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Maclaurin's inequalities for functions whose first derivatives are preinvex

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Abstract

In this paper, using a new identity, we study one of the famous Newton-Cotes three-point quadrature rules. More precisely Maclaurin's quadrature rule, for which we establish the error estimate of this method under the constraint that the first derivatives belong to the class of preinvex functions. We also give some applications to special means as applications. We believe that this new studied inequality and the results obtained in this article will further inspire intrigued researchers.

Keywords: Maclaurin's inequality, preinvex functions, Hölder inequality, power mean inequality
2010 MSC: 26D10, 26D15, 26A51.

1. Introduction

The concept of convexity plays an important and very central role in many areas, such as economics, finances, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions. The significant one is that introduced by Hanson [1], called invex functions. Pini [2], Noor [3, 4], Yang and Li [5] and Weir [6], have studied the basic properties of preinvex functions and their roles in optimization, variational inequalities and equilibrium problems.

It is well known that the concept of convexity has a close relationship in the development of the theory of inequalities, which is an important tool in the study of error estimates of quadrature formulas as well as in the study of the qualitative properties of differential solutions and integral equations.

In the last decades several generalizations, extensions and improvements of certain integral inequalities have been established we refer the reader to [7, 8, 9, 10, 11, 12, 13, 5] and references therein.

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The best known and most used of the three-point quadrature is that of Simpson, which can be described as:

$$\left|\frac{1}{6}\left(\mathcal{U}(\upsilon)+4\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+\mathcal{U}(\tau)\right)-\frac{1}{\tau-\upsilon}\int\limits_{\upsilon}^{\tau}\mathcal{U}(r)\,dr\right|\leqslant\frac{(\tau-\upsilon)^4}{2880}\left\|\mathcal{U}^{(4)}\right\|_{\infty}, \tag{1.1}$$

where f is four-times continuously differentiable function on (υ,τ), and

$$\left\|\mathcal{U}^{(4)}\right\|_{\infty}=\sup_{r\in(\upsilon,\tau)}\left|\mathcal{U}^{(4)}(r)\right|$$

.

However, there are other three-point quadratures like Bullen's, dual Simpson's and Maclaurin's quadrature formulas.

The following inequality is known as the Maclaurin formula see [15], which can be stated as follows

$$\left|\frac{1}{8}\left(3\mathcal{U}\left(\frac{5\upsilon+\tau}{6}\right)+2\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+3\mathcal{U}\left(\frac{\upsilon+5\tau}{6}\right)\right)-\frac{1}{\tau-\upsilon}\int\limits_{\upsilon}^{\tau}\mathcal{U}(r)\,dr\right|\leqslant\frac{7(\tau-\upsilon)^4}{51840}\left\|\mathcal{U}^{(4)}\right\|_{\infty}, \tag{1.2}$$

where f is four-times continuously differentiable function on (υ,τ), and

$$\left\|\mathcal{U}^{(4)}\right\|_{\infty}=\sup_{r\in(\upsilon,\tau)}\left|\mathcal{U}^{(4)}(r)\right|$$

.

In this paper, by adopting a novel approach, governed by a new identity, we establish some new Maclaurin-type inequalities for functions whose absolute value of the first derivatives are preinvex. The paper is organized as follows: In Section 2, we collect some definitions and tools which will be used throughout this paper. In Section 3, we prove some new results of Maclaurin-type inequalities under the preinvexity. In section 4. Some examples to explain the acquired results. In the last section, we summarize our work and suggest future directions.

2. Preliminaries

In this sections we recall some definitions.

Definition 2.1. [14] A function $\mathcal{U}: I \to \mathbb{R}$ is said to be convex, if

$$\mathcal{U}(t\upsilon+(1-t)\tau)\leqslant t\mathcal{U}(\upsilon)+(1-t)\mathcal{U}(\tau)$$

holds for all $\upsilon,\tau\in I$ and all $t\in[0,1]$.

Definition 2.2. [6] A set $K\subseteq\mathbb{R}$ is said an invex with respect to the bifunction $\varphi: K\times K\to\mathbb{R}$, if for all $\upsilon,\tau\in K$, we have

$$\upsilon+t\varphi(\tau,\upsilon)\in K.$$

Weir and Mond [10], gave the concept of preinvex functions which is special case of invexity.

Definition 2.3. [6] A function $\mathcal{U} : K \to \mathbb{R}$ is said to be preinvex with respect to φ, if

$$\mathcal{U}\left(\upsilon + t\varphi\left(\tau,\upsilon\right)\right) \leqslant \left(1-t\right)\mathcal{U}\left(\upsilon\right) + t\mathcal{U}(\tau)$$

holds for all $\upsilon, \tau \in K$ and all $t \in [0,1]$.

Theorem 2.4. *[16, Hölder inequality] Let $f(\varkappa)$ and $g(\varkappa)$ be a real functions defined on $[a,b]$. If $\rho > 1$ and $\frac{1}{\delta}+\frac{1}{\rho}=1$, then*

$$\int\limits_a^b \left|f\left(\varkappa\right)g\left(\varkappa\right)\right| d\varkappa \leqslant \left(\int\limits_a^b \left|f\left(\varkappa\right)\right|^{\delta} d\varkappa\right)^{\frac{1}{\delta}} \left(\int\limits_a^b \left|g\left(\varkappa\right)\right|^{\rho} d\varkappa\right)^{\frac{1}{\rho}}.$$

Theorem 2.5. *[16, Power mean inequality] Let $f(\varkappa)$ and $g(\varkappa)$ be positive continuous functions on $[a,b]$. If $\rho \geqslant 1$, then*

$$\int\limits_a^b \left|f\left(\varkappa\right)g\left(\varkappa\right)\right| d\varkappa \leqslant \left(\int\limits_a^b \left|f\left(\varkappa\right)\right| d\varkappa\right)^{1-\frac{1}{\rho}} \left(\int\limits_a^b \left|f\left(\varkappa\right)\right|\left|g\left(\varkappa\right)\right|^{\rho} d\varkappa\right)^{\frac{1}{\rho}}.$$

Theorem 2.6. *[16, Discrete Power mean inequality] For any $a, b > 0$ and $0 \leqslant \alpha \leqslant 1$, we have*

$$a^{\alpha} + b^{\alpha} \leqslant 2^{1-\alpha}\left(a+b\right)^{\alpha}.$$

3. Main results

The identity given by the following lemma is crucial in the sequel

Lemma 3.1. *Let $\mathcal{U} : [\upsilon, \upsilon+\varphi\left(\tau,\upsilon\right)] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(\upsilon, \upsilon+\varphi\left(\tau,\upsilon\right))$ with $\varphi\left(\tau,\upsilon\right) > 0$, and $\mathcal{U}' \in L^1\left[\upsilon, \upsilon+\varphi\left(\tau,\upsilon\right)\right]$, then we have*

$$\begin{aligned}
&\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8} - \frac{1}{\varphi(\tau,\upsilon)} \int\limits_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)} \mathcal{U}\left(r\right) dr \\
= \;& \frac{\varphi(\tau,\upsilon)}{36}\left(\int\limits_0^1 \left(1-\varkappa\right)\mathcal{U}'\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi\left(\tau,\upsilon\right)\right) d\varkappa\right. \\
&+\int\limits_0^1 \left(\tfrac{3}{2}-4\varkappa\right)\mathcal{U}'\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi\left(\tau,\upsilon\right)\right) d\varkappa \\
&+\int\limits_0^1 \left(\tfrac{5}{2}-4\varkappa\right)\mathcal{U}'\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi\left(\tau,\upsilon\right)\right) d\varkappa \\
&\left.+\int\limits_0^1 \left(\varkappa-1\right)\mathcal{U}'\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi\left(\tau,\upsilon\right)\right) d\varkappa\right).
\end{aligned} \tag{3.1}$$

Proof. Let

$$\begin{aligned} I_1 &= \int\limits_0^1 (1-\varkappa)\,\mathcal{U}'\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa, \\ I_2 &= \int\limits_0^1 \left(\tfrac{3}{2}-4\varkappa\right)\mathcal{U}'\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa, \\ I_3 &= \int\limits_0^1 \left(\tfrac{5}{2}-4\varkappa\right)\mathcal{U}'\left(\upsilon+\tfrac{5+2\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \end{aligned}$$

and

$$I_4 = \int\limits_0^1 (\varkappa-1)\,\mathcal{U}'\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa.$$

Integrating by parts I_1, we get

$$\begin{aligned} I_1 &= -\tfrac{6}{\varphi(\tau,\upsilon)}(1-\varkappa)\,\mathcal{U}\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)\Big|_{\varkappa=0}^{\varkappa=1}-\tfrac{6}{\varphi(\tau,\upsilon)}\int\limits_0^1 \mathcal{U}\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\ &= \tfrac{6}{\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)-\tfrac{6}{\varphi(\tau,\upsilon)}\int\limits_0^1 \mathcal{U}\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\ &= \tfrac{6}{\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)-\tfrac{36}{\varphi^2(\tau,\upsilon)}\int\limits_a^{\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}} \mathcal{U}(r)\,dr. \end{aligned} \tag{3.2}$$

Similarly, we have

$$\begin{aligned} I_2 &= -\tfrac{3}{\varphi(\tau,\upsilon)}\left(\tfrac{3}{2}-4\varkappa\right)\mathcal{U}\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\Big|_{\varkappa=0}^{\varkappa=1} \\ &\quad -\tfrac{12}{\varphi(\tau,\upsilon)}\int\limits_0^1 \mathcal{U}\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\ &= \tfrac{15}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+\tfrac{9}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right) \\ &\quad -\tfrac{12}{\varphi(\tau,\upsilon)}\int\limits_0^1 \mathcal{U}\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\ &= \tfrac{15}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+\tfrac{9}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)-\tfrac{36}{\varphi^2(\tau,\upsilon)}\int\limits_{\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}}^{\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}} \mathcal{U}(r)\,dr, \end{aligned} \tag{3.3}$$

$$\begin{aligned}
I_3 &= -\tfrac{3}{\varphi(\tau,\upsilon)}\left(\tfrac{5}{2}-4\varkappa\right)\mathcal{U}\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\Big|_{\varkappa=0}^{\varkappa=1} \\
&\quad -\tfrac{12}{\varphi(\tau,\upsilon)}\int_0^1 \mathcal{U}\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\
&= \tfrac{9}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+\tfrac{15}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right) \\
&\quad -\tfrac{12}{\varphi(\tau,\upsilon)}\int_0^1 \mathcal{U}\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\
&= \tfrac{9}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+\tfrac{15}{2\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)-\tfrac{36}{\varphi^2(\tau,\upsilon)}\int_{\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}}^{\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}}\mathcal{U}(r)\,dr
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
I_4 &= \tfrac{6}{\varphi(\tau,\upsilon)}(\varkappa-1)\,\mathcal{U}\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)\Big|_{\varkappa=0}^{\varkappa=1}-\tfrac{6}{\varphi(\tau,\upsilon)}\int_0^1 \mathcal{U}\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\
&= \tfrac{6}{\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)-\tfrac{6}{\varphi(\tau,\upsilon)}\int_0^1 \mathcal{U}\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)d\varkappa \\
&= \tfrac{6}{\varphi(\tau,\upsilon)}\mathcal{U}\left(\tfrac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)-\tfrac{36}{\varphi^2(\tau,\upsilon)}\int_{\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr.
\end{aligned} \tag{3.5}$$

Summing (3.2)-(3.5), then multiplying the resulting equality by $\frac{\varphi(\tau,\upsilon)}{36}$, we get the result. The proof is completed. □

Theorem 3.2. *Let $\mathcal{U}:[\upsilon,\upsilon+\varphi(\tau,\upsilon)]\to\mathbb{R}$ be a differentiable map on $(\upsilon,\upsilon+\varphi(\tau,\upsilon))$ such that $\mathcal{U}'\in L^1[\upsilon,\upsilon+\varphi(\tau,\upsilon)]$ with $\varphi(\tau,\upsilon)>0$. If $|\mathcal{U}'|$ is preinvex, then we have*

$$\begin{aligned}
&\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right| \\
&\leqslant \frac{25\varphi(\tau,\upsilon)}{576}\left(\left|\mathcal{U}'(\upsilon)\right|+\left|\mathcal{U}'(\tau)\right|\right).
\end{aligned}$$

Proof. Applying the absolute value in both sides of (3.1) and using the preinvexity of $|f'|$, it yields

$$\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right|$$

$$\begin{aligned}
&\leqslant \tfrac{\varphi(\tau,\upsilon)}{36}\left(\int_0^1 (1-\varkappa)\left|\mathcal{U}'\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)\right| d\varkappa\right.\\
&\quad+\int_0^1 \left|\tfrac{3}{2}-4\varkappa\right|\left|\mathcal{U}'\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\right| d\varkappa\\
&\quad+\int_0^1 \left|\tfrac{5}{2}-4\varkappa\right|\left|\mathcal{U}'\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\right| d\varkappa\\
&\quad\left.+\int_0^1 (1-\varkappa)\left|\mathcal{U}'\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)\right| d\varkappa\right)\\
&\leqslant \tfrac{\varphi(\tau,\upsilon)}{36}\left(\int_0^1 (1-\varkappa)\left(\left(1-\tfrac{1-\varkappa}{6}\right)\left|\mathcal{U}'(\upsilon)\right|+\tfrac{1-\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\right.\\
&\quad+\int_0^1 \left|\tfrac{3}{2}-4\varkappa\right|\left(\left(1-\tfrac{3-2\varkappa}{6}\right)\left|\mathcal{U}'(\upsilon)\right|+\tfrac{3-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\\
&\quad+\int_0^1 \left|\tfrac{5}{2}-4\varkappa\right|\left(\left(1-\tfrac{5-2\varkappa}{6}\right)\left|\mathcal{U}'(\upsilon)\right|+\tfrac{5-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\\
&\quad\left.+\int_0^1 (1-\varkappa)\left(\left(1-\tfrac{5+\varkappa}{6}\right)\left|\mathcal{U}'(\upsilon)\right|+\tfrac{5+\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\right)\\
&= \tfrac{\varphi(\tau,\upsilon)}{36}\left(\int_0^1 (1-\varkappa)\left(\tfrac{5+\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|+\tfrac{1-\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\right.\\
&\quad+\int_0^{\frac{3}{8}} \left(\tfrac{3}{2}-4\varkappa\right)\left(\tfrac{3+2\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|+\tfrac{3-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\\
&\quad+\int_{\frac{3}{8}}^1 \left(4\varkappa-\tfrac{3}{2}\right)\left(\tfrac{3+2\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|+\tfrac{3-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\\
&\quad+\int_0^{\frac{5}{8}} \left(\tfrac{5}{2}-4\varkappa\right)\left(\tfrac{1+2\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|+\tfrac{5-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa\\
&\quad+\int_{\frac{5}{8}}^1 \left(4\varkappa-\tfrac{5}{2}\right)\left(\tfrac{1+2\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|+\tfrac{5-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right) d\varkappa
\end{aligned}$$

$$
\begin{aligned}
&+\int_{0}^{1}(1-\varkappa)\left(\tfrac{1-\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|+\tfrac{5+\varkappa}{6}\left|\mathcal{U}'(\tau)\right|\right)d\varkappa\Bigg)\\
=\;&\tfrac{\varphi(\tau,\upsilon)}{36}\left(\left|\mathcal{U}'(\upsilon)\right|+\left|\mathcal{U}'(\tau)\right|\right)\\
&\times\Bigg(\int_{0}^{1}(1-\varkappa)\left(\tfrac{5+\varkappa}{6}\right)d\varkappa+\int_{0}^{\frac{3}{8}}\left(\tfrac{3}{2}-4\varkappa\right)\left(\tfrac{3+2\varkappa}{6}\right)d\varkappa\\
&+\int_{\frac{3}{8}}^{1}\left(4\varkappa-\tfrac{3}{2}\right)\left(\tfrac{3+2\varkappa}{6}\right)d\varkappa+\int_{0}^{\frac{5}{8}}\left(\tfrac{5}{2}-4\varkappa\right)\left(\tfrac{1+2\varkappa}{6}\right)d\varkappa\\
&+\int_{\frac{5}{8}}^{1}\left(4\varkappa-\tfrac{5}{2}\right)\left(\tfrac{1+2\varkappa}{6}\right)d\varkappa+\int_{0}^{1}(1-\varkappa)\left(\tfrac{1-\varkappa}{6}\right)d\varkappa\Bigg)\\
=\;&\tfrac{25\varphi(\tau,\upsilon)}{576}\left(\left|\mathcal{U}'(\upsilon)\right|+\left|\mathcal{U}'(\tau)\right|\right),
\end{aligned}
$$

where we have used the fact that

$$\int_{0}^{1}(1-\varkappa)\left(\tfrac{5+\varkappa}{6}\right)d\varkappa = \tfrac{4}{9}, \tag{3.6}$$

$$\int_{0}^{1}(1-\varkappa)\left(\tfrac{1-\varkappa}{6}\right)d\varkappa = \tfrac{1}{18}, \tag{3.7}$$

$$\int_{0}^{\frac{3}{8}}\left(\tfrac{3}{2}-4\varkappa\right)\left(\tfrac{3+2\varkappa}{6}\right)d\varkappa = \int_{\frac{5}{8}}^{1}\left(4\varkappa-\tfrac{5}{2}\right)\left(\tfrac{5-2\varkappa}{6}\right)d\varkappa=\tfrac{39}{256}, \tag{3.8}$$

$$\int_{0}^{\frac{3}{8}}\left(\tfrac{3}{2}-4\varkappa\right)\left(\tfrac{3-2\varkappa}{6}\right)d\varkappa = \int_{\frac{5}{8}}^{1}\left(4\varkappa-\tfrac{5}{2}\right)\left(\tfrac{1+2\varkappa}{6}\right)d\varkappa=\tfrac{33}{256}, \tag{3.9}$$

$$\int_{\frac{3}{8}}^{1}\left(4\varkappa-\tfrac{3}{2}\right)\left(\tfrac{3+2\varkappa}{6}\right)d\varkappa = \int_{0}^{\frac{5}{8}}\left(\tfrac{5}{2}-4\varkappa\right)\left(\tfrac{5-2\varkappa}{6}\right)d\varkappa=\tfrac{1375}{2304} \tag{3.10}$$

and

$$\int_{\frac{3}{8}}^{1}\left(4\varkappa-\tfrac{3}{2}\right)\left(\tfrac{3-2\varkappa}{6}\right)d\varkappa=\int_{0}^{\frac{5}{8}}\left(\tfrac{5}{2}-4\varkappa\right)\left(\tfrac{1+2\varkappa}{6}\right)d\varkappa=\tfrac{425}{2304}. \tag{3.11}$$

The proof is over. □

Corollary 3.3. *Choosing* $\varphi(\tau,\upsilon)=\tau-\upsilon$, *then Theorem 3.2 gives*

$$\left|\frac{3\mathcal{U}\left(\frac{5\upsilon+\tau}{6}\right)+2\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+3\mathcal{U}\left(\frac{\upsilon+5\tau}{6}\right)}{8}-\frac{1}{\tau-\upsilon}\int_{\upsilon}^{\tau}\mathcal{U}(r)\,dr\right|\leqslant\frac{25(\tau-\upsilon)}{576}\left(\left|\mathcal{U}'(\upsilon)\right|+\left|\mathcal{U}'(\tau)\right|\right).$$

Theorem 3.4. *Let* $\mathcal{U}:[\upsilon,\upsilon+\varphi(\tau,\upsilon)]\to\mathbb{R}$ *be a differentiable map on* $(\upsilon,\upsilon+\varphi(\tau,\upsilon))$ *such that* $\mathcal{U}'\in L^1[\upsilon,\upsilon+\varphi(\tau,\upsilon)]$ *with* $\varphi(\tau,\upsilon)>0$. *If* $|\mathcal{U}'|^{\rho}$ *is preinvex where* $\rho>1$ *with* $\frac{1}{\delta}+\frac{1}{\rho}=1$, *then we have*

$$\begin{aligned}&\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right|\\&\leqslant\frac{\varphi(\tau,\upsilon)}{36}\left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\left(\left(\frac{11|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{12}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+11|\mathcal{U}'(\tau)|^{\rho}}{12}\right)^{\frac{1}{\rho}}\right.\\&\quad\left.+\frac{1}{2}\left(\frac{3^{\delta+1}+5^{\delta+1}}{8}\right)^{\frac{1}{\delta}}\left(\left(\frac{2|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{3}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+2|\mathcal{U}'(\tau)|^{\rho}}{3}\right)^{\frac{1}{\rho}}\right)\right).\end{aligned} \tag{3.12}$$

Proof. Applying the absolute value in both sides of (3.1), using Hölder's inequality, and the preinvexity of $|\mathcal{U}'|^{\rho}$, we have

$$\begin{aligned}&\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right|\\&\leqslant\frac{\varphi(\tau,\upsilon)}{36}\left(\left(\int_{0}^{1}(1-\varkappa)^{\delta}d\varkappa\right)^{\frac{1}{\delta}}\left(\int_{0}^{1}\left|\mathcal{U}'\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}\right.\\&\quad+\left(\int_{0}^{1}\left|\tfrac{3}{2}-4\varkappa\right|^{\delta}d\varkappa\right)^{\frac{1}{\delta}}\left(\int_{0}^{1}\left|\mathcal{U}'\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}\\&\quad+\left(\int_{0}^{1}\left|\tfrac{5}{2}-4\varkappa\right|^{\delta}d\varkappa\right)^{\frac{1}{\delta}}\left(\int_{0}^{1}\left|\mathcal{U}'\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}\\&\quad\left.+\left(\int_{0}^{1}(1-\varkappa)^{\delta}d\varkappa\right)^{\frac{1}{\delta}}\left(\int_{0}^{1}\left|\mathcal{U}'\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}\right)\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{\varphi(\tau,\upsilon)}{36}\left(\left(\int_0^1 (1-\varkappa)^{\delta}\, d\varkappa\right)^{\frac{1}{\delta}}\left(\int_0^1 \left(\tfrac{5+\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|^{\rho}+\tfrac{1-\varkappa}{6}\left|\mathcal{U}'(\tau)\right|^{\rho}\right) d\varkappa\right)^{\frac{1}{\rho}}\right.\\
&\quad +4\left(\int_0^{\frac{3}{8}} \left(\tfrac{3}{8}-\varkappa\right)^{\delta} d\varkappa+\int_{\frac{3}{8}}^{1} \left(\varkappa-\tfrac{3}{8}\right)^{\delta} d\varkappa\right)^{\frac{1}{\delta}}\\
&\quad \times\left(\int_0^1 \left(\tfrac{3+2\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|^{\rho}+\tfrac{3-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|^{\rho}\right) d\varkappa\right)^{\frac{1}{\rho}}\\
&\quad +4\left(\int_0^{\frac{5}{8}} \left(\tfrac{5}{8}-\varkappa\right)^{\delta} d\varkappa+\int_{\frac{5}{8}}^{1} \left(\varkappa-\tfrac{5}{8}\right)^{\delta} d\varkappa\right)^{\frac{1}{\delta}}\\
&\quad \times\left(\int_0^1 \left(\tfrac{1+2\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|^{\rho}+\tfrac{5-2\varkappa}{6}\left|\mathcal{U}'(\tau)\right|^{\rho}\right) d\varkappa\right)^{\frac{1}{\rho}}\\
&\quad \left.+\left(\int_0^1 (1-\varkappa)^{\delta}\, d\varkappa\right)^{\frac{1}{\delta}}\left(\int_0^1 \left(\tfrac{1-\varkappa}{6}\left|\mathcal{U}'(\upsilon)\right|^{\rho}+\tfrac{5+\varkappa}{6}\left|\mathcal{U}'(\tau)\right|^{\rho}\right) d\varkappa\right)^{\frac{1}{\rho}}\right)\\
&= \frac{\varphi(\tau,\upsilon)}{36}\left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\left(\left(\frac{11|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{12}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+11|\mathcal{U}'(\tau)|^{\rho}}{12}\right)^{\frac{1}{\rho}}\right.\\
&\quad \left.+\frac{1}{2}\left(\frac{3^{\delta+1}+5^{\delta+1}}{8}\right)^{\frac{1}{\delta}}\left(\left(\frac{2|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{3}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+2|\mathcal{U}'(\tau)|^{\rho}}{3}\right)^{\frac{1}{\rho}}\right)\right).
\end{aligned}$$

The proof is completed. □

Corollary 3.5. *Taking* $\varphi(\tau,\upsilon)=\tau-\upsilon$ *in Theorem 5, then we get*

$$\begin{aligned}
&\left|\frac{3\mathcal{U}\left(\frac{5\upsilon+\tau}{6}\right)+2\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+3\mathcal{U}\left(\frac{\upsilon+5\tau}{6}\right)}{8}-\frac{1}{\tau-\upsilon}\int_{\upsilon}^{\tau}\mathcal{U}(r)\, dr\right|\\
&\leqslant \frac{\tau-\upsilon}{36}\left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\left(\left(\frac{11|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{12}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+11|\mathcal{U}'(\tau)|^{\rho}}{12}\right)^{\frac{1}{\rho}}\right.\\
&\quad \left.+\frac{1}{2}\left(\frac{3^{\delta+1}+5^{\delta+1}}{8}\right)^{\frac{1}{\delta}}\left(\left(\frac{2|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{3}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+2|\mathcal{U}'(\tau)|^{\rho}}{3}\right)^{\frac{1}{\rho}}\right)\right).
\end{aligned}$$

Corollary 3.6. *Under the assumptions of Theorem 3.4, we have*

$$\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\, dr\right|$$

$$\leqslant \frac{\varphi(\tau,\upsilon)}{18}\left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\left(1+\frac{1}{2}\left(\frac{3^{\delta+1}+5^{\delta+1}}{8}\right)^{\frac{1}{\delta}}\right)\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{2}\right)^{\frac{1}{\rho}}.$$

Proof. It suffices to apply the discrete power mean inequality to (3.12). ☐

Corollary 3.7. *Taking $\varphi(\tau,\upsilon)=\tau-\upsilon$ in Corollary 3, then we get*

$$\left|\frac{3\mathcal{U}\left(\frac{5\upsilon+\tau}{6}\right)+2\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+3\mathcal{U}\left(\frac{\upsilon+5\tau}{6}\right)}{8}-\frac{1}{\tau-\upsilon}\int_{\upsilon}^{\tau}\mathcal{U}(r)\,dr\right|$$
$$\leqslant \frac{\tau-\upsilon}{18}\left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\left(1+\frac{1}{2}\left(\frac{3^{\delta+1}+5^{\delta+1}}{8}\right)^{\frac{1}{\delta}}\right)\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{2}\right)^{\frac{1}{\rho}}.$$

Theorem 3.8. *Let $\mathcal{U}:[\upsilon,\upsilon+\varphi(\tau,\upsilon)]\to\mathbb{R}$ be a differentiable map on $(\upsilon,\upsilon+\varphi(\tau,\upsilon))$ such that $\mathcal{U}'\in L^{1}[\upsilon,\upsilon+\varphi(\tau,\upsilon)]$ with $\varphi(\tau,\upsilon)>0$. If $|\mathcal{U}'|^{\rho}$ is preinvex where $\rho\geqslant 1$, then we have*

$$\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right|$$
$$\leqslant \frac{\varphi(\tau,\upsilon)}{36}\left(\frac{1}{2}\left(\left(\frac{8|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{9}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+8|\mathcal{U}'(\tau)|^{\rho}}{9}\right)^{\frac{1}{\rho}}\right)\right.$$
$$\left.+\frac{17}{16}\left(\left(\frac{1726|\mathcal{U}'(\upsilon)|^{\rho}+722|\mathcal{U}'(\tau)|^{\rho}}{2448}\right)^{\frac{1}{\rho}}+\left(\frac{722|\mathcal{U}'(\upsilon)|^{\rho}+1726|\mathcal{U}'(\tau)|^{\rho}}{2448}\right)^{\frac{1}{\rho}}\right)\right). \tag{3.13}$$

Proof. Applying the absolute value in both sides of (3.1), using power mean inequality and the preinvexity of $|\mathcal{U}'|^{\rho}$, we have

$$\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right|$$
$$\leqslant \frac{\varphi(\tau,\upsilon)}{36}\left(\left(\int_{0}^{1}(1-\varkappa)\,d\varkappa\right)^{1-\frac{1}{\rho}}\left(\int_{0}^{1}(1-\varkappa)\left|\mathcal{U}'\left(\upsilon+\tfrac{1-\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}\right.$$
$$+\left(\int_{0}^{1}\left|\tfrac{3}{2}-4\varkappa\right|d\varkappa\right)^{1-\frac{1}{\rho}}\left(\int_{0}^{1}\left|\tfrac{3}{2}-4\varkappa\right|\left|\mathcal{U}'\left(\upsilon+\tfrac{3-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}$$
$$+\left(\int_{0}^{1}\left|\tfrac{5}{2}-4\varkappa\right|d\varkappa\right)^{1-\frac{1}{\rho}}\left(\int_{0}^{1}\left|\tfrac{5}{2}-4\varkappa\right|\left|\mathcal{U}'\left(\upsilon+\tfrac{5-2\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}$$
$$\left.+\left(\int_{0}^{1}(1-\varkappa)\,d\varkappa\right)^{1-\frac{1}{\rho}}\left(\int_{0}^{1}(1-\varkappa)\left|\mathcal{U}'\left(\upsilon+\tfrac{5+\varkappa}{6}\varphi(\tau,\upsilon)\right)\right|^{\rho}d\varkappa\right)^{\frac{1}{\rho}}\right)$$

$$\begin{aligned}
&\leqslant \frac{\varphi(\tau,\upsilon)}{36}\left(\left(\tfrac{1}{2}\right)^{1-\frac{1}{\rho}}\left(\left|\mathcal{U}'(\upsilon)\right|^{\rho}\int_{0}^{1}(1-\varkappa)\left(\tfrac{5+\varkappa}{6}\right)d\varkappa+\left|\mathcal{U}'(\tau)\right|^{\rho}\int_{0}^{1}(1-\varkappa)\left(\tfrac{1-\varkappa}{6}\right)d\varkappa\right)^{\frac{1}{\rho}}\right.\\
&\quad+\left(\int_{0}^{1}\left|\tfrac{3}{2}-4\varkappa\right|d\varkappa\right)^{1-\frac{1}{\rho}}\\
&\quad\times\left(\left|\mathcal{U}'(\upsilon)\right|^{\rho}\left(\int_{0}^{\frac{3}{8}}\left(\tfrac{3}{2}-4\varkappa\right)\left(\tfrac{3+2\varkappa}{6}\right)d\varkappa+\int_{\frac{3}{8}}^{1}\left(4\varkappa-\tfrac{3}{2}\right)\left(\tfrac{3+2\varkappa}{6}\right)d\varkappa\right)\right.\\
&\quad\left.+\left|\mathcal{U}'(\tau)\right|^{\rho}\left(\int_{0}^{\frac{3}{8}}\left(\tfrac{3}{2}-4\varkappa\right)\left(\tfrac{3-2\varkappa}{6}\right)d\varkappa+\int_{\frac{3}{8}}^{1}\left(4\varkappa-\tfrac{3}{2}\right)\left(\tfrac{3-2\varkappa}{6}\right)d\varkappa\right)\right)^{\frac{1}{\rho}}\\
&\quad+\left(\int_{0}^{1}\left|\tfrac{5}{2}-4\varkappa\right|d\varkappa\right)^{1-\frac{1}{\rho}}\\
&\quad\times\left(\left|\mathcal{U}'(\upsilon)\right|^{\rho}\left(\int_{0}^{\frac{5}{8}}\left(\tfrac{5}{2}-4\varkappa\right)\left(\tfrac{1+2\varkappa}{6}\right)d\varkappa+\int_{\frac{5}{8}}^{1}\left(4\varkappa-\tfrac{5}{2}\right)\left(\tfrac{1+2\varkappa}{6}\right)d\varkappa\right)\right.\\
&\quad\left.+\left|\mathcal{U}'(\tau)\right|^{\rho}\left(\int_{0}^{\frac{5}{8}}\left(\tfrac{5}{2}-4\varkappa\right)\left(\tfrac{5-2\varkappa}{6}\right)d\varkappa+\int_{\frac{5}{8}}^{1}\left(4\varkappa-\tfrac{5}{2}\right)\left(\tfrac{5-2\varkappa}{6}\right)d\varkappa\right)\right)^{\frac{1}{\rho}}\\
&\quad\left.+\left(\tfrac{1}{2}\right)^{1-\frac{1}{\rho}}\left(\left|\mathcal{U}'(\upsilon)\right|^{\rho}\int_{0}^{1}(1-\varkappa)\left(\tfrac{1-\varkappa}{6}\right)d\varkappa+\left|\mathcal{U}'(\tau)\right|^{\rho}\int_{0}^{1}(1-\varkappa)\left(\tfrac{5+\varkappa}{6}\right)d\varkappa\right)^{\frac{1}{\rho}}\right)\\
&= \frac{\varphi(\tau,\upsilon)}{36}\left(\frac{1}{2}\left(\left(\frac{8|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{9}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+8|\mathcal{U}'(\tau)|^{\rho}}{9}\right)^{\frac{1}{\rho}}\right)\right.\\
&\quad\left.+\frac{17}{16}\left(\left(\frac{1726|\mathcal{U}'(\upsilon)|^{\rho}+722|\mathcal{U}'(\tau)|^{\rho}}{2448}\right)^{\frac{1}{\rho}}+\left(\frac{722|\mathcal{U}'(\upsilon)|^{\rho}+1726|\mathcal{U}'(\tau)|^{\rho}}{2448}\right)^{\frac{1}{\rho}}\right)\right),
\end{aligned}$$

where we have used (3.6)-(3.11) and

$$\int_{0}^{1}\left|\tfrac{3}{2}-4\varkappa\right|d\varkappa=\int_{0}^{1}\left|\tfrac{5}{2}-4\varkappa\right|d\varkappa=\tfrac{17}{16}.$$

The proof is completed. □

Corollary 3.9. *Taking* $\varphi(\tau,\upsilon)=\tau-\upsilon$ *in Theorem 6, then we get*

$$\begin{aligned}&\left|\frac{3\mathcal{U}\left(\frac{5\upsilon+\tau}{6}\right)+2\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+3\mathcal{U}\left(\frac{\upsilon+5\tau}{6}\right)}{8}-\frac{1}{\tau-\upsilon}\int\limits_{\upsilon}^{\tau}\mathcal{U}(r)\,dr\right|\\ \leqslant\ &\frac{\tau-\upsilon}{36}\left(\frac{1}{2}\left(\left(\frac{8|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{9}\right)^{\frac{1}{\rho}}+\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+8|\mathcal{U}'(\tau)|^{\rho}}{9}\right)^{\frac{1}{\rho}}\right)\right.\\ &\left.+\frac{17}{16}\left(\left(\frac{1726|\mathcal{U}'(\upsilon)|^{\rho}+722|\mathcal{U}'(\tau)|^{\rho}}{2448}\right)^{\frac{1}{\rho}}+\left(\frac{722|\mathcal{U}'(\upsilon)|^{\rho}+1726|\mathcal{U}'(\tau)|^{\rho}}{2448}\right)^{\frac{1}{\rho}}\right)\right).\end{aligned}$$

Corollary 3.10. *Under the assumptions of Theorem 3.8, we have*

$$\begin{aligned}&\left|\frac{3\mathcal{U}\left(\frac{6\upsilon+\varphi(\tau,\upsilon)}{6}\right)+2\mathcal{U}\left(\frac{2\upsilon+\varphi(\tau,\upsilon)}{2}\right)+3\mathcal{U}\left(\frac{6\upsilon+5\varphi(\tau,\upsilon)}{6}\right)}{8}-\frac{1}{\varphi(\tau,\upsilon)}\int\limits_{\upsilon}^{\upsilon+\varphi(\tau,\upsilon)}\mathcal{U}(r)\,dr\right|\\ \leqslant\ &\frac{25\varphi(\tau,\upsilon)}{288}\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{2}\right)^{\frac{1}{\rho}}.\end{aligned}$$

Proof. It suffices to apply the discrete power mean inequality to (3.13). ☐

Corollary 3.11. *Taking* $\varphi(\tau,\upsilon)=\tau-\upsilon$ *in Corollary 6, then we get*

$$\left|\frac{3\mathcal{U}\left(\frac{5\upsilon+\tau}{6}\right)+2\mathcal{U}\left(\frac{\upsilon+\tau}{2}\right)+3\mathcal{U}\left(\frac{\upsilon+5\tau}{6}\right)}{8}-\frac{1}{\tau-\upsilon}\int\limits_{\upsilon}^{\tau}\mathcal{U}(r)\,dr\right|\leqslant\frac{25(\tau-\upsilon)}{288}\left(\frac{|\mathcal{U}'(\upsilon)|^{\rho}+|\mathcal{U}'(\tau)|^{\rho}}{2}\right)^{\frac{1}{\rho}}.$$

4. Applications

For arbitrary real numbers k,r we have:
The Arithmetic mean: $A(k,r)=\frac{a+b}{2}$.

The p-Logarithmic mean: $L_p(k,r)=\left(\frac{r^{p+1}-k^{p+1}}{(p+1)(r-k)}\right)^{\frac{1}{p}}$, $k,r>0, k\neq r$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 4.1. *Let* $k,r\in\mathbb{R}$ *with* $0<k<r$, *then we have*

$$\begin{aligned}&\left|3A\left(A^{j}\left(2k,\tfrac{1}{3}A(k,r)\right),A^{j}\left(2k,\tfrac{5}{3}A(k,r)\right)\right)+A^{j}\left(2k,A(k,r)\right)\right.\\ &\left.-4L_{j}^{j}\left(k,k+A(k,r)\right)\right|\\ \leqslant\ &\frac{25A(k,r)}{144}\left(jk^{j-1}+jr^{j-1}\right).\end{aligned}$$

Proof. The assertion follows from Theorem 3.2 with $\varphi(r,k)=A(k,r)$, applied to the function $\mathcal{U}(r)=r^{j}$ with $j\geqslant 3$. ☐

Proposition 4.2. *Let* $k,r\in\mathbb{R}$ *with* $0<k<r$, $\rho>1$, *and* $j\geqslant 2\rho$, *then we have*

$$\begin{aligned}&\left|3A^{\frac{j}{\rho}+1}\left(\left(\frac{5k+r}{6}\right),\left(\frac{k+5r}{6}\right)\right)+A^{\frac{j}{\rho}+1}(k,r)-4L_{\frac{j}{\rho}+1}^{\frac{j}{\rho}+1}(k,r)\right|\\ \leqslant\ &\frac{25(j+\rho)(r-k)}{72\rho}\left(\frac{k^{j}+r^{j}}{2}\right)^{\frac{1}{\rho}}.\end{aligned}$$

Proof. The assertion follows from Corollary 3.10, applied to the function $\mathcal{U}(r)=r^{\frac{j}{\rho}+1}$. ☐

5. Conclusion

The main results of the paper can be summarized as follows:

1. A new three-point identity is proven
2. Some new Maclaurin-type inequalities for functions whose first derivatives are preinvex are discussed.
3. Some special cases are explained.
4. Applications involving arithmetic and p-logarithmic mean are given

The results obtained in this paper can stimulate other research in this interesting field, as well as generalizations in different other type of calculations as fractional calculus and quantum calculus.

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