\|r\| = 0$, or $\|q\|^{-1}q$ is pythagorean-orthogonal to $\|r\|^{-1}r$.

(xiii). Area(1986): either $\|q\|\|r\| = 0$, or they are linearly independent and such that $q, r, -q, -r$ cut the unit ball of their plane independently in four equivalent parts.

(xiv). Diminnie(1983): $\sup\{q(r)s(t) - q(t)s(r) : q, s \in S'\} = \|r\|\|t\|$, S' representing the unit sphere of the space of linear functionals of E.

A bounded linear operator Q on a Hilbert space H is finite if $\|QX - XQ - I\| \geqslant 1$ for each $X \in L(H)$ [28]. Results of [29] showed that the algebra of finite operators involves normal operators, operators that are closed, operators with a uniformly continuous summand, and the Banach algebra with an involution satisfying the properties of adjoint originating from each and every member. The results implied the group of self-commutators is uniformly closed and that the class of operators that have a reducing subspace of finite dimension is non-uniformly dense as seen in [30], [31], [32], [33] and [34]. In [35], the study gave a new class of finite operators using the knowledge of the reducing approximate spectrum of an operator. In this case the concept of completely finite operators was introduced. Those are operators A such that A_E is finite for any orthogonal reducing subspace E of A as asserted by [36] and [37] and [38]. The work of [39] and [40] improved the Du Hong-Ke inequality to $\|QZQ - Z + J\| \geqslant \|Z\|$ for all operators Z. Indeed, it was proved that the Du Hong-Ke inequality is valid for unitary invariant norms and it was shown that the Du Hong-Ke inequality is equivalent to the Anderson inequality [41]. The research in [42] introduced another group "class A" provided by operator inequalities that involves the group of paranormal operators and the group of log-hyponormal operators. It turned out in [43] that their results contained another proof of Ando's results in which every log-hyponormal operator is paranormal. New groups of operators similar to class A operators and paranormal operators were also introduced in [44], [45], [46], [47], [48], [49] and [50]. The author in [51] gave a group of finite operators of the form $S + G$ whereby $S \in L(Z)$ and G is compact whereby it was proved that $w_o(\delta_{S,P}) = c_o\delta(\delta_{S,P})$, where $w_o(\delta_{S,P}), c_o\delta(\delta_{S,P})$ denote respectively the numerical range of $\delta_{S,P}$ and the convex hull of $\delta(\delta_{S,P})$ (the spectrum of $\delta_{S,P}$) for certain operators $S, P \in L(Z)$, $\delta_{S,P}$ is the ant operator on $L(Z)$ defined by $\delta_{S,P} = SZ - ZP$ $Z \in L(Z)$. In [52] the researcher characterized the operators $T \in L(H)$ and proved the range-kernel orthogonality results for the operators $Q, R \in L(H)$ that are non-normal in terms of Birkoff-James and [53] also introduced another notion to characterize Anderson's theorem that is independent of normality through

the Putnam-Fuglede property [54]. In [55], they gave results on orthogonality of dominant operators and log-hyponormal or p-hyponormal operators and studied orthogonality of certain operators. The main goal was to determine the range-kernel orthogonality results of $\delta_{S,R}$ for some operators [56]. In [57] they also proved that the range of $\delta_{S,R}$ is orthogonal to the nullspace of $\delta_{A,B}$ when B^* and A is dominant is log-hyponormal or p-hyponormal. In [58] they proved that paranormal operators are finite and presented some generalized finite operators. An extension of inequality $\|I - AX - XA\| \geqslant 1$ was also given in [59]-[62]. In [63] they presented some properties of finite operators and gave some groups of operators which are in the group of finite operators and found for which condition $A + W$ is a finite operator in $L(H \oplus H)$. Moreover, [64], [65], [66], [67], [68] and [69] presented another set of finite operators which involves the set of paranormal operators and proved that the range and null space of $\delta_{Y,Z}$ are orthogonal for a group of operators involving the group of normal operators while [70], [71], [72] and [73] proved that a paranormal operator is finite and presented properties of finite operators as seen in [74] and [75]. Kapoor and Jagadish Prasad [46] characterized inner product spaces and provided simple results of characterizations same as the existing ones. Also, it was shown that Isosceles orthogonality is unique provided the space is strictly convex and that Pythagorean orthogonality is unique in a normed linear space. Bhatia and Semrl [20] showed that if Q and F are matrices such that $\|Q + zF\| \geqslant \|Q\|$ for all complex numbers z, then in this case Q is orthogonal to F. Important properties for this kind of orthogonality were fund to hold and some characterizations and generalizations were also obtained. In normed spaces, both pythagorean and Isosceles orthogonalities have been discussed and it was found that the homogeneity property holds for the orthogonalities in an inner product space. Koldobsky [42] showed that a bounded linear operator $G : Y \to Y$ is orthogonal provided that there is a product G and a positive constant. The work of [8] studied geometric properties defined in Banach spaces of an orthogonality relation and based on the property of right angles. Jacek [40] defined an approximate Birkhofff orthogonality relation in normed spaces and compared it with that introduced by Dragomir and established few characteristics of approximate Birkhofff orthogonality. In this case, it was shown that approximate Birkhofff orthogonality in smooth space and from the semi-inner product is equivalent to approximate orthogonality. In [28] Dragoljub introduced ψ-Gateaux derivative for operators to be orthogonal to the operator in both spaces C_1 and C_∞ (nuclear and compact operators on a Hilbert spaces). Further, Dragoljub [28] applied these results to prove that there exists a normal derivation δ_A such that $\overline{ran\delta_A} \oplus ker\delta_A \neq C_1$ and a related result concerning C_∞. Fathi [35] adopted the notion of orthogonality and established a characterization for orthogonality in the spaces $L_s^p(\mathbb{C})$, $1 \leqslant P < \infty$ and denoted $L(Q, Z)$ as the group of linear transformations from the normed space Q to the Banach space Z. For the Hilbert spaces Q and Z it was shown that the group of compact operators in $L(Q, Z)$ is the closure $L(Q, Z)$ of the algebra of finite-rank one operators. That gave a more efficient characterization of compact operators. Debmalya [30] found a condition for the existence of conjugate diameters through the points $e_1, e_2 \in S_j$ in a real 2-dimensional strict convex space. For a real strictly convex smooth space of finite dimension the concept of generalized conjugate diameters was introduced. Madjid and Mohammad [51] introduced the notion of orthogonality constant mappings in isosceles orthogonal spaces and established stability of orthogonal constant mappings and the stability of periderized quadratic equation $q(r + s) + h(r + s) = g(r) + g(s)$ was studied. They also dealt with isosceles orthogonality and in their case a normed linear space Z given that the isosceles

orthogonality was referred as an isosceles orthogonal space. Moreover, they investigated some properties of the General orthogonality in Banach spaces, and obtained some results of general orthogonality in Banach spaces similar to orthogonality of Hilbert spaces. The relation between this concept in smooth spaces and sense of Birkhoff-James was also considered. In normed spaces, Jacek [39] considered a class of linear mappings preserving this relation through Birkhoff-James orthogonality. Some related stability problems were stated. In [21] showed that the linear mapping from a normed space Q to a normed space R is isosceles orthogonal given that it is an isometric scalar multiple. In normed spaces, it was shown that the concept of distance that preserve maps originated from the Mazur-Uham theorem. Since Birkhoff-Orthogonality is homogenous and not symmetric whereby Isosceles orthogonality is symmetric and not homogenous, that showed that the two types of orthogonality have different properties in linear normed spaces. In inner product spaces, one could easily yield the concepts of orthogonality that yield the usual orthogonality. Precisely, the orthogonalities coincide provided in an inner product space. Therefore they might have been referred as natural extension of orthogonality to normed spaces. Further, the author investigated that an orthogonal linear map in an inner product space is necessary an isometric scalar multiple, whereby a mapping Q preserves orthogonality provided that p is orthogonal to s means that Q_p is orthogonal to Q_s. The study of [12] obtained the required condition for completely continuous linear operator T to be orthogonal to another completely continuous linear operator A in the sense of James. Also it was shown that if T is orthogonal to A and $0 \notin \sigma_{ap}(A)$ then $\sup\{|(Tu,v)| = \|u\| = 1$ and $(Au,v) = 0\}$. It was proved that the complex scalar λ_0 is characterized by the fact that there exist $\{x_n\}, \|x_n\| = 1$ such that $((T - \lambda_o A)_{x_n}, Ax_n) \to 0$ and $\|(T - \lambda_0 A)_{x_n}\| \to \|T - \lambda_0 A\|$. Dragomir and Kikianty [29] introduced types of orthogonality in terms of 2-HH norms and the properties for those orthogonalities were determined. Inner product spaces and strictly convex spaces were also characterized. They presented two new definition of orthogonality types. One was related to proximity in Banach spaces and other related to contractive projections. The relation between the two types was studied and basic properties of each type were presented. The reflection of such orthogonalities to compact operators was discussed. Cuixia and Senlin [25] studied homogeneity in normed linear spaces of isosceles orthogonality and that was an important notion of orthogonality from the two view point. They related homogeneous isosceles orthogonality to other types of orthogonality which include vectors with isometric reflection and vectors with l_2-summand and it was shown that a Banach space Z is a Hilbert space provided that the interior of the group of isosceles orthogonality with homogeneity property in the unit sphere of Z is not empty. Moreover, a geometric constant NH_Z to determine the non-homogeneity of isosceles orthogonality was introduced. It was shown that $0 \leqslant NH \leqslant 2$ $NH_Z = 0$ provided Z is a Hilbert space and $NH_Z = 2$ given that Z is not a square uniformly. Salah and Hacene [63] minimized the C_∞-norm from $L(H)$ to C_∞ of suitable affine mappings through convex and differentiable analysis as studied in operator theory. The mappings considered generally elementary operator especially the generalized derivations that were the most important. As a consequence, global minima in terms of orthogonality was characterized in Banach spaces. Ali Zamani and Mohammad [10] gave results on approximate Roberts orthogonality and approximate Birkhoff orthogonality and the properties of approximate Roberts orthogonality were also studied. Moreover, the set of linear mappings that preserve approximate Roberts orthogonality of type $\varepsilon \perp R$. It was shown that an ε-isometric scalar multiple is a mapping that preserves approximate

Roberts orthogonality. Justyna [41] showed how different types of orthogonality have been described in functional equationsand introduced aspects of orthogonality, functional equations examples were given for vectors that are orthogonal. Some of their results and some applications were shown. Then, the factors affecting stability of some of functional equations were discussed considering different notions. Also, the author mentioned the orthogonality equation and the challenge that preserve orthogonality. Finally, some open problems regarding those topics were stated. Pawel [56] introduced an approximate and exact orthogonality relation and considered algebra of linear mappings that preserve approximate orthogonality. The author also studied the property of a linear mapping reserving the B-orthogonality and it was proved to be equivalent to the p, p_+-orthogonality (although these orthogonalities need not be equivalent). However, it was shown [68] that every map that is linear with approximate orthogonality is a isometric scalar multiple. It was shown that a linear map which preserve Birkhoff-James orthogonality is a isometric scalar multiple. Later, [62] extended this study and showed that approximate semi-orthogonality and approximate p_+-orthogonality are not comparable unless it is for a smooth normed space. Consequently smooth spaces were characterized in terms of approximate orthogonality. In [37] the notion of approximate Roberts orthogonality set and investigated the properties of the given sets was introduced. To add, they introduced the concept of approximate a-isosceles orthogonality and considered a group of transformations with approximate a-isosceles orthogonality. Chaoqun and Fangyan [26] investigated maps between normed spaces with the orthogonality given by the norm derivative. Those maps were proved to be an isometric scalar multiple. Bhuwan [21] studied two new types of orthogonality from generalized carlsson orthogonality and some properties of orthogonality in Banach spaces were verified as Best implied Birkhoff orthogonality and Birkhoff orthogonality implied Best approximation. It was also shown that Pythagorean orthogonality implies Best approximation. In [38] introduced new geometric constants that differentiates Roberts orthogonality and Birkhoff orthogonality in normed spaces by characterizing Roberts orthogonality in two different ways through bisectors of two points and using certain linear transformations. The main objective was to present two new characterizations of Rorbert orthogonality. One of them was related to segments whose bisectors contain lines, and the other one associated this type of orthogonality to certain symmetries of the unit circle. In [70] the authors studied geometrical structure of bisectors in normal planes and defined constant C_s, which quantifies the maximum symmetry of the unit circle regarding directions which are Birkhoff orthogonal. From a geometric point of view [34] studied two types of approximate Birkhoff-James othogonality in a normed space, and characterized them in the sense of normal cones. The concept of normal cones was characterized and related to approximate Birkhoff-James orthogonality in a Banach space of dimension 2 was explored. Uniqueness theorem was obtained for approximate Birkhoff-James orthogonality in a normed space. Their main aim was to study two different approximation of Birkhoff-James orthogonality, to have a good understanding of the properties of normed spaces. Among other things they exhibited that the two types of approximate Birkhoff-James orthogonality have a close connection with normal cones in a normed space. Thomas [73] combined functional analytic and geometric view points on approximate Birkhoff orthogonality in generalized minkowskis spaces which are finite dimensional vector spaces endowed with a gauge. That was the first approach in those spaces. In a normed space X, [42] related strict convexity to orthogonality of operators in terms of Birkhoff-James in K(X), the space of all completely continuous operators on

X. It was shown that a real reflexive Banach space Z is strictly convex if for $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} Q$ or $Rz = 0$ for some $x \in S_z$ with $\|Qz\| = \|Q\|$, $Q \in K(Z)$. It was shown that if Z is a real Hilbert space of infinite dimension then for every $R \in L(Z)$ $R \perp_B Q \Rightarrow Q \perp_B R$ if Q is the zero operator. It was then proved that $R \perp_B Q \Rightarrow Q \perp_B R$ for a real Hilbert space Z, $Q \perp_B R \Rightarrow R \perp_B Q$ for every $R \in L(Z)$ if Q is the zero operator. Debmalya [30] studied Birkhoff-James orthogonality defined on a real Banach space of finite dimension for bounded linear operators. The main reason for the study was in two ways, to determine Birkhoff-James orthogonality of linear transformations on a real Banach space of finite dimension and to characterize the symmetric properties of Birkhoff-James orthogonality of linear transformations defined on Z. Considering the obtained results,the author also studied the left symmetric properties of Birkhoff-James orthogonality of linear operators defined on $L(l_p^2)$ ($p \geqslant 2$. Letting $F, \|.\|$ to be a Banach space of finite dimension and $G_F = f \in F : \|f\| \leqslant 1$ and $G_F = f \in F : \|f\| = 1$ to be the unit ball and the unit sphere of the Banach space defined by the usual operator norm respectively. Further, the author introduced a particular notion motivated by geometric observations to determine Birkhoff-James orthogonality of vector space homomorphism for j, k in a vector space Z of which a norm is defined on a real Banach space of finite dimension, $k \in Z^+$ if $\|j + \lambda k\| = \|j\|$ for every $\lambda \geqslant 0$, also $k \in Z^-$ if $\|j + \lambda k\| = \|j\|$ for every $\lambda \geqslant 0$. The symmetric property of Birkhoff-James orthogonality of linear transformations on a real complete vector space Z on which a norm of finite dimension is defined was considered. So,the author considered this property in Banach spaces and proved some results similar to the symmetric property of of linear transformations on a real complete vector space Z on which a norm of finite dimension is defined. It was shown that there is nonzero liner operators $Q \in L(Z)$ such that Q is left symmetric in $L(Z)$. Lastly, using some of the results obtained, the study proved that $Q \in L_r^2$ $r \geqslant 2$, $r \neq 0$ is left symmetric given that Q is the zero operator. It was proved that $Q \in L(l_r^2)$ ($r \geqslant 2, r \neq \infty$) is left symmetric in relation to Birkhoff-James orthogonality given that Q is the zero operator. Lastly, the author concluded that the result holds for a strictly convex of any finite dimension and smooth real Banach spaces l_r^n ($r > 2, r \neq \infty$). It had been shown that $Q \perp_B G \Rightarrow G \perp_B Q$ for all operators G on $(\mathbb{R}^n, \|.\|_1)$ given that Q obtains a norm at the extreme point, image that is left symmetric point of $(\mathbb{R}^n, \|.\|_1)$ and images of other extreme points are zero. It was also proved that $G \perp_B Q \Rightarrow Q \perp_B G$ for all operators G provided that Q obtains norm the extreme points given that the images of extreme points are scalar multiples of extreme points. A necessary condition was obtained for an operator Q to be left symmetric. It was proved that $Q = q_{ij}$ is right symmetric given that for every $i \in \{1, 2, ...\}$ exactly one term $q_{i1}, q_{i2}...q_{in}$ is non-zero and of the same magnitude proved that Q is a left symmetric provided Q is the zero operator when the dimension is more than two. It was also proved that if Q is a linear operator $(\mathbb{R}^2, \|.\|_1)$ then Q is left symmetric given that Q attains norms at only one extreme point say e, Q_e is symmetric and the other extreme point is zero. While Birkhoff-James orthogonality was characterized for bounded linear operators defined on a Hilbert space or a finite dimensional Banach space, the problem of of characterizing Birkhoff-James orthogonality on normed linear spaces of infinite dimension for linear mappings that are bounded remained unsolved. Motivated by the result on rotund bounded linear mappings, Birkhoff-James orthogonality of rotund points in the space of bounded linear operators was obtained. In order to obtain the desired characterization for rotund points and for general bounded linear operators. Moreover, these researchers introduced a new definition which was essentially geometric in nature and hence in this

manner a Birkhoff-James orthogonality for linear operators that are bounded of general normed spaces was characterized. ϵ-orthogonality was decomposed to completely characterize bounded linear mappings that are bounded in the sense of Birkhoff-James. As a consequence, Birkhoff James orthogonality on a real normed linear space for linear functionals that are bounded was characterized provided the dual space is strictly convex. The authors required conditions for smoothness of linear that are bounded on a normed linear space of infinite dimension was provided. In [50] the wresearch characterized Birkhoff-James orthogonality of bounded linear mappings on complex complete vectors spaces on which a norm is defined and obtained a complete characterization of the same. By means of introducing new definitions, it was illustrated that it is possible in the complex spaces, to introduce orthogonality of linear mappings similar to the real paces. Furthermore, earlier operator theoretic characterization of Birkhoff-James orthogonality in the real case could be obtained as simple corollaries to their present study. In fact, Birkhoff-James orthogonality of completely continuous operators was characterized in the complex case in order to distinguish the complex case from the real case. The left symmetric linear operators on complex two-dimensional l_p space if and only if J is the zero operator was also studied. In [58] Sanati and Kardel characterized the class of operators that preserve orthogonality on Hilbert space H of infinite dimension as a scalar multiple of unitary operators of H and the subspaces of H that are closed. For an orthogonal preserving operator, it was shown that the spectrum is any circle that is centred at the origin. The research of [73] studied Birkhoff-James orthogonality for vector spaces in which a norm is defined for completely continuous operators. Their main aim was to determine Birkhoff-James orthogonality of completely continuous operators defined on a normed linear space. Using the concept of semi-inner-products and the similar ideas in normed spaces, some of the recent results were generalized and improved. In particular, Euclidean spaces was characterized and it was also proved that there is a possibility of retrieving the norm of a completely continuous operator in the terms of Birkhoff-James orthogonality set. Certain best approximation type results were also presented in the space of linear operator that are bounded. In [30] the study introduced the Bhatia-Semrl theorem for completely continuous operators on a Hilbert space on infinite dimension and also characterized Euclidean spaces for all Banach spaces of finite dimension. The concept of inner product spaces was correlated with the notions of r^+ and r^-. This enabled them to get the norm of a completely continuous linear operators in relation to its interaction with Birkhoff-James orthogonality set. Finally, some best approximation results were presented in Hilbert spaces and Banach spaces In [50] Kallol presented results on Birkhoff-James and smoothness of operators in normed spaces. Kallol [4] explored the orthogonality relation between elements in Banach spaces Z of operators $L(Z)$ that are linear and bounded. Smoothness of the space of operators that are linear and bounded was also studied. In [22] Bhuwan and Prakash applied orthogonality in the best approximation in normed linear spaces. Hence, it was shown that Birkhoff orthogonality means best approximation and best approximation means Birkhoff orthogonality. It was also proved that for ϵ-orthogonality, ϵ-best approximation means ϵ-orthogonality. The authors finally showed how pythagorean orthogonality and best approximation, isosceles orthogonality and ϵ-best approximation are related in normed spaces. In [12] Ali Zamani generalized operators for a semi-inner product on a Hilbert space in the sense of Birkhoff-James. Given that P and Q are linear transformations on a complex Hilbert space Z, the relation $P \perp_J^K Q$ was defined if P and Q are bounded with a semi-norm endowed with a positive operator J that satisfy $\|P + \gamma Q\|_J \geqslant \|P\|_J$ for

a complex γ. This study proved that $P \perp_J^K Q$ given that there exist a sequence $\{z_n\}$ with a norm of 1 in Z such that $\lim_{n\to\infty} \|Pz_n\|_J = \|P\|_J$ and $\lim\langle Pz_n, Qz_n\rangle_J = 0$. Some distance formulas in Semi-Hilbert spaces were also Provided. In [13] Birkhoff-James orthogonality for linear transformations was characterized and proved to be a vector space of operators on arbitrary Banach spaces. Arbitrary Banach spaces were characterized and some conditions were obtained. They also studied orthogonality in space of operators $L(Z)$ on arbitrary Hilbert space Z, both in relation to operator norm and numerical radius norm. Birkhoff-James orthogonality of linear operators in Banach spaces was also obtained. Their main goal was to determine Birkhoff-James orthogonality of the operator $T \in L(Z, W)$ to the subspace of $L(Z, W)$ in an arbitrary Banach spaces Z and W set up. Arpita and Kallol [13] first characterized $Q \perp R$ where $Q \in L(W, Z)$ and R is a subspace of $L(W, Z)$ of finite dimension and W is a reflexive Banach space given that Z is a Banach space of finite dimension. For arbitrary Banach spaces W and Z of $L(W, Z)$ and for an arbitrary subspace W, $Q \perp_B R$ under suitable conditions. They also characterized $T \in L(W, Z)$ to a subspace of $L(H)$ in the sense of Birkhoff-James on a Hilbert space H of infinite dimension. Later, it was discovered that in order to characterize orthogonality of operators, there was need for the operators to attain norms. In [36] determined the orthogonalities of Birkhoff-James and isosceles for operators defined on Hilbert spaces and Banach spaces. There was no other universal concepts of orthogonality in a Banach space unlike in Hilbert spaces. Then, it was found that there is a possibility of having several types of orthogonality in such a space, in which each characterizes certain particular concept of orthogonality in Hilbert spaces. Since lack of a standard orthogonality led to the differences of Hilbert spaces and Banach spaces, the authors explored linear operators in terms of Birkhoff-James in a different aspect and discussed some applications to this regard. A study on Isosceles orthogonality of mappings that are linear and bounded on a Hilbert space was done and related properties were determined, and properties of disjoint support were also included. It was shown that for bounded linear operators between Banach space of infinite dimension, Bhatia-Semrl theorem verbatim of finite dimension was extended under some additional assumptions. The author in [14] studied the properties of the set $O_{P,A} = \{x \in S_X : P_x \perp P_y\}$ for any $P \in L(X, Y)$ and characterized the Hilbert space of finite-dimension in relation to the new introduced concept. They focused on orthogonality of operators that were positive and those that were linear defined on a Hilbert space. Isosceles orthogonality was generalized for two positive bounded linear operators and some remarks between Birkhoff-James orthogonality and Isosceles orthogonality were discussed. Properties of Isosceles orthogonality and Birkhoff orthogonality were further explored in Banach spaces. They concluded by establishing that Rorbert's orthogonality is more agreeable than that of either Birkhoff-James and Isosceles orthogonality. In [23] Bhuwan and Prakash enlisted properties of Birkhoff-Orthogonality and Carlsson orthogonality and introduced two new particular cases of Carlsson orthogonality and checked some properties of orthogonality in relation to these particular cases in normed spaces. They showed how isosceles, Rorbert and Pythagorean orthogonalities can be derived from the carlsson orthogonality and obtained two new orthogonality relations for the Carlsson. In [55] Priyanka and Sushil gave the known properties of Birkhoff-James orthogonality in Banach space. Concepts of orthogonality, the Gateaux derivative and the sub-differential set of norm function were discussed and important distance formulas that were determined by characterizing Birkhoff-James orthogonality. This work also mentioned the relation between orthogonality and geometric properties of normed

spaces. This lead to the determination of different related concepts like characterization of smooth points and extreme points, sub differential sets and ψ-Gateaux derivative sets. Moreover, the authors also characterize symmetric property of orthogonality. Generalizations of orthogonality in different Banach spaces were determined together with their applications. The characterizations obtained were used to determine distance formulas in certain Banach spaces. In [69] Tanaka and Debmalya characterized the left and right symmetric points in the terms of Birkhoff orthogonality in $L(G, R)$ and $K(G, R)$ where G, R are complex Hilbert space and $L(G, R)$ $(K(G, R))$ is the space of all compact bounded mappings from G into R. Their main aim was to improve the notion of local symmetry for a strong version of Birkhoff orthogonality. It was shown that an element J in $L(G, R)$ $(K(G, R))$ is left symmetric for $\perp_{L(G)}$ $(\perp_{K(G)})$ in $L(G, R)$ $(K(G, R))$ provided that J is rank one operator, it turned out that $J \in L(G, R)$ given that J is right invertible such that Q is of infinite dimension or $\dim G > \dim R$ and J is an isometric scalar multiple where G is of infinite dimension and $\dim G < \dim R$ while $J \in K(G, R)$ is right symmetric for $\perp_{K(G)} \in K(G, R)$ if J has the dense range. Debmalya, Ray and Kallol [30] explored the relation between the orthogonality of bounded linear operators and the elements in the ground space. It was shown that if $Q, R \in L(W, Z)$ satisfy $Q \perp_B R$, and that there exists $w \in W$ so that $Q_w \perp R_z$ with $\|z\| = 1, \|Q_W\| = \|T\|$, given that W, Z are normed linear spaces. The concept of property D_n for a Banach space was introduced and its relation with orthogonality of a bounded linear operator on Banach spaces was illustrated. Moreover, the property D_n for various polyhedra Banach spaces were characterized. Their aim was to study Bhatia-Semri(BS) property in polyhedral Banach spaces for bounded linear operators. For orthogonality of elementary operators, orthogonality of range and kernel of normal derivations was determined by many authors. For instance, Anderson [6] showed that if J and P are operators in $L(Z)$ such that J is normal and $JP = PJ$ then for every $Y \in L(Z)$, $\|\delta_J(Y) + P\| \geqslant \|P\|$ where $\|.\|$ is the usual operator norm. Anderson [4] showed that if Q is isometric or is normal then the range of δ_Q is orthogonal to its nullspace. Also Anderson [6] proved that if Q is normal and has infinite number of points then the closed linear space of the range and null space of δ_Q is not all of $L(Z)$. Kittaneh [48] extended the study and showed that given J and P are operators in $L(Z)$ such than J is normal, P is a Hilbert Schmidt operator and $P \in \{J\}$ then for all $Y \in L(Z)$, $\|\delta_J(Y) + P\|_2^2 \geqslant \|\delta_J(Y)\|_2^2 + \|P\|_2^2$ where $\|.\|_2$ is the Hilbert Schmidt operator norm. Therefore, the range of δ_J if orthogonal to the kernel of δ_J for the Hilbert Schmidt operators in the usual sense. In the schatten p-norms Kittaneh [49] used the Gateaux differentiability and the usual operator norm to determine the range and kernel orthogonality of elementary operators in relation to p-norms. In [32] Duggal considered an elementary operator δ_{ab} in which the operators a, b, x are hyponormal, the operators a_1, b_2 are normal and a_1 commutes with b_2. Turnsek [71] considered a normed algebra A and a linear operator $\phi : A \rightarrow A$ and proved that the range $\phi - 1$ is orthogonal to its kernel if $\|\phi\| \leqslant 1$. This could also be applied to the case when $\phi; L(Z) \rightarrow L(Z)$ is an arbitrary elementary operator defined by $\phi(Z) = \sum_{i=1}^k A_i Z B_i$. Dragoljub [27] proved the orthogonality of an important elementary operator in relation to the unitary invariant norms and their association with the norm ideals of operators. The group consisted the mapping $Q : L(Z) \rightarrow L(Z)$, $Q(V) : FVH + JVP$ where $L(Z)$ denotes the group of all bounded operators and F, H, J and P are normal operators so that $FJ = JF$, $HP = PH$ and $KerF \cap KerJ = KerH \cap KerP = \{0\}$. Moreover, the author established this set in sense in which an orthogonality result holds. Bachir and Hashem [18]

presented a new class of finite operators and extended orthogonality results to some finite operators and some commutativity results were also generalized. Their main goal was to investigate the orthogonality of $\mathrm{Ran}\delta_{A,B}$ and $\mathrm{Ker}\delta_{A,B}$ for certain finite operators. It was proved that $\mathrm{Ran}(\delta_{A,B})$ is orthogonal to $\mathrm{Ker}(\delta_{A,B})$ where A is dominant and B^* is M-hyponormal. Duggal and Harste [33] studied orthogonality and range closure properties for some elementary operators as proved for hyponormal operators or contractions on Hilbert spaces. Okelo and Agure [53] presented various types and aspects of orthogonality in normed spaces. Indeed, the range and kernel orthogonality results for elementary operators were given and the operators that characterize them were then provided. In [14] Bouali and Bouhafsi exhibited pair (Q, R) of operators such that orthogonality of $\delta_{Q,R}$ is valid for the usual operator norm. Range and nullspace of $\delta_{Q,R}$ results were obtained in relation to the group of unitarily invariant norms. Bachir and Nawal [17] studied and characterized the range-kernel orthogonality of the points $C_1(H)$, the trace class operators in nonsmoothness case and gave a counter example. In [52] Okelo characterized orthogonality of elementary operators in norm attainable classes and gave conditions for operators to be norm attainable in Hilbert spaces. Lastly, the range-kernel orthogonality results were given for elementary operators in norm-attainable classes.

2. Preliminaries

Definition 2.1 ([9], Definition 3.1)**.** Let X be a linear space over $\mathbb{F}$ .Then a norm on X is a non-negative real-valued function $\|.\| : X \to \mathbb{R}$ such that $\forall\, w, z \in X$ and $\eta \in \mathbb{F}$ the following properties are satisfied:

  (i). $\|w\| \geqslant 0$ and $\|w\| = 0$, if and only if $w = 0$.
 (ii). $\|\eta w\| = |\eta| \|w\|$.
(iii). $\|w + z\| \leqslant \|w\| + \|z\|$.

The ordered pair $(X, \|.\|)$ is called a normed space.

Definition 2.2 ([16], Definition 3.5)**.** Suppose Z is a vector space with norm $\|.\|$ and $d : Z \times Z \to \mathbb{R}$ is a metric defined by $d(w.z) = \|w - z\|$, then d is called the metric associated with the norm.

Definition 2.3 ([32], Definition 3.18)**.** Let Z be a real or complex vector space. An inner product on Z is a function $\langle .,.\rangle : Z \times Z \to \mathbb{J}$ such that $\forall\, w,\ z,\ k \in Z$ and $\lambda, \beta \in \mathbb{J}$; if it satisfy:

  (i). $\langle w, w\rangle \geqslant 0$ and $\langle w, w\rangle = 0$, if and only if $w = 0$.
 (ii). $\langle \alpha w + \beta z, k\rangle = \alpha\langle w, k\rangle + \beta\langle z, k\rangle$.
(iii). $\langle \lambda w, z\rangle = \lambda\langle w, z\rangle$.
(iv). $\langle w, z\rangle = \overline{\langle z, w\rangle}$.

The ordered pair $(X, \langle .,.\rangle)$ is called an inner product space.

Example 2.4 ([15], Example 2)**.** Let $X = \mathbb{F}^n$ for $w = (w_1 ... w_n)$ and $z = (z_1 ... z_n)$ in X define $\langle w, z\rangle = \sum_{i=1}^{n} w_i \overline{z_i}$.

Example 2.5 ([30], Example 1)**.** Let $X = l_0$ the space of finitely non-zero sequences of real or complex numbers. For $w = (w_1 ... w_n)$ and $z = (z_1 ... z_n)$ in X define $\langle w, z\rangle = \sum_{i=1}^{\infty} w_i \overline{z_i}$.

Example 2.6 ([7], Example 4). Let $Z = l_2$ the space of all sequences $w = (w_1, w_2...)$ of real or complex numbers for $\sum_{i=1}^{\infty} |w_i|^2 < \infty$. For $w = (w_1...w_n)$ and $z = (z_1...z_n)$ in X define $\langle w, z \rangle = \sum_{i=1}^{\infty} w_i \overline{z_i}$.

Example 2.7 ([8], Example 1). Let $Z = C[q, s]$ the space of all continuous complex valued function on $C[q, g]$ for $q, g \in Z$ define $\langle q, g \rangle = \int_q^g q_t \overline{g_t} dt$.

Definition 2.8 ([7], Definition 3.6-1). An operator P is said to be linear if, for every pair of vectors w and z and scalar λ, $P(l + d) = P(l) + P(d)$ and $P(\lambda l) = \lambda P(l)$.

Definition 2.9 ([36], Definition 3.2-1). Two vectors $w, z \in H$ are called orthogonal, denoted by $w \perp z$ if $\langle w, z \rangle = 0$.

Definition 2.10 ([52], Section 1). Consider a normed space $\mathcal{D}$ and let $T: \mathcal{D} \to \mathcal{D}$. T is said to be an elementary operator if it can be represented in the following form $T(X) = \Sigma_{i=1}^n S_i X P_i$ for all $X \in \mathcal{D}$ where S_i and P_i are fixed in $\mathcal{D}$.

Example 2.11. Let $S = L(Z)$ for $S, P \in L(Z)$ we define particular elementary operator.

(i). The left multiplication operator $L_S : L(Z) \to L(Z)$ by $L_S(X) = SX, \forall X \in L(Z)$.
(ii). The right multiplication operator $R_P : L(Z) \to L(Z)$ by $R_P(X) = XP, \forall X \in L(Z)$.
(iii). The generalized derivation by $\delta_{S,P} = L_S - R_P$.
(iv). The basic elementary operator by $M_{S,P}(X) = SXP, \forall X \in L(Z)$.
(iv). The Jordan elementary operator by $\mu_{S,P}(X) = SXP + PXS, \forall X \in L(Z)$.

Definition 2.12 ([53], Section 1). The range of an operator $P : L(H) \to L(H)$ is defined as $Ran(T) = \{y \in L(H) : y = T(x) \forall x \in L(H)\}$.

Definition 2.13 ([53], Section 1). The kernel of an operator $T : L(H) \to L(H)$ is defined as $Ker(T) = \{x \in L(H) : T(x) = 0 \forall x \in L(H)\}$.

Definition 2.14 ([74], Section 2). A bounded linear operator S on a Hilbert space H is called finite if $\|I - SX - XS\| \geqslant 1$ for each $X \in L(H)$.

Definition 2.15 ([14], Section 2). A proper two sided ideal J in $L(H)$ is called a norm ideal if there is a norm on J possessing the following properties:

(i). $(J, \|\|.\|\|)$ is a Banach space.
(ii). $\|\|SVP\|\| \leqslant \|S\|\|\|V\|\|\|P\|$, for every $S, P \in L(H)$ and for every $V \in J$.
(iii). $\|\|V\|\| = \|V\|$, for V a rank one operator.

3. Main Results

In this section, we discuss the results of our study. We consider finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James orthogonality for finite elementary operators.

Proposition 3.1. *Let Ω be a normed space, then for $S \in \Omega$, $\sigma_p(S) \neq \emptyset$ if S is normaloid.*

Proof. Let $S \in \Omega$ be normaloid, then $\|S\| = r(S)$. This implies that there exist $\lambda \in \sigma_p(S)$ such that $|\lambda| = \|S\|$. It is known that $\sigma_p(S) \subseteq \sigma_{ap}(S) \subseteq \sigma(S)$. Therefore, $\sigma_p(S) = \sigma_{ap}(S)$. But λ is in the boundary of $\sigma_p(S)$ and since this is a subset of the approximate point spectrum of S, we have that $\lambda \in \sigma_p(S) = \sigma_{ap}(S)$. But for a sequence $\{x_n\}_{n \in N}$ of unit vectors we have, $\|(S - \lambda I)x_n\| \to 0$. So $0 \in \sigma_p(S)$ and hence $\sigma_p(S) \neq \emptyset$. $\square$

Proposition 3.2. *Every normaloid operator is finite.*

Proof. From Proposition 3.1 we have that $\sigma_p(S) \neq \emptyset$ if S is normaloid. To show that every normaloid operator is finite, we let S to be a normaloid operator, i.e $\|S\| = r(S)$. Hence, there exist $\lambda \in \sigma_p(S)$ such that $|\lambda| = \|S\|$. By definition, an operator S in a normed space Ω is finite if $\|SX - XS - I\| \geqslant 1$, for all $X \in \Omega$. But $\|(S - \lambda I)x_n\| \to 0$ with $\|x_n\| = 1$. From Gram schmidt procedure $\{x_n\}$ is a normalized sequence and hence we have,

$$
\begin{aligned}
\|(SX - XS) - I\| &= \|((S - \lambda I)X - X(S - \lambda I)) - I\| \\
&\geqslant |\langle (S - \lambda I)X_{x_n, x_n} \rangle - \langle X(S - \lambda I)_{x_n, x_n} \rangle - I| \\
&\geqslant |\langle (S - \lambda I)X - X(S - \lambda I) \rangle_{x_n, x_n} - I| \\
&\geqslant |\langle (SX - XS)_{x_n, x_n} \rangle - I|.
\end{aligned}
$$

Letting $n \to \infty$ we obtain $\|(SX - XS) - I\| \geqslant 1$. $\qquad \square$

Lemma 3.3. *Let $S \in \Omega$ be normaloid and $S_o \in \Omega$ be norm-attainable such that $SS_o = S_oS$. Then for every $\eta \in \sigma_p(S_o)$, $\|S_o - (SX - XS)\| \geqslant |\eta| \; \forall X \in \Omega$.*

Proof. From [52], if $S_o \in \Omega$ is norm-attainable, then it is normal. So, we let $\eta \in \sigma_p(S_o)$ and M_η be the eigenspace associated with η. Since $SS_o = S_oS$, we have $SS_o^* = S_o^*S$ by Fuglede Putnam's Theorem [36]. Hence M_η reduces both S and S_o. According to the decomposition of $H = M_\eta \oplus M_\eta^\perp$, we write S, S_o and X as follows:

$$
S = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, S_o = \begin{pmatrix} \eta & 0 \\ 0 & S_2 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.
$$

We have,

$$
\begin{aligned}
\|S_o - (SX - XS)\| &= \left\| \begin{pmatrix} \eta - (S_1X_1 - X_1S_1) & * \\ * & * \end{pmatrix} \right\| \\
&\geqslant \|\eta - (S_1X_1 - X_1S_1)\| \\
&\geqslant |\eta| \left\| 1 - \left(\left(\frac{S_1X_1}{\eta} \right) - \left(\frac{X_1S_1}{\eta} \right) \right) \right\| \\
&\geqslant |\eta|.
\end{aligned}
$$

$\qquad \square$

Lemma 3.4. *Every paranormal operator in a unital C^* algebra Ω is finite.*

Proof. Let S be a paranormal operator, then S is normal i.e $S^*S = SS^*$. By Berberian theorem, it is known that, there exist a $*$-isometric isomorphism $\psi : \Omega \to \Omega$ that preserves order such that,

$$
\begin{aligned}
\|S\|^2 = \|SS^*\| = 1 &\leqslant \|(SX - XS) - I\| \\
&\leqslant \|\psi(SX - XS) - I\| \\
&\leqslant \|(\psi(S)\psi(X) - \psi(X)\psi(S)) - I\|.
\end{aligned}
$$

If $S \in \Omega$ is an element of $F(H)$ such that $\sigma_p(S) \neq \emptyset$ then it results from Proposition 3.2 that $\psi(S) \in \Omega$ is finite i.e.

$$
\|(SX - XS) - I\| = \|(\psi(S)\psi(X) - \psi(X)\psi(S)) - I\| \geqslant 1.
$$

$\qquad \square$

Theorem 3.5. *Let $S \in \Omega$ be norm-attainable. Then $J = S + P$ is finite where P is compact in a C^*-algebra Ω.*

Proof. Let S be norm-attainable, since Ω is a unital C^*-algebra, it follows that $J = S + P$ is finite. Indeed from Lemma 3.4 and Proposition 3.2 we have,

$$
\begin{aligned}
\|J\|^2 = \|JJ^*\| = 1 \;\; &\leqslant \;\; \|(JX - XJ) - I\| \\
&\leqslant \;\; \|(SX - XS) - I\| \\
&\leqslant \;\; \|(SX + PP^{-1} - XS + P^{-1}P) - I\| \\
&\leqslant \;\; \|(S + P)(X + P^{-1}) - (X + P^{-1})(S + P) - I\|.
\end{aligned}
$$

For $Y = X + P^{-1}$ we have, $\|(S + P)Y - Y(S + P) - I\| \geqslant 1$. This proves that $J = S + P$ is a finite operator. $\qquad\square$

Corollary 3.6. *Let $S \in \Omega$ be log-hyponormal and S^*be p-hyponormal then $\|J - (SX - XS_o)\| \geqslant \|J\|$, for all $X \in \Omega$ and for all $J \in \ker\delta_{S,S_o}$.*

Proof. If $J \in \mathrm{Ker}\delta_{S,S_o}$, then also $J \in \mathrm{Ker}\delta_{S^*S_o^*}$ by Putnam-Fuglede's theorem in [36]. Therefore, $SJJ^* = JS_o^* = JJ^*S$. Since S is log-hyponormal, JJ^* is normal and $S(JJ^*) = (JJ^*)S$. Since $X \in \Omega$, we deduce that

$$
\begin{aligned}
\|J\|^2 = \|JJ^*\| \;\; &= \;\; \|JJ^* - SXJ^* - XJ^*S\| \\
&\leqslant \;\; \|JJ^* - SXJ^* - XS_oJ^*\| \\
&\leqslant \;\; \|J^*\|\|J - (SX - XS_o)\|
\end{aligned}
$$

By Cauchy-Schwarz inequality [27], $\|J\|^2 = \|J\|\|J^*\|$.
This implies that $\|J\|^2 = \|J\|\|J^*\| \leqslant \|J^*\|\|J - (SX - XS_o)\|$.
Dividing both sides by $\|J^*\|$ we obtain,
$\|J\| \leqslant \|J - (SX - XS_o)\|$. $\qquad\square$

Remark 3.7. At this point, we characterize finiteness of elementary operators in a general set up. Let $\mathfrak{C}_n(S, S_o)$ be the set of all $(S, S_o) \in \Omega \times \Omega$ such that S and S_o have an n-dimensional reducing subspace $J_n(S, S_o)$ satisfying $S \mid J_n(S, S_o) = S_o \mid J_n(S, S_o)$.

Now, we characterize finiteness in the cartesian product of $\Omega \times \Omega$ in the next proposition.

Proposition 3.8. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then, the following inequality holds i.e $\|(SX - XS_o) - I\| \geqslant 1$.*

Proof. Let $\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$ be the matrix representation of S and S_o respectively relative to the decomposition $H = H_1 \oplus H_1^\perp$ where H_1 is an n-dimensional reducing subspace of S and S_o i.e $H_1 = J_n(S, S_o)$. For any operator X on H has a representation

$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$. It follows that,

$$
\begin{aligned}
\|(SX - XS_o) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1 - X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1 - X_1S_2) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\geqslant \|(S_1X_1 - X_1S_2) - I_1\|.
\end{aligned}
$$

This implies that

$$
\|(SX - XS_o) - I\| \geqslant \|(S_1X_1 - X_1S_2) - I_1\| \geqslant \|I_1\| = \|I\|.
$$

Hence, $\|(SX - XS_o) - I\| \geqslant 1$. □

Proposition 3.9. *Let* $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. *Then the following inequality holds i.e* $\|(SXS_o) - I\| \geqslant 1$.

Proof. Let S, S_o, X and I have the following representation:
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$
From Proposition 3.8, it follows that,

$$
\begin{aligned}
\|(SXS_o) - I\| &= \left\| \left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1S_2) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|. \\
&\geqslant \|(S_1X_1S_2) - I_1\|.
\end{aligned}
$$

This implies that
$$
\|(SXS_o) - I\| \geqslant \|(S_1X_1S_2) - I_1\| \geqslant \|I_1\| = \|I\|.
$$

Hence, $\|(SXS_o) - I\| \geqslant 1$. □

Theorem 3.10. *Let* $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. *Then the following inequality holds i.e* $\|(SXS_o + S_oXS) - I\| \geqslant 1$.

Proof. Let S, S_o, X, and I have the following representation[decomposition]. $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$,
$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$.

From Proposition 3.8 and Proposition 3.9 we have,

$$
\begin{aligned}
\|(SXS_o + S_oXS) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_2X_1S_1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1S_2 + S_2X_1S_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1S_2 + S_2X_1S_1) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\geqslant \|(S_1X_1S_2 + S_2X_1S_1) - I_1\|.
\end{aligned}
$$

This implies that

$$
\|(SXS_o + S_oXS) - I\| \geqslant \|(S_1X_1S_2 + S_2X_1S_1) - I_1\| \geqslant \|I_1\| = \|I\|.
$$

Hence, $\|(SXS_o + S_oXS) - I\| \geqslant 1$. □

Theorem 3.11. *Let* $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. *Then the following inequality holds i.e* $\|(SXS_o + CXC_o) - I\| \geqslant 1$.

Proof. Let S, S_o, C, C_o, X, and I have the following representation[decomposition].
$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}, S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, C_o = \begin{pmatrix} C_2 & 0 \\ 0 & 0 \end{pmatrix}$$
and $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$.
From Theorem 3.10 we have,

$$
\begin{aligned}
\|(SXS_o + CXC_o) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_1X_1C_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1S_2 + C_1X_1C_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1S_2 + C_1X_1C_2) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\geqslant \|(S_1X_1S_2 + C_1X_1C_2) - I_1\|.
\end{aligned}
$$

This implies that

$$
\|(SXS_o + CXC_o) - I\| \geqslant \|(S_1X_1S_2 + C_1X_1C_2) - I_1\| \geqslant \|I_1\| = \|I\|.
$$

Hence, $\|(SXS_o + CXC_o) - I\| \geqslant 1$. □

Theorem 3.12. *Let* $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. *Then the following inequality holds i.e* $\|\sum_{i=1}^n S_iXC_i - I\| \geqslant 1$.

Proof. Let S_i, X, C_i and I have the following representation.
$$S_i = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}, C_i = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From Theorem 3.11 it follows that,

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} S_i X C_i - I \right\| &= \left\| \sum_{i=1}^{n} \left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \sum_{i=1}^{n} \begin{pmatrix} S_1 X_1 C_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \sum_{i=1}^{n} \begin{pmatrix} (S_1 X_1 C_1) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\geqslant \left\| \sum_{i=1}^{n} (S_1 X_1 C_1) - I_1 \right\|.
\end{aligned}
$$

This implies that

$$
\| \sum_{i=1}^{n} S_i X C_i - I \| \geqslant \| \sum_{i=1}^{n} (S_1 X_1 C_1) - I_1 \| \geqslant \| I_1 \| = \| I \|.
$$

Hence, $\| \sum_{i=1}^{n} S_i X C_i - I \| \geqslant 1$. $\qquad\square$

Remark 3.13. It is known from [59] that there exists a compact operator C such that $R(\delta_c) = K(H)$. As a consequence we deduce that the dist$(I, K(H)) = 1$, where dist$(I, K(H))$ is the distance from I to $K(H)$. Therefore if S, S_o are compact operators, then we have that dist$(I, R(\delta_{S,S_o})) = 1$.

At this juncture, we characterize orthogonality for finite elementary operators. Let Ω denote a Complex Banach algebra with identity I and let $\sigma_r(\Omega)$, $\sigma_l(\Omega)$ denote, respectively the right spectrum and the left spectrum of Ω. Recall from [14] that

$S^n X - X S^n = \sum_{i=0}^{n-i-1} S^{n-i-1} (SX - XS) S^i$ for all $X \in \Omega$.

If $SJ = JS$ we have,

$$
nJS^{n-1} = S^n X - X S^n - \sum_{i=0}^{n-i-1} S^{n-i-1} ((SX - XS) - J) S^i \text{ for all } X \in \Omega.
$$

Proposition 3.14. *Let* $S \in \Omega$, x_n *be an increasing sequence of positive integers and* S^{x_n} *converge to* $Z \in \Omega$, *with* $0 \notin \sigma_r(Z) \cap \sigma_l(Z)$. *If there exist a constant* λ *such that* $\| S^n \| \leqslant \lambda$ *for all integers* n *and if* S_o *is the left or right inverse of* Z *then*

$$
\lambda^2 \| S_o \| \| (SX - XS) - J \| \geqslant \| J \| \text{ for all } X \in \Omega \text{ and for all } J \in \mathrm{Ker}\delta_S.
$$

Proof. Let $X \in \Omega$, since

$$
nJS^{n-1} = S^n X - X S^n - \sum_{i=0}^{n-i-1} S^{n-i-1} ((SX - XS) - J) S^i \text{ for } SJ = JS.
$$

We can write

$$
\begin{aligned}
(x_n + 1)JS^{x_n + 1 - 1} &= S^{x_n + 1} X - X S^{x_n + 1} - \sum_{i=0}^{x_n + 1 - i - 1} S^{x_n + 1 - i - 1} ((SX - XS) - J) S^i \\
&= S^{x_n + 1} X - X S^{x_n + 1} - \sum_{i=0}^{x_n - i} S^{x_n - i} ((SX - XS) - J) S^i.
\end{aligned}
$$

Dividing both sides by $x_n + 1$ and taking norms we obtain,

$\|JS^{x_n}\| \leqslant \frac{1}{x_n+1}\|\|S^{x_n+1}\| + |S^{x_n+1}\|\|\|X\| + \frac{1}{x_n+1}\sum_{i=0}^{x_n-i}\|S^{x_n-i}\|\|\|(SX-XS)-J\|\|\|S^i\|$

Since $\|S^n\| \leqslant \lambda$ we have that $\|S^{x_n+1}\| \leqslant \lambda$ and hence we obtain,

$$\|JS^{x_n}\| \leqslant \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|(SX-XS)-J\|.$$

Letting $n \to \infty$ we obtain,

$$\|JS^{x_n}\| \leqslant \lambda^2\|(SX-XS)-J\|.$$

But S^{x_n} converges to Z, so we have,

$$\|JZ\| \leqslant \lambda^2\|(SX-XS)-J\|.$$

Now, since S_o is in the right or the left of Z we have,

$$\|J\| \leqslant \|S_o\|\lambda^2\|(SX-XS)-J\|.$$

$\square$

Remark 3.15. Let $S \in L(H)$ and x_n be an increasing sequence of positive integers. Assume that there is a constant λ such that $\|S^n\| \leqslant \lambda$ for all integers n

(i) . If $S^{x_n} \to P$, with $0 \notin \sigma_r(P) \cap \sigma_l(P)$, then
   $\lambda^2\|(SX-XS)-J\| \geqslant \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$.

(ii) . If $S^{x_n} \to P + K$, with K compact and $0 \notin \sigma_r(P) \cap \sigma_l(P)$, then
   $\lambda^2\|(SX-XS)-J-K\| \geqslant \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$.

Theorem 3.16. *Let* $S \in L(H)$ *such that* $S^n = I$ *for some integer* n. *Then* $\lambda^2\|(SX-XS)-J\| \geqslant \|J\|$ *for all* $X \in L(H)$ *and for all* $J \in Ker\delta_S$.

Proof. Since $S^n = \{I, S, S^2, ...S^{m-1}\}$ for all integers n, $\|S^n\| \leqslant \lambda$, $n \in \mathbb{N}$ and $S^{x_n} = I$, where $x_n = nm$, $n \in \mathbb{N}$. It is known from [49] that

$nJS^{n-1} = S^nX - XS^n - \sum_{i=0}^{n-i-1}S^{n-i-1}((SX-XS)-J)S^i$, for all $X \in L(H)$.

From Proposition 4.14 we have that

$$(x_n + 1)JS^{x_n+1-1} = S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n+1-i-1}S^{x_n+1-i-1}(SX-XS-J)S^i.$$

$$(x_n + 1)JS^{x_n} = S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n-i}S^{x_n-i}((SX-XS)-J)S^i.$$

Dividing both sides by $x_n + 1$ and taking the norms we obtain,

$\|JS^{x_n}\| \leqslant \frac{1}{x_n+1}\|\|S^{x_n+1}\| + |S^{x_n+1}\|\|\|X\| + \frac{1}{x_n+1}\sum_{i=0}^{x_n-i}\|S^{x_n-i}\|\|\|(SX-XS)-J\|\|\|S^i\|$

Since $\|S^n\| \leqslant \lambda$ we have that $\|S^{x_n+1}\| \leqslant \lambda$ and hence we obtain,

$$\|JS^{x_n}\| \leqslant \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|(SX-XS)-J\|.$$

Since $S^{x_n} = I$ we have,

$$\|J\| \leqslant \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|(SX-XS)-J\|.$$

Letting n tend to infinity, we get

$$\|J\| \leqslant \lambda^2 \|(SX - XS) - J\|.$$

Hence, $\|J\| \leqslant \lambda^2 \|(SX - XS) - J\|$. □

Corollary 3.17. *Let* $S_1, S_o \in L(H)$ *such that* $S_1^m = I$ *and* $S_o^m = I$ *for some integer* m. *Then*

$$\|(S_1 X - X S_o) - J\| \geqslant \|J\| \text{ for all } X \in L(H) \text{ and for all } J \in Ker\delta_{S_1,S_o}.$$

Proof. Consider the operators P, S and Y defined on $H \oplus H$.
$P = \begin{pmatrix} S_1 & 0 \\ 0 & S_o \end{pmatrix}$, $S = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$.
Then P is normal on $H \oplus H$ and it is clear that $P^m = 1$, $PS = SP$ i.e $S \in Ker\delta_P$.
Since $PY - YP = \begin{pmatrix} 0 & S_1 X \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & X S_o \\ 0 & 0 \end{pmatrix}$

$$\|(PY - YP) - S\| = \left\| \begin{pmatrix} 0 & S_1 X - X S_o \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 0 & (S_1 X - X S_o) - J \\ 0 & 0 \end{pmatrix} \right\|.$$

Then, it follows that

$$\|PY - YP - S\| \geqslant \|S\|.$$

Consequently, from Theorem 3.12 we obtain,

$$\|(S_1 X - X S_o) - J\| \geqslant \|(PY - YP) - S\| \geqslant \|S\| = \|J\|.$$

□

Proposition 3.18. *Let* $S, S_o \in F(H)$. *If* $S_o \in [F(H)]^{-1}$ *and* $\|S\|\|S_o^{-1}\| \leqslant 1$, *then* $\|\delta_{S,S_o} + J\| \geqslant \|J\|$ *for all* $X \in F(H)$ *and* $J \in Ker\delta_{S,S_o}$.

Proof. Let $J \in F(H)$ such that $SJ = JS_o$. Therefore, $SJS_o^{-1} = J$. But $\|S\|\|S_o^{-1}\| = 1$. It follows from [32] that

$$\|SYS_o^{-1} - Y + J\| \geqslant \|J\|, \forall Y \in F(H).$$

If we set $X = YS_o^{-1}$ then we obtain,

$$\|(SX - XS_o) + J\| \geqslant \|J\| \text{ for all } X \in F(H).$$

But $\delta_{S,S_o}(X) = SX - XS_o$.
Hence, $\|\delta_{S,S_o}(X) + J\| \geqslant \|J\|$, for all $J \in Ker\delta_{S,S_o}$ and for all $X \in F(H)$. □

Remark 3.19. If $(J, \|\|.\|\|)$ is a norm ideal then the norm $\|\|.\|\|$ is unitarily invariant in the sense that $\|\|SXP\|\| = \|\|T\|\|$ for all $T \in J$ and for all unitary operators.

Remark 3.20. Let $(J, \|\|.\|\|)$ be a norm ideal and $S, P \in L(H)$. If S is isometric and P contractive, then

$$\|\|\delta_{S,P}(X) + T\|\| \geqslant \|\|T\|\| \text{ for all } X \in J \text{ and for all } T \in Ker\delta_{S,P}.$$

Proposition 3.21. *Let* $(J, |||.|||)$ *be a norm ideal and* $S \in F(H)$. *Suppose that* $f(S)$ *is a cyclic subnormal operator, where* f *is a nonconstant analytic function on an open set containing* $\sigma(S)$. *Then*

$$|||\delta_S(X) + T||| \geqslant |||T||| \text{ for all } X \in J \text{ and for all } T \in \{S\} \cap J.$$

Proof. Let $T \in J$ such that $ST = TS$. This implies that $Tf(S) = f(S)T$ and $Sf(S) = f(S)S$. Since $f(S)$ is a cyclic subnormal operator, it follows from [75] that S and T are subnormal. But every subnormal operator is hyponormal [24]. Therefore, T is normal. Consequently, $ST = TS$ implies that $ST^* = T^*S$ by Putnam-Fuglede Theorem. Hence, $\overline{Ran(T)}$ and $Ker(T)^\perp$ reduces S and $S|_{\overline{R(T)}}$ and $S|_{Ker(T)^\perp}$ are normal operators. Let $T_o x = T_x$ for each $x \in Ker(T)$, it results that $\delta_{S,P}(T_o) = \delta_{S^*,P^*}(T_o) = 0$. Let $S = S_1 \oplus S_2$ with respect to $H = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $P = P_1 \oplus P_2$ with respect to $H = Ker(T)^\perp \oplus Ker(T)$. Then we can write S, T and X as follows

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$|||(SX - XS) + T||| = \left|\left|\left| \begin{pmatrix} S_1 X_1 - X_1 S_1 + T_1 & 0 \\ 0 & 0 \end{pmatrix} \right|\right|\right|.$$

This implies that

$$|||(SX - XS) + T||| \geqslant |||S_1 X_1 - X_1 S_1 + T_1||| \geqslant |||T_1||| = |||T|||.$$

Hence, $|||\delta_S(X) + T||| \geqslant |||\delta_{S_1}(X) + T_1||| \geqslant |||T_1||| = |||T|||.$ $\square$

Proposition 3.22. *Let* $S, P \in F(H)$ *such that the pair* (S, P) *possesses the* PF *property. Then,* $|||\delta_{S,P} + T||| \geqslant |||T|||$ *for all* $X \in J$ *and* $T \in Ker\delta_{S,P}$.

Proof. Let $T \in J$, since the pair S, P satisfies PF property. Then, $\overline{Ran(T)}$ reduces S and $Ker(T)^\perp$ reduces P and $S|_{\overline{Ran(T)}}$ and $P|_{Ker(T)^\perp}$ are normal operators. Let $T_o : Ker(T)^\perp \to \overline{Ran(T)}$ be the quasi affinity defined by setting $T_o x = T_x$ for each $x \in Ker(T)$, it results that $\delta_{S,P}(T_o) = \delta_{S^*,P^*}(T_o) = 0$. Let $S = S_1 \oplus S_2$ with respect to $H = \overline{Ran(T)} \oplus \overline{Ran(T)}^\perp$ and $P = P_1 \oplus P_2$ with respect to $H = Ker(T)^\perp \oplus Ker(T)$. Let S, P, T and X have the following representation.

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

From Proposition 3.2 we have,

$$|||(SX - XP) + T||| = \left|\left|\left| \begin{pmatrix} (S_1 X_1 - X_1 P_1) + T_1 & 0 \\ 0 & 0 \end{pmatrix} \right|\right|\right|.$$

This implies that

$$|||(SX - XP) + T||| \geqslant |||(S_1 X_1 - X_1 P_1) + T_1||| \geqslant |||T_1||| = |||T|||.$$

Hence, $|||\delta_{S,P}(X) + T||| \geqslant |||\delta_{S_1,P_1}(X) + T_1||| \geqslant |||T_1||| = |||T|||.$ $\square$

Proposition 3.23. *Let* $S, P \in F(H)$ *be normal operators such that* $SP = PS$ *and* $S^*S + P^*P > 0$. *For an elementary operator* $E(X) = SXP - PXS$, $|||E(X) + J||| \geqslant |||J|||$ *for all* $J \in KerE$.

Proof. Assume that $P^{-1} \in L(H)$, then from $SP = PS$ and $SJP = PJS$ we get, $SP^{-1}J = JP^{-1}S$. Hence applying theorem AK [38] to the operators SP^{-1}, $P^{-1}S$ and J we get,

$$|||(SX - XS) + J||| \geqslant |||(SP^{-1}PXP - PXP^{-1}S) + J||| \geqslant |||J|||.$$

Consider now the case when P is injective i.e $KerP = 0$. Let $\sigma_n = \{\lambda \in \mathbb{C} : \lambda \leqslant \frac{1}{n}\}$ and let $E_P(\sigma_n)$ be the corresponding spectral projector. If we put $P_n = I - E_P(\sigma_n)$. The subspace $P_n H$ reduces both S and P (since they commute and are normal). Hence, with respect to the decomposition $H = (I - P_n)H \oplus P_n(H)$

$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_1^{(n)} \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_1^{(n)} \end{pmatrix} J = \begin{pmatrix} J_{11}^{(n)} & J_{12}^{(n)} \\ J_{21}^{(n)} & J_{22}^{(n)} \end{pmatrix} \text{ and } X = \begin{pmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{pmatrix}.$$

It is easy to see that $P_1^{(n)}$ acting on $P_n(H)$ is invertible. It follows that

$$
\begin{aligned}
|||SXP - PXS + J||| &\geqslant |||P_n(SXP - PXS + J)P_n||| \\
&= |||S_1^{(n)}X_{22}^{(n)}P_1^{(n)} - P_1^{(n)}X_{22}^{(n)}S_1^{(n)} + J_{22}||| \\
&\geqslant |||J_{22}||| = |||P_n J P_n|||
\end{aligned}
$$

Therefore, we have $|||SXP - PXS + J||| \geqslant |||P_n J P_n|||$.
Applying Lemma 3 in [62] we obtain $|||SXP - PXS + J||| \geqslant |||J|||$.
Now, we assume that $KerS \cap KerP = \{0\}$. Let S, P, J and X have the following representation with respect to the space decomposition $H = KerP \oplus H_o(H_o \ominus KerP)$.

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \text{ and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Operators S_1 and P_2 are injective and we have,

$$(SXP - PXS) = \begin{pmatrix} 0 & S_1 X_{12} P_2 \\ -P_2 X_{21} S_1 & S_2 X_{22} P_2 - P_2 X_{22} S_2 \end{pmatrix}.$$

Since $SJP = PJS = 0$, then $S_2 J_{22} P_2 = P_2 J_{22} S_2$ and $S_1 J_{12} P_2 = P_2 J_{21} S_1 = 0$ since S_1 and P_2 are injective and their ranges are dense. We have,

$$
\begin{aligned}
|||SXP - PXS + J||| &= \left|\left|\left| \begin{pmatrix} 0 & S_1 X_{12} P_2 \\ -P_2 X_{21} S_1 & S_2 X_{22} P_2 - P_2 X_{22} S_2 \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right|\right|\right| \\
&= \left|\left|\left| \begin{pmatrix} J_{11} & S_1 X_{12} P_2 \\ -P_2 X_{21} S_1 & S_2 X_{22} P_2 - P_2 X_{22} S_2 + J_{22} \end{pmatrix} \right|\right|\right|.
\end{aligned}
$$

Since P_2 is injective, we have already shown that

$$|||S_2 X_{22} P_2 - P_2 X_{22} S_2 + J_{22}||| \geqslant |||J_{22}|||$$

Applying Lemma GK in [62] we have

$$
\begin{aligned}
|||S_2 X_{22} P_2 - P_2 X_{22} S_2 + J_{22}||| &\geqslant \left|\left|\left| \begin{pmatrix} J_{11} & 0 \\ 0 & S_2 X_{22} P_2 - P_2 X_{22} S_2 + J_{22} \end{pmatrix} \right|\right|\right| \\
&\geqslant \left|\left|\left| \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} \right|\right|\right| = |||J|||.
\end{aligned}
$$

$\square$

Theorem 3.24. *Let* $S, P \in L(H)$ *be normal operators such that* $PS = SP$ *and* $E(X) = SXP - PXS$. *If* $J \in KerE$ *then*

$$|||E(X) + J||| \geqslant 3^{-1}|||J||| \tag{3.1}$$

and

$$\|E(X) + J\|_p \geqslant 2^{|1 - \frac{2}{p}|}\|J\|_p, \tag{3.2}$$

where $\|.\|_P$ is the C_P norm.
In particular, for the Hilbert Schmidt-norm we have

$$\|E(X) + J\|_2^2 \geqslant \|J\|_2^2 + \|E(X)\|_2^2. \tag{3.3}$$

Proof. Let S, P, J and X have the following representation with respect to the space decomposition $H = H_1 \oplus H_2$, where $H_1 = KerS \cap KerP)$ and $H_2 = H \ominus H_1$
$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \text{ and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$
We have that $KerS_2 \cap KerP_2 = \{0\}$ in H_2. Applying Proposition 3.23 we have,

$$
\begin{aligned}
\||SXP - PXS + J\|| &= \left\|\left|\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix}\right|\right\| \\
&\geqslant \||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}\|| \\
&\geqslant 2^{-1}\||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}\|| \\
&\geqslant 2^{-1}\||S_2X_{22}P_2 - P_2X_{22}S_2\|| \\
&\geqslant 2^{-1}\||SXP - PXS\||.
\end{aligned}
$$

For us to prove Inequality 3.2 we start with the same inequalities before and then we apply Lemma K in [6] twice and Proposition 3.23. For $1 \leqslant p \leqslant 2$ we have,

$$
\begin{aligned}
\|SXP - PXS + J\|_p^p &= \left\|\begin{pmatrix} 0 & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}\right\|_p^p \\
&= \left\|\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix}\right\|_p^p \\
&\geqslant 2^{p-2}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p \\
&\quad + \|S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22}\|_p^p) \\
&\geqslant 2^{p-2}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p + \|J_{22}\|_p^p) \\
&\geqslant 2^{p-2}\|J\|_p^p.
\end{aligned}
$$

and for $2 \leqslant p < \infty$ we have,

$$
\begin{aligned}
\|SXP - PXS + J\|_p^p &= \left\|\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22} \end{pmatrix}\right\|_p^p \\
&\geqslant 2^{2-p}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p \\
&\quad + \|S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22}\|_p^p) \\
&\geqslant 2^{2-p}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p + \|J_{22}\|_p^p) \\
&\geqslant 2^{2-p}\|J\|_p^p.
\end{aligned}
$$

Hence, $\|SXP - PXS + J\|_p^p \geqslant 2^{|1-\frac{2}{p}|}\|J\|_p^p$ and this proves Inequality 3.2 Now, if $p = 2$ Inequality 3.2 becomes $\|E(X) + J\|_2 \geqslant \|J\|_2$ and this implies Inequality 3.3. $\qquad \square$

Corollary 3.25. *Let $S, P \in L(H)$ be normal, then for every operator J satisfying $SJP = J$, $\|SXP - PXS + J\| \geqslant \|J\|$ for all $X \in L(H)$.*

Proof. Let $SJP = J$, then $SJ = JP^{-1}$. Since $SJP = J$ we have that $SJP = PJS$ which implies that $SP^{-1}J = JP^{-1}S$. Applying theorem AK [4] to the operators SP^{-1}, $P^{-1}S$ and J and from Proposition 3.8 we get

$$\|SXP - PXS + J\| = \|SP^{-1}PXP - PXPP^{-1}S + J\| = \|J\|.$$

Now, suppose P is not injective with respect to the decomposition $H = Ker(P)^{\perp} \cap KerP$. Using the condition $SJP = J$ we have,

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \text{ and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

where S_1 is injective, from Proposition 4.23 it follows that

$$\begin{aligned} \|SXP - PXS + J\| &= \left\| \begin{pmatrix} S_1X_1P_1 - P_1X_1S_1 & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\| \\ &\geqslant \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\| \\ &\geqslant \|J\|. \end{aligned}$$

$\square$

Corollary 3.26. *If the assumptions of Theorem 3.24 hold, then* $\overline{ranE} \cap KerE = \{0\}$ *where the closure can be taken in the more uniform norm. Hence* $E(E(X)) = 0$ *implies that* $E(X) = 0$.

Proof. If $S \in \overline{ranE} \cap KerE$, then $S = \lim_{n \to \infty} E(x_n)$ and $E(S) = 0$. From Theorem 2.1 in [60] we have that

$$\|E(x_n) - S\| \geqslant c\|S\|.$$

Hence,

$$\|S - S\| \geqslant c\|S\|.$$

Therefore,

$$S = 0.$$

$\square$

At this point, we characterize Birkhoff-James orthogonality for finite elementary operators. It is known from [52] that for any examples elementary defined in Section 2 (inner derivation, generalized derivation, basic elementary operator, Jordan elementary operator) the following implication hold for a general bounded linear operator S on a normed linear space Ω. i.e $Ran(S) \perp KerS \Rightarrow \overline{Ran(S)} \cap KerS = 0 \Rightarrow Ran(S) \cap KerS = 0$, where $\overline{Ran(S)}$ denotes the closure of the Range of S and $KerS$ denotes the Kernel of S and $Ran(S) \perp KerS$ means Range of S is orthogonal to the Kernel of S in the sense of Birkhoff.

Proposition 3.27. *Let* $S \in L(H)$ *be isometric, then* $Ran\delta_S \perp Ker\delta_S$.

Proof. From Proposition we know,

$$S^nX - XS^n = \sum_{i=0}^{n-i-1}(SX - XS)S^i \text{ for all } X \in L(H).$$

Therefore if $SJ = JS$ we have,

$$nJS^{n-1} = S^nX - XS^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS) - J)S^i \text{ for all } X \in L(H).$$

Dividing both sides by n and taking norms we obtain,

$$\|JS^{n-1}\| \leqslant \tfrac{1}{n}\|S^nX + XS^n\| + \tfrac{1}{n}\sum_{i=0}^{n-i-1}\|S^{n-i-1}\|\|((SX - XS) - J)\|\|S^i\|.$$

Since S is isometric we have,

$$\|J\| \leqslant \tfrac{2}{n}\|X\| + \|((SX - XS) - J)\|.$$

Letting $n \to \infty$ we obtain,
$\|(SX - XS) - J\| \geqslant \|J\|$ and hence, $\mathrm{Ran}\delta_S \perp \mathrm{Ker}\delta_S$. $\qquad\square$

Corollary 3.28. *Let* $S, S_o \in L(H)$ *be contractive such that* $\delta_{S,S_o}(J) = 0$ *for some* $J \in L(H)$. *Then*

$$\|\delta_{S,S_o} + J\| \geqslant \|J\| \text{ for all } X \in L(H).$$

Proof. Given $J \in L(H)$ and from Proposition 3.27 we have,

$$nJS_o^{n-1} = S^nX - XS_o^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS_o) - J)S_o^i \text{ for all } X \in L(H).$$

Dividing both sides by n and taking norms we obtain,
$\|JS_o^{n-1}\| \leqslant \tfrac{1}{n}\|S^nX + XS_o^n\| + \tfrac{1}{n}\sum_{i=0}^{n-i-1}\|S^{n-i-1}\|\|((SX - XS_o) - J)\|\|S_o^i\|$. But S and S_o are contractive i.e $\|S^n\| \leqslant 1$ and $\|S_o^n\| \leqslant 1$. This implies that $\|S^{n-1}\| \leqslant 1$ and $\|S_o^{n-1}\| \leqslant 1$ and hence we have,

$$\|J\| \leqslant \tfrac{2}{n}\|X\| + \|((SX - XS_o) - J)\|.$$

Letting $n \to \infty$ we obtain,

$$\|(SX - XS_o) - J\| \geqslant \|J\|. \text{ Therefore, } \mathrm{Ran}\delta_{S,S_o} \perp \mathrm{Ker}\delta_{S,S_o}.$$

$\qquad\square$

Lemma 3.29. *Let* $S, P \in L(H)$, *such that the pair* (S, P) *satisfies* (PF) *property, then* $\mathrm{Ran}\delta_{S,P} \perp \mathrm{Ker}\delta_{S,P}$.

Proof. Suppose $X \in \mathrm{Ker}\delta_{S,P}$, then $SX - XP \in \mathrm{Ran}\delta_{S,P} \cap \mathrm{Ker}\delta_{S,P}$. For $J \in \mathrm{Ker}\delta_{S,P}$, we have that the $\overline{\mathrm{Ran}(J)}$ reduces S and $\mathrm{Ker}(J)^\perp$ reduces P and $S\mid_{\overline{\mathrm{Ran}(J)}}$ and $P\mid_{\mathrm{Ker}(J)^\perp}$ are normal operators. Let S, P, J and X have the following representation with respect to the decompositions $H = H_1 = \overline{R(J)} \oplus \overline{R(J)}^\perp$, $H = H_2 = \mathrm{Ker}(J)^\perp \oplus \mathrm{Ker}(J)$.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.
From Proposition 4.8 we have,

$$\|(SX - XP) + J\| = \left\| \begin{pmatrix} (S_1X_1 - X_1P_1) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|.$$

This implies that

$$\|(SX - XP) + J\| \geqslant \|(S_1X_1 - X_1P_1) + J_1\| \geqslant \|J_1\| = \|J\|.$$

Hence, $\|\delta_{S,P}(X) + J\| \geqslant \|\delta_{S_1,P_1}(X) + J_1\| \geqslant \|J_1\| = \|J\|$.
Therefore, $\mathrm{Ran}\delta_{S,P} \cap \mathrm{Ker}\delta_{S,P} = 0$. $\qquad\square$

Remark 3.30. Let $S \in L(H)$ be quasihyponormal and T^* be injective hyponormal operator, if $ST = TS$ for some $X \in L(H)$. Then $S^*T = T^*S$, $\mathrm{Ran}J$ reduces S, $\mathrm{Ker}J^\perp$ reduces T and $S\mid_{\overline{\mathrm{Ran}(J)}}$ and $T\mid_{\mathrm{Ker}(J)^\perp}$ are unitarily equivalent normal operators.

Theorem 3.31. *Let* $S \in L(H)$ *be quasihyponormal and* T^* *be injective hyponormal operator in* $L(H)$, *then* $Ran\delta_{S,T} \perp Ker\delta_{S,T}$.

Proof. The pair (S,T) has the $PF_{(LH)}$ property by Remark 3.30. Let $J \in L(H)$ be such that $SJ = JT$. Let S, T, J and X have the following representation with respect to the decompositions $H = K = \overline{Ran(J)} \oplus \overline{Ran(J)}^{\perp}$, $H = L = Ker(J)^{\perp} \oplus Ker(J)$.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$. where T_1 and S_1 are normal operators on K to L, then we have,

$$\|(SX - XT) + J\| = \left\| \begin{pmatrix} (S_1X_1 - X_1T_1) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|.$$

Thus, from Lemma 3.29 it follows that

$$\|(SX - XT) + J\| \geqslant \|(S_1X_1 - X_1T_1) + J_1\| \geqslant \|J_1\| = \|J\|.$$

Hence, $Ran\delta_{S,T} \perp Ker\delta_{S,T}$. $\qquad\qquad\square$

Let $E(X) = SXS_o - S_oXS$, then we have the following theorem.

Theorem 3.32. *Let* S, $S_o \in L(H)$ *be normal operators such that* $SS_o = S_oS$. *Then* $\|(SXS_o - S_oXS) + J\|_p \geqslant \|J\|_p$, *for all* $X \in C_p$ *and for all* $J \in KerE \cap C_p$ $(1 \leqslant p < \infty)$.

Proof. It suffices to take the Hilbert space $H \oplus H$ and the operators.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$. It follows that

$$\|(SXS_o - S_oXS) + J\|_p = \left\| \begin{pmatrix} (S_1X_1S_2 - S_2X_1S_1) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_p.$$

Thus, from Theorem 3.31 we have

$$\|(SXS_o - S_oXS) + J\|_p \geqslant \|(S_1X_1S_2 - S_2X_1S_1) + J_1\|_p \geqslant \|J_1\|_p = \|J\|_p.$$

Hence, $RanE \perp KerE$. $\qquad\qquad\square$

Let $\varphi(X) = SXS_o - PXP_o$, then we have the following corollary.

Corollary 3.33. *Let* S, S_o, P, $P_o \in L(H)$ *be normal operators such that* $SP = PS$ *and* $S_oP_o = P_oS_o$. *Then* $\|(SXS_o - PXP_o) + J\|_p \geqslant \|J\|_p$, *for all* $X \in C_p$ *and for all* $J \in Ker\varphi \cap C_p$ $(1 \leqslant p < \infty)$.

Proof. On $H \oplus H$ consider the operators S, S_o, P, P_o, J and X defined by
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, P_o = \begin{pmatrix} P_2 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.
It follows that

$$\|(SXS_o - PXP_o) + J\|_p = \left\| \begin{pmatrix} (S_1X_1S_2 - P_1X_1P_2) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_p.$$

Thus, from Theorem 3.32 we have

$$\|(SXS_o - PXP_o) + J\|_p \geqslant \|(S_1X_1S_2 - P_1X_1P_2\|_p \geqslant \|J_1\|_p = \|J\|_p.$$

Hence, $Ran\varphi \perp Ker\varphi$. $\qquad\qquad\square$

4. Conclusion

In this work, we have given a detailed survey on characterization of orthogonality of elementary operators in normed spaces. We considered these operators when they are finite and unveiled new conditions which are necessary and sufficient for their orthogonality. Lastly, we have characterized Birkhoff-James orthogonality for this class of operators. We have shown that finite elementary operators satisfy orthogonality in the sense of Birkhoff-James if they are bounded, isometric and normal.

Acknowledgement

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