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## Uniqueness of continuous solution to $q$ -Hilfer fractional hybrid integro-difference equation of variable order

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• Received: 09 October 2021 • Accepted: 05 November 2021 • Published Online: 30 December 2021

### Abstract

In this paper, the authors introduced a novel definition based on Hilfer fractional derivative, which name  $q$ -Hilfer fractional derivative of variable order. And the uniqueness of solution to  $q$ -Hilfer fractional hybrid integro-difference equation of variable order of the form (1.1) with  $0 < \alpha(t) < 1$ ,  $0 \leq \beta \leq 1$ , and  $0 < q < 1$  is studied. Moreover, an example is provided to demonstrate the result.

Keywords:  $q$ -calculus, Hilfer fractional derivative, Hybrid integro-difference equation, Variable-order.  
2010 MSC: 26A33, 34A08, 34A38.

### 1. Introduction

Fractional calculus caught much attention towards mathematical worlds (see [1, 2, 3, 4, 5, 6, 7, 8, 13, 14, 15, 22]). In fact, fractional calculus is a branch of mathematical analysis, which separate itself from normal calculus, with non-integers order of derivatives and integrals as special characteristics. The development of fractional calculus started from the first-order derivative such that

$$\frac{d}{dt} f(t) = D^1 f(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}.$$

In this case, it is said that the discrete version of such operator is called  $h$ -derivative, which is

$$D_h f(t) = \frac{f(t+h) - f(t)}{h}.$$

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Fractional calculus is developed towards time, and various experts propose many definitions of fractional derivatives. The two famous senses that caught the most attention in the differential equation are Caputo fractional derivative and Riemann-Liouville fractional derivative. Subsequently, Hilfer developed the general definition of fractional derivative by interpolating such operators motivated by these two derivatives. Determine  $n - 1 < \alpha < n$  and  $\beta \in [0, 1]$ , the three visualizations of Caputo, Riemann-Liouville and Hilfer derivatives are given as follows. Firstly, The left Riemann-Liouville fractional derivative of order  $\alpha$  for the function  $f(t)$  is defined by

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t (t - s)^{n - \alpha - 1} f(s) ds.$$

Secondly, The left Caputo fractional derivative of order  $\alpha$  is defined by

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} f^{(n)}(s) ds.$$

Lastly, The Hilfer fractional derivative [18] is defined by

$${}_a D_t^{\alpha, \beta} f(t) = {}_a I_t^{\alpha(n - \beta)} D^n {}_a I_t^{(1 - \beta)(n - \alpha)} f(t).$$

As the consequences, these common definitions lead to further enormous generalization of fractional derivatives such as fractional derivatives of a function with respect to another function [10, 27], fractional proportional derivative, variable-order fractional derivatives [9, 21, 29], etc. Also, there are several methods used to illustrate the existence and uniqueness of solution such as Banach fixed point theorem, Schaefer fixed point theorem, Schauder fixed point theorem, etc. (see [16, 17, 26, 12, 31, 20])

In 1909, Jackson [19] introduced the new branch of calculus by defining  $q$ -derivative with  $0 < q < 1$  as

$$D_q f(t) = \frac{f(qt) - f(t)}{qt - t},$$

and  $q$ -integral operator such that

$$I_q f(t) = \int_0^t f(s) d_q s = (1 - q) \sum_{n=0}^{\infty} t q^n f(t q^n).$$

Moreover, the definition of  $q$ -derivative and  $q$ -integral is studied and gradually developed by many researchers (see [11, 23, 24, 28]). The definitions of  $q$ -derivative and  $q$ -integral are developed, which are based on the  $q$ -Riemann-Liouville fractional integral.

In this work, motivated by [17, 26, 29, 30, 21], and the Hilfer operator in [18], the authors will introduce a novel definition based on Hilfer fractional derivative, which name  $q$ -Hilfer fractional derivative of variable order. Also, the main purpose of this paper is to study  $q$ -Fractional Hybrid Integro-Difference Equation of Variable Order ( $q$ -FHIDEVO) of the form

$$\begin{aligned} {}_q D_t^{\alpha(t), \beta} [x(t) - f(t, x(t))] &= g(t, x(t), {}_q I_t^\beta x(t)), \quad t \in (0, T) \\ {}_q I_t^{1 - \gamma(t)} x(0) &= x_0, \quad {}_q I_t^{1 - \gamma(t)} f(0, x(0)) = f_0, \quad \gamma(t) = \alpha(t) + \beta - \alpha(t)\beta \end{aligned} \tag{1.1}$$

where  $0 < \alpha(t) < 1$ ,  $0 \leq \beta \leq 1$  and  $0 < q < 1$ . Our result illustrates the uniqueness of the solution.

This paper is constructed as follows. In section 2, the notation and concept of  $q$ -fractional calculus will be introduced. In section 3, the concept of variable order and essential conditions to display the uniqueness and stability of the solution to  $q$ -FHIDEVO will be displayed. In sections 4 and 5, the uniqueness of solution in subinterval and the uniqueness of continuous solution will be presented, respectively. Lastly, in section 6, the example will be illustrated.

## 2. Preliminaries and Framework

The preliminaries section will introduce the necessary definition of operator, space, and concept of  $q$ -difference equation.

**Definition 2.1.** [11] For any  $p \geq 1$ , the space  $L_q^p(a, b)$  is the space of the functions such that

$$\left( \int_a^b |f(t)|^p d_q t \right)^{\frac{1}{p}} < \infty$$

For  $p = 1$  it can be denoted the space as  $L_q(a, b)$ .

**Definition 2.2.** [11] For any  $p \in \mathbb{R}^+$ , the space  $\mathbb{L}_q^p[a, b]$  is the space of the functions on interval  $(a, b]$ . The space  $\mathbb{L}_q^p[a, b]$  is a Banach space with the supremum norm  $\|\cdot\|_p$  defined by

$$\|f\|_p = \sup_{t \in (a, b]} \left( \int_a^b |f(t)|^p d_q t \right)^{\frac{1}{p}} < \infty$$

For  $p = 1$  it can be denoted the space as  $\mathbb{L}_q(a, b)$ .

**Definition 2.3.** [28] Let,  $q \in (0, 1)$  and  $\alpha > 0$ , then the  $q$ -Riemann-Liouville fractional integral is defined as

$${}_q I_t^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s) d_qs.$$

Where

$$(n - m)^{(k)} = \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{i+k}}, \quad n \neq 0, \quad k \in \mathbb{R},$$

and

$$\Gamma_q(t) = \frac{(1 - q)^{(t-1)}}{(1 - q)^{t-1}}, \quad t \in \mathbb{R} - \{0, -1, -2, \dots\}$$

,where  $\Gamma_q(t + 1) = [t]_q \Gamma_q(t)$  with

$$[m]_q = \frac{1 - q^m}{1 - q}, \quad m \in \mathbb{R}.$$

Also, let  $\alpha, \beta \geq 0$  and  $f(t)$  is a function on  $[0, T]$ , then there are following properties

- (1)  ${}_q I_t^\alpha {}_q I_t^\beta f(t) = {}_q I_t^{\alpha+\beta} f(t)$
- (2)  ${}_q D_t^\alpha {}_q I_t^\alpha f(t) = f(t)$

**Definition 2.4.** [11] Let  $n - 1 < \alpha < n$ , the  $q$ -Riemann-Liouville fractional derivative of the function  $f(t)$  is defined by  ${}_q D_t^\alpha f(t) = D_q^n I_t^{n-\alpha} f(t)$

**Definition 2.5.** [11] Let  $n - 1 < \alpha < n$ , the  $q$ -Caputo fractional derivative of the function  $f(t)$  is defined by  ${}_q^C D_t^\alpha f(t) = I_t^{n-\alpha} D_q^n f(t)$

Motivated by definition 2.4 and definition 2.5, based on Hilfer fractional derivative, authors shall introduce the operator of the  $q$ -Hilfer fractional derivative as follows.

**Definition 2.6.** Let  $0 < \alpha < 1$ ,  $0 \leq \beta \leq 1$  and  $0 < q < 1$  then, the  $q$ -Hilfer fractional derivative of the function  $f(t)$  is defined by

$${}_q D_t^{\alpha,\beta} f(t) = {}_q I_t^{\beta(1-\alpha)} D_{q,q} I_t^{(1-\beta)(1-\alpha)} f(t) = {}_q I_t^{\gamma-\alpha} {}_q D_t^\gamma f(t), \gamma = \alpha + \beta - \alpha\beta$$

### 3. Variable approach and mild solution

**Definition 3.1.** [11] The space  $C_q^n[a, b]$  is a space of a continuous function on  $[a, b]$  such that  $D_q^{n-1} f(t) \in C[a, b]$ . Also,  $C_q^n[a, b]$  is a banach space with supremum norm  $\|\cdot\|$  such that

$$\|f\| = \sup_{t \in [a,b]} \sum_{i=0}^{n-1} |D_q^i f(t)| < \infty$$

For  $n = 1$  it can be noted the space as  $C[a, b]$ , and for  $q = 1$  as  $C^n[a, b]$ .

**Definition 3.2.** [11] Let  $AC_q[a, b]$  be a space of the absolutely continuous functions on  $[a, b]$ , then  $f \in AC_q[a, b]$  if and only if there exists an arbitrary constant  $\omega \in \mathbb{R}$  and the function  $\psi(t) \in \mathbb{L}_q^p[a, b]$  such that

$$f(t) = \omega + \int_a^b \psi(t) d_q s.$$

For  $q = 1$ , it can be noted the space as  $AC[a, b]$ .

**Definition 3.3.** [11] The space  $AC_q^{(n)}[a, b]$  is a space of function on  $[a, b]$  such that  $D_q^{n-1} f(t) \in AC_q[a, b]$ . For  $q = 1$ , it can be denoted as  $AC^{(n)}[a, b]$

**Theorem 3.4.** [11] Suppose  $n - 1 < \alpha < n$ ,  $f \in \mathbb{L}_q[0, T]$  with  ${}_q I_t^{n-\alpha} f(t) \in AC_q^{(n)}[0, T]$ , then

$${}_q I_t^\alpha {}_q D_t^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} {}_q I_t^{1+i-\alpha} f(0) \frac{t^{\alpha-i-1}}{\Gamma_q(\alpha-i)}$$

where

$${}_q I_t^{1+i-\alpha} f(0) = \lim_{t \rightarrow 0^+} {}_q I_t^{1+i-\alpha} f(t).$$

**Theorem 3.5.** Suppose  $0 < \alpha < 1$ ,  $0 \leq \beta \leq 1$ ,  $f \in \mathbb{L}_q[0, T]$  with  ${}_q I_t^{1-\gamma} f(t) \in AC_q^{(n)}[0, T]$ , where  $\gamma = \alpha + \beta - \alpha\beta$  then,

$${}_q I_t^\alpha {}_q D_t^{\alpha,\beta} f(t) = f(t) - {}_q I_t^{1-\gamma} f(0) \frac{t^{\gamma-1}}{\Gamma_q(\gamma)}$$

*Proof.* The proof is trivial. By property (1) pursuant to the definition 2.3 and the definition 2.6, we obtain  ${}_q I_t^\alpha {}_q D_t^{\alpha, \beta} f(t) = {}_q I_t^\gamma {}_q D_t^\gamma f(t)$ . Subsequently, applies theorem 3.4 with  $n = 1$ , we will obtain the illustrated result.  $\square$

Moving into the variable concept, the authors define the  $q$ -Hilfer derivative with order  $0 < \alpha(t) < 1$  and  $0 \leq \beta \leq 1$ , and the  $q$ -fractional integral of variable order as follows.

**Definition 3.6.** Let,  $q \in (0, 1)$  and  $\alpha(t) > 0$ , then the  $q$ -Riemann-Liouville fractional integral of variable order is defined as

$${}_q I_t^{\alpha(t)} f(t) = \frac{1}{\Gamma_q(\alpha(t))} \int_0^t (t - qs)^{(\alpha(t)-1)} f(s) d_qs.$$

**Definition 3.7.** Let  $0 < \alpha(t) < 1$ ,  $0 \leq \beta \leq 1$  and  $0 < q < 1$  then, the  $q$ -Hilfer variable order fractional derivative of the function  $f(t)$  is defined by

$${}_q D_t^{\alpha(t), \beta} f(t) = {}_q I_t^{\beta(1-\alpha(t))} D_{qq} I_t^{(1-\beta)(1-\alpha(t))} f(t)$$

It is obvious that when  $\alpha(t) = \alpha$ , the operator is the same as definition 2.6.

In this work, the fractional order hybrid integro-difference equation with initial condition given by (1.1), where  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and initial data  $x_0, f_0 \in \mathbb{R}$ , will be analysed.

Firstly, Let  $P = \{(0, T_1], (T_1, T_2], (T_2, T_3], \dots, (T_{N-1}, T]\}$  where  $P_k \in P$  is the  $k^{th}$  sub-interval of  $P$  and let  $\alpha : (0, T] \rightarrow (0, 1)$  be a continuous function.

Secondly, we define the  $\alpha$ -approximation function  $\tilde{\alpha}(t) : (0, T] \rightarrow (0, 1)$  as piecewise continuous function respect to  $P$ . The function  $\tilde{\alpha}$  is written by

$$\tilde{\alpha}(t) = \sum_{k=1}^N \alpha(t_k) I_k(t) = \sum_{k=1}^N \alpha_k I_k(t) = \begin{cases} \alpha_1 & , t \in (0, T_1] \\ \alpha_2 & , t \in (T_1, T_2] \\ \alpha_3 & , t \in (T_2, T_3] \\ \vdots & \\ \alpha_N & , t \in (T_{N-1}, T] \end{cases} , \quad t_k \in P_k \subset (0, T] \quad (3.1)$$

where  $I_k$  is the indicator on  $P_k$ . In other words,  $I_k(t) = 1$  for  $t \in P_k$ . Otherwise,  $I_k(t) = 0$ . Consequently, the function  $\alpha(t) = \lim_{N \rightarrow \infty} \tilde{\alpha}(t)$ , as  $|\alpha_k - \alpha_{k-1}| \rightarrow 0$  for any  $|t_k - t_{k-1}| \rightarrow 0$ . Hence (1.1) can be represented by

$$\begin{aligned} \sum_{k=1}^{\infty} I_k(t) {}_q D_t^{\alpha_k, \beta} [x(t) - f(t, x(t))] &= g(t, x(t), {}_q I_t^\beta x(t)), \quad t \in (0, T] \\ {}_q I_t^{1-\gamma_k} x(0) &= x_0, \quad {}_q I_t^{1-\gamma_k} f(0, x(0)) = f_0, \quad \gamma_k = \alpha_k + \beta - \alpha_k \beta \end{aligned} \quad (3.2)$$

Now, we present the definition of solution to problem (1.1), which is fundamental to this article. From the theorem 3.5 and the equation (3.1), the integral represent solution  $x_k(t)$  in subinterval  $P_k$  is written by

$$x_k(t) = \frac{Ct^{\gamma_k-1}}{\Gamma_q(\gamma_k)} + f(t, x_k(t)) + \frac{1}{\Gamma_q(\alpha_k)} \int_0^t (t - qs)^{(\alpha_k-1)} g(s, x_k(s), {}_q I_t^\beta x_k(s)) d_qs, \quad (3.3)$$

where  $C = x_0 - f_0 \in \mathbb{R}$  for  $t \in P_k$ .

Moreover, the continuous mild solution  $x(t) = \sum_{k=1}^{\infty} I_k(t)x_k(t)$  is written by

$$x(t) = \frac{Ct^{\gamma(t)-1}}{\Gamma_q(\gamma(t))} + f(t, x(t)) + \frac{1}{\Gamma_q(\alpha(t))} \int_0^t (t - qs)^{(\alpha(t)-1)} g(s, x(s), {}_qI_t^\beta x(s)) d_qs. \quad (3.4)$$

#### 4. Uniqueness of solution in subinterval

In this part, the authors will illustrate the uniqueness of solution according to the  $k^{th}$ -subinterval.

**Theorem 4.1.** [11] Suppose  $\phi : (0, a] \rightarrow \mathbb{R}$  is a function, if  $\phi \in \mathbb{L}_q[0, a]$ , then  ${}_qI_t^\alpha \phi \in \mathbb{L}_q[0, a]$ , and  $\|{}_qI_t^\alpha \phi\|_1 \leq \frac{a^\alpha}{\Gamma_q(\alpha+1)} \|\phi\|_1$ .

**Theorem 4.2.** The equation (1.1) has a solution in  $\mathbb{L}_q[0, T]$ , if there exist  $x_1 \in \mathbb{L}_q[0, T_1]$ ,  ${}_qI_t^{1-\gamma_1} x(0) = x_0$  and  ${}_qI_t^{1-\gamma_1} f(0, x(0)) = f_0$  satisfying (3.3);  $x_2 \in \mathbb{L}_q[0, T_2]$ ,  ${}_qI_t^{1-\gamma_2} x(0) = x_0$  and  ${}_qI_t^{1-\gamma_2} f(0, x(0)) = f_0$  satisfying (3.3);  $x_3 \in \mathbb{L}_q[0, T_3]$ ,  ${}_qI_t^{1-\gamma_3} x(0) = x_0$  and  ${}_qI_t^{1-\gamma_3} f(0, x(0)) = f_0$  satisfying (3.3);  $x_i \in \mathbb{L}_q[0, T_i]$ ,  ${}_qI_t^{1-\gamma_i} x(0) = x_0$  and  ${}_qI_t^{1-\gamma_i} f(0, x(0)) = f_0$  satisfying (3.3) where  $i = 3, 4, \dots, N$ .

To display the uniqueness of solution, we state the essential assumptions as follows:

(A0) There exists positive constant  $M_f$  such that

$$\|f(t, u) - f(t, v)\| \leq M_f \|u - v\|$$

for all  $u, v \in \mathbb{L}_q[0, T]$ .

(A1) There exist positive constants  $L_1, L_2, M_g$  such that

$$\|g(t, u_1, u_2) - g(t, v_1, v_2)\| \leq L_1 \|u_1 - v_1\| + L_2 \|u_2 - v_2\| \leq M_g \|u - v\|$$

for all  $u_1, u_2, v_1, v_2 \in \mathbb{L}_q[0, T]$ .

**Theorem 4.3.** Suppose the assumptions (A0)-(A1) are satisfied, then the (3.3) is a unique solution in  $\mathbb{L}_q[0, T_k]$  if there exist a contraction constant  $M_f + \frac{M_g T_k^{\alpha_k+1}}{\Gamma_q(\alpha_k+1)} < 1$

*Proof.* For each  $k = 1, 2, \dots$ , we define the contraction mapping  $Q : \mathbb{L}_q[0, T_k] \rightarrow \mathbb{L}_q[0, T_k]$  by  $Qx_k = x_k$ , we get

$$Qx_k(t) = \frac{Ct^{\gamma_k-1}}{\Gamma_q(\gamma_k)} + f(t, x_k(t)) + \frac{1}{\Gamma_q(\alpha_k)} \int_0^t (t - qs)^{(\alpha_k-1)} g(s, x_k(s), {}_qI_t^\beta x_k(s)) d_qs.$$

Then,

$$\begin{aligned} \|Qx_k - Qy_k\|_1 &\leq \|f(t, x_k(t)) - f(t, y_k(t))\|_1 \\ &\quad + \|{}_q I_t^{\alpha_k} 1\|_1 \|g(t, x_k(t), {}_q I_t^\beta x_k(t)) - g(t, y_k(t), {}_q I_t^\beta y_k(t))\|_1 \\ &\leq M_f \|x_k - y_k\|_1 + \frac{\Gamma_k^{\alpha_k+1}}{\Gamma_q(\alpha_k+1)} (L_1 \|x_k - y_k\|_1 + L_2 \|{}_q I_t^\beta x_k - {}_q I_t^\beta y_k\|_1) \\ &\leq M_f \|x_k - y_k\|_1 + \frac{\Gamma_k^{\alpha_k+1}}{\Gamma_q(\alpha_k+1)} \left( L_1 + \frac{L_2 \Gamma_k^{\beta+1}}{\Gamma_q(\beta+1)} \right) \|x_k - y_k\|_1 \\ &= \left( M_f + \frac{M_g \Gamma_k^{\alpha_k+1}}{\Gamma_q(\alpha_k+1)} \right) \|x_k - y_k\|_1. \end{aligned}$$

By Banach contraction theorem, since  $M_f + \frac{M_g \Gamma_k^{\alpha_k+1}}{\Gamma_q(\alpha_k+1)} < 1$  then  $x_k$  is unique solution on  $\mathbb{L}_q[0, T_k]$ . The proof is completed. □

### 5. uniqueness of continuous solution

In this part, the uniqueness of (3.4) will be displayed.

**Theorem 5.1.** [25] For any  $0 < q < 1$  and  $0 < s < 1$ , the inequality of  $q$ -gamma function for any  $z > 0$  holds,

$$\left( \frac{1 - q^{z+\frac{s}{2}}}{1 - q} \right)^{1-s} < \frac{\Gamma_q(z+1)}{\Gamma_q(z+s)} < \left( \frac{1 - q^{z+s}}{1 - q} \right)^{1-s}.$$

**Theorem 5.2.** For any  $0 < q < 1$  and  $0 < \alpha(t) < 1$ , the inequality of  $q$ -gamma function holds,

$$(1 - q)^{\alpha(t)} \leq \frac{1}{\Gamma_q(\alpha(t) + 1)} < \left( \frac{1 - q}{1 - q^{\frac{\alpha(t)+1}{2}}} \right)^{\alpha(t)}.$$

*Proof.* Since  $0 < \alpha(t) < 1$ , the value of  $z$  pursuant to theorem 5.1 is lying between the interval  $(1, 2)$ . Then, consider

$$\Gamma_q(x) = (1 - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}},$$

we obtain  $1 - q^{n+1} < 1 - q^{n+x}$  for any  $x \in (1, 2)$ . Consequently, it is clear that

$$\frac{1 - q^{n+1}}{1 - q^{n+x}} < 1.$$

Thus,

$$\prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}} < 1.$$

From this point, it is obvious that

$$\Gamma_q(x) \leq (1 - q)^{1-x}, \quad x \in (1, 2).$$

Let  $x = \alpha(t) + 1$  we obtain the inequality

$$\Gamma_q(\alpha(t) + 1) \leq (1 - q)^{-\alpha(t)}.$$

Next, suppose  $z = \alpha(t)$  and  $s = 1 - \alpha(t)$  into inequality of theorem 5.1, we get

$$\left( \frac{1 - q^{\alpha(t) + \frac{1-\alpha(t)}{2}}}{1 - q} \right)^{\alpha(t)} < \Gamma_q(\alpha(t) + 1) < 1.$$

Now, combining inequalities together, the new inequality holds

$$\left( \frac{1 - q^{\frac{\alpha(t)+1}{2}}}{1 - q} \right)^{\alpha(t)} < \Gamma_q(\alpha(t) + 1) \leq (1 - q)^{-\alpha(t)}.$$

Rearrange the inequality, the inequality holds

$$(1 - q)^{\alpha(t)} \leq \frac{1}{\Gamma_q(\alpha(t) + 1)} < \left( \frac{1 - q}{1 - q^{\frac{\alpha(t)+1}{2}}} \right)^{\alpha(t)}.$$

The proof is completed. □

**Theorem 5.3.** Suppose  $x_k$  is unique on  $\mathbb{L}_q[0, T_k]$ , then  $x(t)$  is unique solution on  $\mathbb{L}_q[0, T]$  if there exist a contraction function  $\varphi : (0, T] \rightarrow (0, 1)$  such that

$$\varphi(t) = M_f + M_g T^{\alpha(t)+1} \left( \frac{1 - q}{1 - q^{\frac{\alpha(t)+1}{2}}} \right)^{\alpha(t)} < 1$$

*Proof.* Generating the approximation contraction function  $\tilde{\varphi}$  on  $(0, T]$  by aggregate the contraction constant in each subinterval  $P_k$  we get

$$\tilde{\varphi}(t) = M_f + \sum_{k=1}^N I_k(t) \left( \frac{M_g T_k^{\alpha_k+1}}{\Gamma_q(\alpha_k + 1)} \right), \quad t \in (0, T].$$

Thus, by take limit  $N \rightarrow \infty$ , the fundamental contraction function  $\varphi^*(t)$  is displayed as

$$\varphi^*(t) = M_f + \sum_{k=1}^{\infty} I_k(t) \left( \frac{M_g T_k^{\alpha_k+1}}{\Gamma_q(\alpha_k + 1)} \right) = M_f + \frac{M_g T^{\alpha(t)+1}}{\Gamma_q(\alpha(t) + 1)}, \quad t \in (0, T]$$

According to the theorem 5.2, it is obvious that

$$\varphi^*(t) = M_f + \frac{M_g T^{\alpha(t)+1}}{\Gamma_q(\alpha(t) + 1)} < M_f + M_g T^{\alpha(t)+1} \left( \frac{1 - q}{1 - q^{\frac{\alpha(t)+1}{2}}} \right)^{\alpha(t)} = \varphi(t).$$

Since there exists the contraction function  $\varphi(t) < 1$ , the continuous solution  $x(t)$  is a unique solution on  $\mathbb{L}_q[0, T]$ . The proof is completed. □



**6. Example**

In this section, we give an example of q-FHIDEVO to illustrate our result. Consider the following equation where  $t \in (0, 1]$  and  $q = e^{-\pi}$ .

$$\begin{aligned}
 {}_qD_t^{\frac{1}{10\cos(\cos(\frac{t}{2}))} \cdot \frac{1}{2}} \left[ x(t) - \frac{x(t)}{100(x^2(t) + 1)} \right] &= \frac{\tan^{-1} x(t)}{100} + \frac{\tan^{-1} {}_qI_t^{\frac{1}{2}} x(t)}{100} \\
 {}_qI_t^{1-\gamma(t)} x(0) = 0, \quad {}_qI_t^{1-\gamma(t)} f(0, x(0)) &= 0, \quad \gamma(t) = \frac{1}{20} \sec \left( \cos \left( \frac{t}{2} \right) \right) + \frac{1}{2}
 \end{aligned} \tag{6.1}$$

It can be seen that  $L_1 = L_2 = \frac{1}{100}$ , and  $M_g$  has following value.

$$M_g = \frac{1}{100} + \frac{e^{\frac{7\pi}{16}} (1 - e^{-\pi}) \pi^{\frac{3}{4}}}{50 \sqrt[16]{2} \sqrt[4]{1 + \sqrt{2}} (1 - e^{-\frac{\pi}{2}}) \sqrt{e^\pi - 1} \Gamma(\frac{1}{4})}$$

For the assumption on  $f$ , we can see that  $M_f = \frac{1}{100}$  for all  $x \in \mathbb{R}$ . By mean value theorem, we get

$$\begin{aligned}
 |f(t, x) - f(t, y)| &= \left| \frac{x}{100(x^2 + 1)} - \frac{y}{100(y^2 + 1)} \right| \\
 &\leq \frac{1}{100} \|x - y\|.
 \end{aligned}$$

This mean the contraction function  $\varphi(t)$  is written as

$$\frac{1}{100} + \left( \frac{1}{100} + \frac{e^{\frac{7\pi}{16}} (1 - e^{-\pi}) \pi^{\frac{3}{4}}}{50 \sqrt[16]{2} \sqrt[4]{1 + \sqrt{2}} (1 - e^{-\frac{\pi}{2}}) \sqrt{e^\pi - 1} \Gamma(\frac{1}{4})} \right) \left( \frac{1 - e^{-\pi}}{1 - e^{-\frac{\pi}{20} \sec(\cos(\frac{t}{2})) + \frac{\pi}{2}}} \right)^{\frac{1}{10\cos(\cos(\frac{t}{2}))}}$$

which the function  $\varphi(t) < 1$ . Thus, according to the theorem 5.3, the equation (6.1) has unique solution in  $\mathbb{L}_q[0, 1]$ .

**7. Conclusion**

In this work, the authors introduce novel operators in quantum calculus, which are q-Hilfer fractional derivative and q-Hilfer fractional derivative of variable order. Also, we present the novel proof of the uniqueness of continuous solutions to q-FHIDEVO. The uniqueness of solution is proved by using Banach fixed point theorem under Lipschitz conditions for nonlinear terms.

**Acknowledgement**

The authors would like to thanks referees for useful comments and suggestions which lead to an extensive improvement of the manuscripts.

## References

- [1] Ahmed I, Baba IA, Yusuf A, Kumam P and Kumam W (2020). *Analysis of Caputo fractional-order model for COVID-19 with lockdown*, Adv. Differ. Equ. **2020**: 394. <https://doi.org/10.1186/s13662-020-02853-0>
- [2] Ahmed I, Gouf EFD, Yusuf A, Kumam P, Chaipanya P, Nonlaopon K (2021). *An epidemic prediction from analysis of a combined hiv-covid-19 co-infection model via ABC-fractional operator*, Alex. Eng. J. **2021**: 60, 2979–2995. <https://doi.org/10.1016/j.aej.2021.01.041>.
- [3] Ahmed I, Kumam P, Abubakar J, Borisut P, Sitthithakerngkiet K (2020). *Solutions for impulsive fractional pantograph differential equation via generalized anti-periodic boundary condition*, Adv. Differ. Equ. **2020**: 477. <https://doi.org/10.1186/s13662-020-02887-4>
- [4] Ahmed I, Kumam P, Abdeljawad T, Jarad F, Borisut P, Demba MA, Kumam W (2020). *Existence and uniqueness results for  $\phi$ -Caputo implicit fractional pantograph differential equation with generalized anti-periodic boundary condition*, Adv. Differ. Equ. **2020**: 555. <https://doi.org/10.1186/s13662-020-03008-x>
- [5] Ahmed I, Kumam P, Jarad F, Borisut P, Jirakitpuwapat W (2020). *On Hilfer generalized proportional fractional derivative*, Adv. Differ. Equ. **2020**: 329. <https://doi.org/10.1186/s13662-020-02792-w>
- [6] Ahmed I, Kumam P, Jarad F, Borisut P, Sitthithakerngkiet K, Ibrahim A (2020). *Stability analysis for boundary value problems with generalized nonlocal condition via Hilfer–Katugampola fractional derivative*, Adv. Differ. Equ. **2020**: 225. <https://doi.org/10.1186/s13662-020-02681-2>
- [7] Ahmed I, Kumam P, Shah K, Borisut P, Sitthithakerngkiet K, Demba MA (2020). **Stability Results for Implicit Fractional Pantograph Differential Equations via  $\phi$ -Hilfer Fractional Derivative with a Nonlocal Riemann-Liouville Fractional Integral Condition**, Mathematics **2020**: 8(1), 94. <https://doi.org/10.3390/math8010094>
- [8] Ahmed I, Yusuf A, Sani MA, Jarad F, Kumam W, Thounthong P (2021). *Analysis of a caputo hiv and malaria co-infection epidemic model*, Thai J. Math. **2021**: 19(3), 897–912.
- [9] Almeida R, Tavares D, Torres DFM (2019). *The Variable-Order Fractional Calculus of Variations*, Springer-Briefs in Applied Sciences and Technology, Springer, Cham. **2019**. <https://doi.org/10.1007/978-3-319-94006-9>
- [10] Ameen R, Jarad F, Abdeljawad T (2018). *Ulam stability of fractional differential equations with a generalized Caputo derivative*, Filomat: 32(15), 5265-5274. <https://doi.org/10.2298/FIL1815265A>
- [11] Annaby MH, Mansour ZS (2012). *q-Fractional Calculus and Equations*, Lecture Notes in Mathematics, vol 2056. Springer, Berlin, Heidelberg. <https://doi.org/10.1007/978-3-642-30898-7>
- [12] Borisut P, Khammahawong K, Kumam P (2018). *Fixed Point Theory Approach to Existence of Solutions with Differential Equations*, IntechOpen: London, UK. [10.5772/intechopen.74560](https://doi.org/10.5772/intechopen.74560)
- [13] Borisut P, Kumam P, Ahmed I, Sitthithakerngkiet K (2019). *Nonlinear Caputo Fractional Derivative with Nonlocal Riemann-Liouville Fractional Integral Condition Via Fixed Point Theorems*, Symmetry **2019**: 11(6), 829. <https://doi.org/10.3390/sym11060829>
- [14] Borisut P, Kumam P, Ahmed I, Sitthithakerngkiet K (2020). *Positive Solution for Nonlinear Fractional Differential Equation with Nonlocal Multi-Point Condition*, Fixed Point Theory **2020**: 21(2), 427–440. [10.24193/fpt-ro.2020.2.30](https://doi.org/10.24193/fpt-ro.2020.2.30)
- [15] Borisut P, Kumam P, Ahmed I, Jirakitpuwapat W (2021). *Existence and uniqueness for  $\psi$ -Hilfer fractional differential equation with nonlocal multi-point condition*, Math. Meth. Appl. Sci. **2021**: 44, 2506-2520. <https://doi.org/10.1002/mma.6092>
- [16] Dhage BC (2002). *On a Fixed Point Theorem of Krasnoselskii-Schaefer Type*, Electron. J. Qual. Theory Differ. Equ. **2002**: 6, 1-9. <https://doi.org/10.14232/ejqtde.2002.1.6>
- [17] Dhage BC, Dhage SB, Ntouyas SK (2014). *Approximating solutions of nonlinear hybrid differential equations*, Appl. Math. Lett. **2014**: 34, 76-80. <https://doi.org/10.1016/j.aml.2014.04.002>
- [18] Hilfer R (2000). *Applications of Fractional Calculus in Physics*, World Scientific, Singapore. <https://doi.org/10.1142/3779>
- [19] Jackson FH, (1909). *On q-functions and a certain difference operator*, Transactions of the Royal Society of Edinburgh **1909**: 46(2), 253-281. <https://doi.org/10.1017/S0080456800002751>
- [20] Khan RA, Shah K (2015). *Existence and uniqueness of positive solutions to fractional order multi-point boundary value problems*, Commun. Appl. Anal. **2015**: 19, 515–526.
- [21] Limpanukorn N, Ngiamsunthorn PS (2020). *Existence and Ulam stability of solution to fractional order hybrid differential equation of variable order*, Thai J. Math. **2020**: 18(1), 453-463.

- [22] Mallah I, Ahmed I, Akgul A, Jarad F, Alha S (2022). *On  $\psi$ -Hilfer generalized proportional fractional operators*, AIMS Mathematics **2022**: 7(1), 82-103. [10.3934/math.2022005](https://doi.org/10.3934/math.2022005)
- [23] Neang P, Nonlaopon K, Tariboon J, Ntouyas SK (2021). *Fractional  $(p,q)$ -Calculus on Finite Intervals and Some Integral Inequalities*, Symmetry **2021**: 13(3), 504. <https://doi.org/10.3390/sym13030504>
- [24] Neang P, Nonlaopon K, Tariboon J, Ntoyas SK, Agarwal P (2021). *Some trapezoid and midpoint type inequalities via fractional  $(p,q)$ -calculus*, Adv. Differ. Equ. **2021**: 333(2021). <https://doi.org/10.1186/s13662-021-03487-6>
- [25] Peng G (2011). *Some Monotonicity Properties of Gamma and  $q$ -Gamma Functions*, International Scholarly Research Notices **2011**. <https://doi.org/10.5402/2011/375715>
- [26] Somjaiwang D, Ngiamsunthorn PS (2016). *Existence and approximation of solutions to fractional order hybrid differential equations*, Adv. Differ. Equ. **2016**: 278(2016). <https://doi.org/10.1186/s13662-016-0999-8>
- [27] Suechoei A, Ngiamsunthorn PS (2020). *Existence uniqueness and stability of mild solutions for semilinear  $\psi$ -Caputo fractional evolution equations*, Adv. Differ. Equ. **2020**: 114(2020). <https://doi.org/10.1186/s13662-020-02570-8>
- [28] Tariboon J, Ntouyas SK, Agarwal P (2015). *New concepts of fractional quantum calculus and applications to impulsive fractional  $q$ -difference equations*, Adv. Differ. Equ. **2015**: 18(2015). <https://doi.org/10.1186/s13662-014-0348-8>
- [29] Zhang S (2018). *The uniqueness result of solutions to initial value problems of differential equations of variable-order*, RACSAM **2018**: 112(2), 407-423. <https://doi.org/10.1007/s13398-017-0389-4>
- [30] Zhang S, Hu L (2019). *Unique Existence Result of Approximate Solution to Initial Value Problem for Fractional Differential Equation of Variable Order Involving the Derivative Arguments on the Half-Axis*, Mathematics **2019**: 7(3), 286. <https://doi.org/10.3390/math7030286>
- [31] Zhao Y, Sun S, Han Z, Li Q (2011). *Theory of fractional hybrid differential equations*, Comput. Math. Appl. **2011**: 62, 1312-1324. <https://doi.org/10.1016/j.camwa.2011.03.041>