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D-precompact Sets in D-Metric Spaces

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Abstract

The aim of this paper is to define and emphasize a strong form of D-compact sets in generalized metric spaces, namely D-precompact sets. Also with other sets, we shall study the relationships. Furthermore, we give the notions of sequentially D-precompact sets.

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1. Introduction

The metric space is the fundamental concept in functional analysis and topology. Dhage, [1], introduced some results in D–metric spaces obtained. In addition, the notions of D-open and D-closed balls are introduced. For some fixed point results applied in D-metric spaces, see [2, 3]. In, [4], Al-shami introduced the notions on somewhere dense sets and ST_1−spaces. The class of somewhere dense sets contains all preopen, regular open, semi open, α-open, β-open, b-open and β-open sets with the exception of the empty set. Al-shami and Noiri, [5], applied to define new types of continuous maps. Hussain and Saif, [6], introduced the class of D-precontinuous functions as a weak form of the class of D-continuous functions in D-metric spaces. Recently, Al-shami, [7], introduced a topological method to produce new rough set models. This method is based on the idea of somewhat open sets which is one of the celebrated generalizations of open sets.

The remainder of this manuscript is organized as follows: In section 2, we recall some definitions and properties of D–metric spaces that help the reader to well understand this manuscript. In section 3, we introduce the notion of D-precompact sets via utilizing the D-preopen sets. Section 4 introduces the notions of sequentially D-precompact sets. Finally, we give some conclusions and make a plan for future works in Section 5.

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2. Preliminaries

Definition 2.1. [8]. A nonempty set X, together with a function $D : X \times X \times X \to [0, \infty)$ is said to be a D-metric space, denoted by (X, D), if for $x, y, z, u \in X$ D satisfies the following $x, y, z, u \in X$:

1. $D(x, y, z) = 0 \Longrightarrow x = y = z$ (coincidence);
2. $D(x, y, z) = D(p(x, y, z))$, where p is a permutation of x, y, z (symmetry);
3. $D(x, y, z) \leqslant D(x, y, u) + D(x, u, z) + D(u, y, z)$ (tetrahedral inequality).

$O^D_\delta(y)$ denotes the D-open ball with radius $\delta > 0$ and center y,

$$O^D_\delta(y) = \{x \in X : D(y, x, x) < \delta\}.$$

$C^D_\delta(y)$ denotes the D-closed ball with radius $\delta > 0$ and center y,

$$C^D_\delta(y) = \{x \in X : D(y, x, x) \leqslant \delta\}.$$

A subset M of (X, D) is said to be a D-open set if for every $y \in M$, there exist $\delta > 0$ where $O^D_\delta(y) \subseteq M$. The set M is said to be a D-closed set if $X - M$ is a D-open set.

Theorem 2.2. [1]. Every D-ball $O^D_\delta(y)$, $y \in X$, $\delta > 0$ is a D-open set in X.

A subset M of (X, D) is said to be a D-preopen set, [9] if for every $y \in M$, there exist $\varepsilon > 0$ where for every $x \in O^D_\varepsilon(y)$, $O^D_\delta(x) \cap M \neq \emptyset$ for every $\delta > 0$. The set $M \subseteq X$ is said to be a D-preclosed set if $X - M$ is a D-preopen set.

Theorem 2.3. [9]. A subset M of (X, D) is a D-preclosed set if and only if $Cl^D_p(M) = M$.

Definition 2.4. [8]. Let (X, D) be D–metric space. A subset M of X is called bounded if these exists constant $K > 0$ such that, $D(x, y, z) \leqslant K$, for all $x, y, z \in M$ and N is aid to be D–bound of M.

Theorem 2.5. [10]. The arbitrary unioin of D-open sets is D-open set.

Definition 2.6. [11]. A sequence $\{x_m\}$ in (X, D) is called D-convergent if there exists an element $y \in X$ such that $\lim D(x_m, x_n, y) = 0$ as $m, n \to \infty$, i.e. for any $\delta > 0$, there exists $I \in N$ such that $D(x_n, x_m, y) < \delta$ for all $m, n \geqslant I$. Write $\{x_m\} \longrightarrow y$ and here $\{x_m\}$ is called convergent to y and y is a limit point of $\{x_m\}$.

Definition 2.7. [11]. A sequence $\{x_m\}$ in (X, D) is called Cauchy (or D-Cauchy) if for any $\delta > 0$ there exists $I \in N$ where $D(x_m, x_n, x_k) \leqslant \delta$ for all $m, n, k \geqslant I$.

Definition 2.8. [12]. A D-metric space (X, D) is said to be complete (or D-complete) if every D-Cauchy sequence in X is D-convergent in X.

Remark 2.9. [10]. Let (X, D) be D–metric space. A subset B of X and if $x \in B$, then $\{x_m\} \longrightarrow x$, where $\{x_m\} = x$ for $m \in N$, because $\lim D(x_m, x_n, x) = 0$ as $m, n \to \infty$, Thus $B \subseteq cl_D(B)$.

Theorem 2.10. [1]. If $\{x_m\}$ a D-Cauchy in (X, D) contains a D-convergent subseq. $\{x_{m_k}\}$ as $k \to \infty$ then the $\{x_m\}$ is D-convergent.

Definition 2.11. [1]. Let $\delta > 0$. A finite subset M of (X, D) is called an δ-net for X if for every $x \in X$, there is $x \in M$ where $x \in O_\delta^D(x)$. That is, M is an δ-net for X if M is finite and $X = \bigcup\{O_\delta^D(x) : x \in M\}$. An (X, D) is called totally D-bounded if X has an δ-net for every $\delta > 0$.

Recall [1] that if every open cover of X has a finite subcover then we say that X is D-compact space.

Theorem 2.12. [1]. Every sequentially D-compact is totally D-bounded.

Theorem 2.13. [9]. For $M, G \subseteq X$ in (X, D), the following hold:

1. If $M \subseteq G$ then $Cl_p^D(M) \subseteq Cl_p^D(G)$
2. $Cl_p^D(M \cap G) \subseteq Cl_p^D(M) \cap Cl_p^D(G)$.
3. $Cl_p^D(M) \cup Cl_p^D(G) \subseteq Cl_p^D(M \cup G)$.
4. $Cl_p^D(M) \subseteq Cl_D(M)$.

Theorem 2.14. [1]. In (X, D) , the following statement are equivalent.

1. X is D-compact space,
2. X is countably D-compact space,
3. X is sequentially D-compact space.

Theorem 2.15. [13]. For $M \subseteq X$ in (X, D), the following hold:

1. If M is a D-compact set then M is a D-closed set and D-bounded set,
2. (X, D) is a D-compact if and only if it is totally D-bounded and complete,
3. a M of a complete (X, D) is D-compact if and only if is totally D-bounded set and D-closed set.

3. D-precompact sets

Definition 3.1. Let (X, D) be a D-metric space and $A \subseteq X$. A is called a D-precompact set in (X, D) if for every D-preopen cover $\{G_\lambda : \lambda \in I\}$ of A has finite D-preopen subcover $\{G_{\lambda_k} : k = 1, 2, ..., n\}$ of A such that $A \subseteq \cup_{k=1}^n G_{\lambda_k}$.
Similar, X is called a D-precompact space if $X = \cup_{k=1}^n G_{\lambda_k}$.

Example 3.2. Let (X, D) be D-metric space with any nonempty finite set X, where

$$D(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\},$$

where (X, d) is any metric space. Since $\{\{x\} : x \in X\}$ is D-preopen cover of X. Any subset of X has finite D-preopen subcover $\{\{x\} : x \in X\}$. Note that any subset of X is D-precompact set.

Theorem 3.3. Every D-precompact set is D-compact set.

Proof. The proof of the theorem is cle. □

Theorem above converse need not be true.

Example 3.4. Let $(\mathbb{R}, D)$ be D-metric space given by

$$D(x, y, z) = 0.$$

Note that $\{\{x\} : x \in \mathbb{R}\}$ is D-preopen cover of $\mathbb{R}$ which has no finite subcover. So $\mathbb{R}$ is not D-precompact space but D-compact space.

4. Sequentially D-precompact sets

Definition 4.1. An (X, D) is called sequentially D-precompact if every sequence has in X a D-convergent subsequence.

Definition 4.2. An (X, D) is said to have the finite intersection property for D-preclosed sets if every decreasing sequence of nonempty D-preclosed sets has a nonempty intersection.

We mean by F dense in E that $Cl_P^D F = E$.

Theorem 4.3. An (X, D) is sequentially D-precompact if and only if it has the finite intersection property for D-preclosed sets.

Proof. Suppose that X is sequentially D-precompact space. Given a decreasing sequence of nonempty D-prclosed sets $\{F_n\}$, choose $x_n \in F_n$ for each $n \in N$. Then (x_n) has a D-convergent subsequence (x_{n_k}) with $x_{n_k} \to x$ as $k \to \infty$. Since $x_{n_k} \in F_n$ for all $n_k \geqslant n$, F_n is D-preclosed sets and $x \in F_n$ for every $n \in N$ then $x \in \cap_{n=1}^{\infty} F_n$, that is, and $\cap_{n=1}^{\infty} F_n \neq \emptyset$. Conversely, suppose that X has the finite intersection property. Let (x_n) be a sequence in X and define

$$F_n = Cl_P^D T_n, \quad T_n = \{x_k : k > n\}.$$

Then (F_n) is a decreasing sequence of non-empty D-preclosed sets, so there exists

$$x \in \cap_{n=1}^{\infty} F_n.$$

Choose a subsequence (x_{n_k}) of (x_n) as follows. For $k = 1$, there exists $(x_{n_1}) \in T_1$ such that $D(x_{n_1}.x_{n_2}, x) < 1$, since $x \in F_1$ and T_1 is dense in F_1. Similarly, since $x \in F_{n_1}$ and T_{n_1} is dense in F_{n_1}, there exists $x_{n_2} \in T_{n_1}$ with $n_2 > n_1$ such that $D(x_{n_1}.x_{n_2}, x) < \frac{1}{2}$. Continuing in this way (or by induction), given x_{n_k} we choose $x_{n_{k+1}} \in T_{n_k}$, where $n_{k+1} > n_k$, such that $D(x_{n_{k+1}}, x_{n_k}, x) < 1/(k+1)$. Then $x_{n_k} \to x$ as $k \to \infty$, that is, X is sequentially D-precompact space. □

Theorem 4.4. Every sequentially D-precompact in (X, D) is totally D-bounded.

Proof. by contradiction, say (X, D) is not totally D-bounded. Then $\exists\, \delta > 0$ where X has no $\delta-$net. Let $y_1 \in X$. $\exists$ must exists the points $y_1, y_2, y_3 \in X$ not necessary distinct, such that $D(y_1, y_2, y_3) \geqslant \delta$, for otherwise, $\{y_1\}$, would be an $\delta-$net for X. Again $\exists$ a point $y_4 \in X$ such that $D(y_2, y_3, y_4) \geqslant \delta$, for otherwise $\{y_1, y_2\}$ would be an $\delta-$net for X. Continuing this process, we get a sequence $\{y_1, y_2, ...\}$ having the property that $D(y_i, y_j, y_k) \geqslant \delta$, $i \neq j$ or $j \neq k$ or $k \neq i$. That the sequence $\{y_n\}$ cannot contain any D-convergent subsequence. Hence (X, D) is not sequentially D-precompact space. □

Lemma 4.5. A subset Y of (X, D) is totally D-bounded if and only if every sequence in Y contains a D-Cauchy subseq.

Theorem 4.6. Let (X, D) be a D-precompact. Then (X, D) is complete.

Proof. Let (X, D) be a D-precompact by contradiction, say (X, D) is not complete. So we found a D-Cauchy sequence $\{y_m\}_{m\geqslant 1}$ in (X, D) does not have a limit in X. Let $x \in X$; since $\{y_m\}_{m\geqslant 1}$ does not D-converge to x, $\exists$ an $\delta_0 > 0$ such that

$$D(y_m, y_n, x) \geqslant \delta_0. \tag{4.1}$$

Since $\{y_m\}_{m\geqslant 1}$ is D-Cauchy, $\exists$ an integer n_0 such that $m, n, p \geqslant n_0$ then

$$D(y_m, y_n, y_p) \geqslant 3\delta_0$$

select $n, k > n_0$ for which $D(y_k, y_n, y) \geqslant \delta_0$ (by inequality (4.1)). Then

$$D(y_k, y_n, y_p) \leqslant D(y_k, y_n, x) + D(y_k, x, y_p) + D(x, y_n, y_p)$$

then

$$\begin{aligned} D(y_k, y_n, x) &\geqslant D(y_k, y_n, y_p) - D(y_p, y_m, x) - D(x, y_n, y_p) \\ &> 3\delta_0 - \delta_0 - \delta_0 = \delta_0 \end{aligned}$$

for all $n, k \geqslant n_0$. So, the D-open ball $O^D_{\delta_0}(x)$ contains y_n. In this way, we can assign with each $x \in X$ a D-ball $O^D_{\delta_0(x)}(x)$, where $\delta_0(x)$ is a positive number that depends on y, and the D-ball $O^D_{\delta_0(x)}(x)$ contains y_m . Observe that

$$X = \cup\{O^D_{\delta_0(x)}(x) : x \in X\}$$

Since X is D-precompact, there is $O^D_{\delta_0(x_j)}(x_j)$, $j = 1, 2, ..., n$, of X. So

$$X = \cup_{j=1}^{n} O^D_{\delta_0(x_j)}(x_j).$$

Hence, (X, D) be complete. □

Theorem 4.7. Let (X, D) be complete a totally D-bounded. Then (X, D) is D-precompact.

Proof. Let (X, D) be complete totally D-bounded and not D-precompact. Then $\exists$ a D-preopen covering $\{G_\lambda : \lambda \in I\}$ of X that does not admit a finite. Since (X, D) is totally D-bounded, then for some $\delta > 0$ and some $y_0 \in X$, we have $O^D_\delta(y_0) \subseteq X$. Observe that $X \subseteq O^D_\delta(y_0)$ then $X = O^D_\delta(y_0)$. Let $\varepsilon_n = \frac{\delta}{2^n}$. Since X is totally D-bounded, it can be covered by a finite number of D-balls with radius ε_1 because there is $O^D_{\varepsilon_1}(y_1)$ which cannot be covered by a finite number of sets G_λ. In this way, a sequence $\{y_n\}_{n\geqslant 1}$ $O^D_{\varepsilon_n}(y_n)$ cannot be covered by a finite number of sets G_λ and $y_{n+1} \in O^D_{\varepsilon_n}(y_n)$. We to show that the sequence $\{y_n\}_{n\geqslant 1}$ is D-convergent. Since $y_{n+1} \in O^D_{\varepsilon_n}(y_n)$, it follows that $D(y_n, y_{n+1}y_{n+2}) < \varepsilon_n$ and hence,

$$D(y_n, y_{n+1}, y_{n+p}) \leqslant D(y_n, y_{n+1}, y_{n+2}) + D(y_{n+1}, y_{n+2}, y_{n+3}) + ...$$

$$+D(y_{n+p-2}, y_{n+p-1}, y_{n+p}) < \varepsilon_n + \varepsilon_{n+1} + ...\varepsilon_{n+p-2} < \frac{\delta}{2^{n-2}}$$

So $\{y_n\}_{n\geqslant 1}$ is a D-Cauchy and since X is complete, it D-converges to $x \in X$. Since $x \in X$, there is $\lambda_0 \in I$ such that $y \in G_{\lambda_0}$. Because G_{λ_0} is D-open(D-preopen), it contains $O^D_\delta(x)$ for some $\delta > 0$. Select n so large that $D(y_n, y_n, x) < \frac{r}{3}$ and $\varepsilon_n < \frac{r}{3}$. Then, for any $y \in X$ such that $D(y, y, y_n) < \varepsilon_n$, it follows that

$$D(y, y, x) \leqslant D(y, y, y_n) + D(y, y_n, x) + D(y_n, y, x)$$

$$< \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r$$

so that $O^D_{\varepsilon_n}(y_n) \subseteq O^D_r(x)$. Therefore, $O^D_{\varepsilon_n}(y_n)$ has a finite subcover and this is contradiction. □

Theorem 4.8. A D-metric space is D-precompact if and only if it is complete and totally D-bounded.

Lemma 4.9. Every infinite set in (X, D) has at least one limit point in X if and only if every infinite sequence in (X, D) contains a D-convergent subseq.

Proof. Let the first case be hold. Let $\{y_n\}_{n\geqslant 1}$ be in X. If the set $\{y_1, y_2, y_3, ...\}$ is finite, then there is y_{j_0} where $y_{j_0} = y_i$ for $i \in N$. Hence $\{y_{j_0}\}$ is a subsequence of $\{y_n\}_{n\geqslant 1}$. Assume that the set $\{y_1, y_2, y_3, ...\}$ is infinite. Hence there is a limit point of $\{y_1, y_2, y_3, ...\}$, say $y \in X$. Let n_1 and n_2 are belong to numbers integer such that $D(y_{n_1}, y_{n_2}, y) < 1$. Having defined n_m, let n_{m+1} and n_{m+2} be the smallest integer such that n_{m+2}, $n_{m+1} > n_m$ and $D(y_{n_{m+1}}, y_{n_{m+2}}, y) < \frac{1}{m+1}$. Then the sequence $\{x_{n_m}\}_{m\geqslant 1}$ D-converges to y. Let the second case be hold. Let $Y \subset X$ infinite. Then $\exists$ a sequence $\{x_n\}_{n\geqslant 1}$ in X of distinct terms. As (2), $\{x_n\}_{n\geqslant 1}$ contains a subsequence $\{x_{n_j}\}_{j\geqslant 1}$ that D-converges to $x \in X$. Hence every D-open ball with centre x contains an infinite number of the D-convergent subsequence $\{x_{n_j}\}_{j\geqslant 1}$. □

Theorem 4.10. An (X, D) is D-precompact if and only if every sequence has a converted subseq. in X.

Proof. Let X is D-precompact and $\{y_n\}_{n\geqslant 1}$ be any sequence in X. Since X is totally D-bounded, using Lemma (4.5), then $\{y_n\}_{n\geqslant 1}$ contains $\{y_{n_j}\}_{j\geqslant 1}$. But $\{y_{n_j}\}_{j\geqslant 1}$ is D-converges to $y \in X$ because X is complete.
Conversely, it depends on Lemma (4.5) that X is totally D-bounded. By assumption, $\{y_n\}_{n\geqslant 1}$ has subsequence $\{y_{n_j}\}_{j\geqslant 1}$ that D-converges to $y \in X$. We shall show that $\lim_n y_n = y$. Let $\delta > 0$. Since $\lim_j y_{n_j} = y$, there is j_0 where $j \geqslant j_0$ then

$$D(y_{n_j}, y_{n_{j-1}}, y) < \frac{1}{3}\delta \tag{4.2}$$

Since $\{y_n\}_{n\geqslant 1}$ is D-Cauchy, there is n_0 where $n, m, p \geqslant n_0$ then

$$D(y_n, y_m, y_p) < \frac{1}{3}\delta \tag{4.3}$$

If $j \geqslant j_0$ and $n_j \geqslant n_0$, then using (4.2) and (4.3), we have

$$D(y_n, y_m, y) \leqslant D(y_n, y_m, y_{n_j}) + D(y_{n_j}, y_m, y) + D(y_n, y_{n_j}, y) < \frac{1}{3}\delta + \frac{1}{3}\delta + \frac{1}{3}\delta = \delta$$

whenever $n, m \geqslant n_0$. □

Theorem 4.11. Let (X, D) be a D-metric space. The following statements are equivalent:

1. (X, D) is D-precompact;
2. (X, D) is totally D-bounded complete;
3. every infinite set in X has at least one limit point;
4. every $\{x_n\}$ in X has a D-convergent subseq.

Let X be a D-preclosed subset of the D-precompact (Y, D). Then (X, D_X) is D-precompact.

Proof. Let $\{x_n\}_{n\geqslant 1}$ be in X.Then $\{x_n\}_{n\geqslant 1}$ also in (Y, D) has a subseq. D-converging to $y \in Y$ by Theorem (4.11). But then $y \in X$ since X is D-preclosed, by Theorem (2.13). Thus by Theorem (4.11), X is D-precompact. □

Corollary 4.12. Let X be a subset of (Y, D). If (X, D_X) is D-precompact, then X is a D-preclosed subset of (Y, D).

5. Conclusion

We introduce the notions of sequentially D-precompact sets and the relationships among this form with totally D-bounded sets, and complete and D-preclosed sets. In an upcoming paper, we plan to use the idea of D-precompact sets to study new types of locally D-precompact sets and countably D-precompact sets in D-metric spaces.

References

[1] Dhage BC (2000). *Generalized Metric Space and Topological Structure I.* An. gtiint. Univ. A1.I. Cuza Iasi. Mat(N.S), **47**: 3-24.

[2] Mlaiki N, Rizk D and Azmi F (2021). *Fixed points of* $(\psi, \phi)-$ *contractions and Fredholm type integral equation*. Journal of Mathematical Analysis and Modeling, **2**(1): 91-100.https://doi.org/10.48185/jmam.v2i1.194.

[3] Afshari H and Aleomraninejad SMA (2021). *Some fixed point results of F-contraction mapping in D-metric spaces by Samet's method*. Journal of Mathematical Analysis and Modeling, **2**(3): 1-8.https://doi.org/10.48185/jmam.v2i3.299.

[4] Al-shami TM (2017). *Somewhere dense sets and* ST_1-*spaces*. Punjab Univ. J. Math. (Lahore), **49**(2): 101–111.

[5] Al-shami TM and Noiri T (2019). *More notions and mappings via somewhere dense sets*. Afrika Matematika, **30**(7): 1011–1024. https://doi.org/10.1007/s13370-019-00700-4.

[6] Hussain W and Saif A (2021). *Continuous Functions via D-preopen Sets in D-Metric Spaces*. International Journal of Advances in Applied Mathematics and Mechanics, **8**3: 1-9.

[7] Al-shami TM (2022). *Topological approach to generate new rough set models*. Complex & Intelligent Systems, **8**: 4101–4113. https://doi.org/10.1007/s40747-022-00704-x.

[8] Bele C and Dolhare U (2017). *An Extension of Common Fixed Point Theorem in D-Metric Space*. International Journal of Mathematics and its Applications, **5**: 13-18.

[9] Hussain W and Saif A (2021). *On D-preopen Sets in D-Metric Spaces*. International Journal of Computer Applications, **174**: 1-5.

[10] Fora A, Massa'deh M and Bataineh M (2017). *A New Structure and Contribution in D-Metric Spaces*. British Journal of Mathematics and Computer Science, **22**(1): 1-9. DOI:10.9734/BJMCS/2017/32791.

[11] Dhage BC, Pathan A and Rhoades B (2000). *A General Existence Principle for Fixed Point Theorems in D-Metric Spaces*. Int. J. Math. Math. Sci., **23**: 441-448. https://doi.org/10.1155/S0161171200001587.

[12] Dhage BC (1992). *Generalized Metric Space and Mapping With Fixed Point*. Bull. Cal. Math. Soc., **84**: 329-336.

[13] Dhage BC (1994). *On Generalized Metric Spaces and Topological Structure 11*. Pure. Appl. Math. Sci., **40**: 37-41.