

Recurrence Decompositions in Finsler Space

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Abstract

Finsler geometry is a kind of differential geometry originated by P. Finsler. Indeed, Finsler geometry has several uses in a wide variety and it is playing an important role in differential geometry and applied mathematics of problems in physics relative, manual footprint. It is usually considered as a generalization of Riemannian geometry. In the present paper, we introduced some types of generalized W^h -birecurrent Finsler space, generalized W^h -birecurrent affinely connected space and we defined a Finsler space F_n for Weyl's projective curvature tensor W_{jkh}^i satisfies the generalized-birecurrence condition with respect to Cartan's connection parameters Γ_{kh}^{*i} , such that given by the condition (2.1), where $|m|n$ is h-covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^m and x^n , successively, λ_{mn} and μ_{mn} are non-null covariant vectors field and such space is called as a generalized W^h -birecurrent space and denoted briefly by GW^h - BRF_n . We have obtained some theorems of generalized W^h -birecurrent affinely connected space for the h-covariant derivative of the second order for Weyl's projective torsion tensor W_{kh}^i , Weyl's projective deviation tensor W_h^i in our space. We have obtained the necessary and sufficient condition for some tensors in our space.

Keywords: Generalized W^h -birecurrent affinely connected space, Generalized W^h -Birecurrent space, Weyl's projective curvature tensor W_{jkh}^i , Finsler space F_n .
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1. Introduction

The recurrent for different curvature tensors have been discussed by Al-Qashbari [1], Dikshit [2], Matsumoto [3], Pandey [4], Pandey, et al. [5], Qasem [6], Qasem et al. [7, 8] and others. Ahsanand Ali [9] who discussed a recurrent curvature tensor on some properties of W-curvature tensor of Weyl's projective curvature tensor W_{jkh}^i is skew-symmetric, it is indices k and h. The generalized birecurrent space was studied by Hadi [10], Qasem and Hadi [11]. Also, W-generalized birecurrent space studied by Qasem and Saleem [12] and others.

An n-dimensional Finsler space, Figure 1, equipped with the metric function f satisfies the requisite conditions [13]. Consider the components of the corresponding metric tensor

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g_{ij} , Cartan's connection parameters $\Gamma_{jk}^*{}^i$ and Berwald's connection parameters G_{jk}^i (the indices i, j, k, \dots assume positive integral values from 1 to n). These are symmetric in their lower indices.

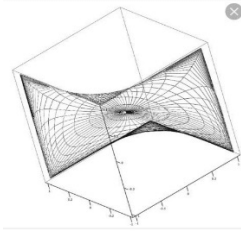


Figure 1: Finsler Space as a Locally Minkowskian Space

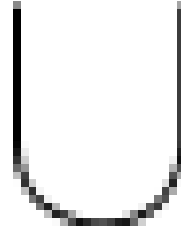


Figure 2: Metric Tensor g_{ij}

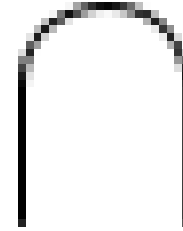


Figure 3: Metric Tensor g^{ij}

The vectors y_i and y^i satisfy the following relations [13]:

$$y_i = g_{ij} y^j, \quad (1.1)$$

$$y_i y^i = F^2, \quad (1.2)$$

$$\partial_i y_j = g_{ij}, \quad (1.3)$$

$$\partial_j y^i = \delta_j^i. \quad (1.4)$$

The metric tensor g_{ij} Figure 2, the metric tensor g^{ij} Figure 3 and the vector y^i are covariant constant with respect to the above process.

$$g_{ij|k} = 0, \quad (1.5)$$

$$y_{|k}^i = 0, \quad (1.6)$$

$$g_{|k}^{ij} = 0., \quad (1.7)$$

The h -covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j , successively, we get

$$X_{|k|j}^i = \partial_j \left(X_{|k}^i \right) - \left(X_{|r}^i \right) \Gamma_{kj}^{*r} + \left(X_{|k}^r \right) \Gamma_{rj}^{*i} - \partial_r \left(X_{|k}^i \right) \Gamma_{js}^{*r} y^s \quad (1.8)$$

Taking skew-symmetric part with respect to the indices k and j , we get the commutation formula for h -covariant differentiation as follows [13]:

$$X_{|k|j}^i - X_{|j|k}^i = X^r K_{rkj}^i - (\partial_r X^i) K_{skj}^r y^s, \quad (1.9)$$

where

$$K_{rkj}^i := \partial_j \Gamma_{kr}^{*i} + (\partial_l \Gamma_{rj}^{*i}) G_k^l + \Gamma_{mj}^{*i} \Gamma_{kr}^{*m} - \frac{j}{k}. \quad (1.10)$$

where the indices i, j, k, \dots are positive integral values from 1 to n , and $-\frac{j}{k}$ means the subtraction from the former term by interchanging the indices k and j .

The tensor K_{rkj}^i as defined above is called Cartan's fourth curvature tensor.

The process of h-covariant differentiation, with respect to x^k , commute with partial differentiation with respect to y^j for arbitrary vector field X^i , according to [13],

$$\partial_j (X_{|k}^i) = (\partial_j X^i)_{|k} + X^r (\partial_j \Gamma_{rk}^{*i}) - (\partial_r X^i) P_{jk}^r \quad (1.11)$$

and

$$\partial_j \Gamma_{rk}^{*i} = \Gamma_{rkj}^{*i}. \quad (1.12)$$

The tensor W_{jkh}^i is known as projective curvature tensor (generalized Wely's projective curvature tensor), the tensor W_{jk}^i is known as projective torsion tensor (Wely's torsion tensor) and the tensor W_j^i is known as projective deviation tensor (Wely's deviation tensor) are defined by

$$\begin{aligned} W_{jkh}^i &= H_{jkh}^i + \frac{2\delta_j^i}{n+1} H_{hk} + \frac{2y^i}{n+1} \partial_j H_{kh} + \frac{\delta_k^i}{n^2-1} (nH_{jh} + H_{hj} + y^r \partial_j H_{hr}) \\ &\quad - \frac{\delta_h^i}{n^2-1} (nH_{jk} + H_{kj} + y^r \partial_j H_{kr}), \end{aligned} \quad (1.13)$$

$$W_{jk}^i = H_{jk}^i + \frac{y^i}{n+1} H_{jk} + \frac{2\partial_j}{n^2-1} (nH_{|k} - y^r H_{k|r}), \quad (1.14)$$

and

$$W_j^i = H_j^i - H \delta_j^i - \frac{1}{n+1} (\partial_r H_j^r - \partial_j H) y^i, \quad (1.15)$$

respectively.

The tensors W_{jkh}^i , W_{jk}^i and W_j^i are satisfying the following identities [13]

$$W_{jkh}^i y^j = W_{kh}^i, \quad (1.16)$$

$$W_{jk}^i y^j = W_k^i, \quad (1.17)$$

$$\partial_j W_{kh}^i = W_{jkh}^i, \quad (1.18)$$

$$\partial_k W_h^i = W_{kh}^i. \quad (1.19)$$

The projective curvature tensor W_{jkh}^i is skew-symmetric in its indices k and h see the definition in (1.13).

An affinely connected space has some properties as follows:

$$G_{jkh}^i = 0, \text{ and } C_{ijk|h} = 0 \quad (1.20)$$

Remark 1.1. An affinely connected space or Berwald space characterized by any one of the above two equivalent conditions. Also, we have the following properties.

The connection parameters Γ_{jk}^{*i} of Cartan and G_{jk}^i of Berwald coincide in an affinely connected space and they are independent of the directional arguments [13], i.e.

$$G_{jkh}^i = \partial_j G_{kh}^i = 0, \quad (1.21)$$

$$\partial_j \Gamma_{kh}^{*i} = 0, \quad (1.22)$$

$$y_r G_{ijk}^r = -2 C_{ijk|h} y^h = -2 P_{ijk} = 0. \quad (1.23)$$

2. On Necessary and Sufficient Condition of Generalized W^h -Birecurrent

A Finsler space F_n for which Weyl's projective curvature tensor W_{jkh}^i satisfies the recurrence property with respect to Cartan's coefficient connection parameters Γ_{jk}^{*i} which is characterized by the condition [13]

$$W_{jkh|m}^i = a_m W_{jkh}^i + b_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}), \quad W_{jkh}^i \neq 0, \quad (2.1)$$

where $|m$ is h -covariant derivative of first order (Cartan's second kind covariant differential operator) with respect to x^m , the quantities a_m and b_m are non-null covariant vectors field. Its space is called as a generalized W^h -recurrent space and is denoted briefly by GW^h - RF_n .

Taking the h -covariant derivative for (2.1) with respect to x^n and using (1.5), we get

$$W_{jkh|m|n}^i = a_{m|n} W_{jkh}^i + a_m W_{jkh|n}^i + b_{m|n} (\delta_h^i g_{jk} - \delta_k^i g_{jh})$$

where $g_{jk|n} = 0$.

In view of (2.1), the above equation yields

$$W_{jkh|m|n}^i = \lambda_{mn} W_{jkh}^i + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}), \quad (2.2)$$

where $\lambda_{mn} = a_{m|n} + a_m a_n$, $\mu_{mn} = a_m a_n + b_{m|n}$ and $|m|n$ is h -covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^m and x^n , successively, λ_{mn} and μ_{mn} are non-null covariant vectors field and such space is called as a generalized W^h -birecurrent space and denoted briefly by GW^h - $BIRF_n$.

Result Every generalized W^h -recurrent space is generalized W^h -birecurrent space.

Transvecting the condition (2.2) by y^j and by y^k , successively, using (1.6), (1.16), (1.17), (1.1) and (1.2), we get

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\delta_h^i y_k - \delta_k^i y_h), \quad (2.3)$$

$$W_{h|m|n}^i = \lambda_{mn} W_h^i + \mu_{mn} (\delta_h^i F^2 - y_h y^i), \quad (2.4)$$

Theorem 2.1. In GW^h - $BIRF_n$, the h -covariant derivative of the second order for Weyl's projective torsion tensor W_{kh}^i and Weyl's projective deviation tensor W_h^i are given by (2.3) and (2.4), respectively.

Differentiating partially of (2.3) with respect to y^j , using (1.23) and (1.3), we get

$$\begin{aligned} \partial_r (W_{kh|m|n}^i) &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &\quad + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned}$$

Using the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $W_{kh|m}^i$,

in the above equation, we get

$$\begin{aligned}
 & \left\{ \dot{\partial}_r W_{kh|m}^i \right\}_{|n} + W_{kh|m}^r (\dot{\partial}_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\dot{\partial}_j \Gamma_{kn}^{*r}) \\
 & - W_{kr|m}^i (\dot{\partial}_j \Gamma_{hn}^{*r}) - W_{kh|r}^i (\dot{\partial}_j \Gamma_{mn}^{*r}) - \left(\dot{\partial}_r W_{kh|m}^i \right) P_{jn}^r \\
 = & (\dot{\partial}_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\dot{\partial}_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\
 & + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) .
 \end{aligned} \tag{2.5}$$

Again, applying the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor (W_{kh}^i), in the equation (2.5) and using (1.18), we obtain

$$\begin{aligned}
 & \left\{ W_{jkh|m}^i + W_{kh}^r (\dot{\partial}_j \Gamma_{rm}^{*i}) - W_{rh}^i (\dot{\partial}_j \Gamma_{km}^{*r}) - W_{kr}^i (\dot{\partial}_j \Gamma_{hm}^{*r}) - (\dot{\partial}_r W_{kh}^i) P_{jm}^r \right\}_{|n} \\
 & + W_{kh|m}^r (\dot{\partial}_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\dot{\partial}_j \Gamma_{kn}^{*r}) - W_{kr|m}^i (\dot{\partial}_j \Gamma_{hn}^{*r}) \\
 & - W_{kh|r}^i (\dot{\partial}_j \Gamma_{mn}^{*r}) - \left(\dot{\partial}_r W_{kh|m}^i \right) P_{jn}^r \\
 = & (\dot{\partial}_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\dot{\partial}_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\
 & + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) ,
 \end{aligned}$$

which can be written as

$$\begin{aligned}
 & W_{jkh|m|n}^i + \left\{ W_{kh}^r (\dot{\partial}_j \Gamma_{rm}^{*i}) - W_{rh}^i (\dot{\partial}_j \Gamma_{km}^{*r}) - W_{kr}^i (\dot{\partial}_j \Gamma_{hm}^{*r}) - (\dot{\partial}_r W_{kh}^i) P_{jm}^r \right\}_{|n} \\
 & + W_{kh|m}^r (\dot{\partial}_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\dot{\partial}_j \Gamma_{kn}^{*r}) - W_{kr|m}^i (\dot{\partial}_j \Gamma_{hn}^{*r}) \\
 & - W_{kh|r}^i (\dot{\partial}_j \Gamma_{mn}^{*r}) - \left(\dot{\partial}_r W_{kh|m}^i \right) P_{jn}^r \\
 = & (\dot{\partial}_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\dot{\partial}_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\
 & + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) .
 \end{aligned} \tag{2.6}$$

This shows that

$$W_{jkh|m|n}^i = \lambda_{mn} W_{jkh}^i + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh})$$

if and only if

$$\begin{aligned}
 & \left\{ W_{kh}^r (\dot{\partial}_j \Gamma_{rm}^{*i}) - W_{rh}^i (\dot{\partial}_j \Gamma_{km}^{*r}) - W_{kr}^i (\dot{\partial}_j \Gamma_{hm}^{*r}) - (\dot{\partial}_r W_{kh}^i) P_{jm}^r \right\}_{|n} \\
 & + W_{kh|m}^r (\dot{\partial}_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\dot{\partial}_j \Gamma_{kn}^{*r}) - W_{kr|m}^i (\dot{\partial}_j \Gamma_{hn}^{*r}) \\
 & - W_{kh|r}^i (\dot{\partial}_j \Gamma_{mn}^{*r}) - \left(\dot{\partial}_r W_{kh|m}^i \right) P_{jn}^r - (\dot{\partial}_j \lambda_{mn}) W_{kh}^i \\
 & - (\dot{\partial}_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) = 0.
 \end{aligned} \tag{2.7}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 2.2. In GW^h - $BIRF_n$, Weyl's projective curvature tensor W_{jkh}^i is generalized birecurrent, if and only if the equation (2.7) is equal to zero.

Differentiating partially of (2.4) with respect to y^k , using (1.19) and (1.3), we obtain

$$\begin{aligned} \partial_k (W_{h|m|n}^i) &= (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) \\ &\quad + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h). \end{aligned}$$

Using the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $W_{h|m}^i$, in the above equation, we have

$$\begin{aligned} &(\partial_k W_{h|m}^i)_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) - W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) \\ &- W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\ &= (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) \\ &\quad + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h). \end{aligned} \quad (2.8)$$

Again, apply the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor (W_h^i) , in the equation (2.8) and using (1.19), we get

$$\begin{aligned} &\left\{ W_{kh|m}^i + W_h^r (\partial_j \Gamma_{hm}^{*r}) - W_r^i (\partial_k \Gamma_{hm}^{*r}) - (\partial_r W_h^i) P_{km}^r \right\}_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) \\ &- W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\ &= (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i \\ &\quad + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h), \end{aligned}$$

which can be written as

$$\begin{aligned} &W_{kh|m|n}^i + \left\{ W_h^r (\partial_j \Gamma_{hm}^{*r}) - W_r^i (\partial_k \Gamma_{hm}^{*r}) - (\partial_r W_h^i) P_{km}^r \right\}_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) \\ &- W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\ &= (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i \\ &\quad + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h). \end{aligned} \quad (2.9)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h) \quad (2.10)$$

if and only if

$$\begin{aligned} &\left\{ W_h^r (\partial_j \Gamma_{hm}^{*r}) - W_r^i (\partial_k \Gamma_{hm}^{*r}) - (\partial_r W_h^i) P_{km}^r \right\}_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) \\ &- W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\ &= (\partial_k \lambda_{mn}) W_h^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i). \end{aligned} \quad (2.11)$$

Therefore, it is concluded the following.

Theorem 2.3. In $GW^h - BIRF_n$, Wely's projective torsion tensor W_{kh}^i is given by (2.10), if and only if the equation (2.11) holds good.

3. Affinely Connected Space on Generalized W^h -Birecurrent

Let us introduce definition of $GW^h - BIRF_n$ to be affinely connected space as follows:

Definition 3.1. The generalized W^h -birecurrent space which is an affinely connected space [satisfies any one of the conditions (1.21), (1.22) and (1.23)] will be called generalized W^h -birecurrentaffinely connected space and will denote it briefly by $GW^h - BIR$ -affinely connected space.

Remark 3.2. It will be sufficient to call the tensor which satisfies the condition of $GW^h - BIR$ -affinely connected space as generalized h -birecurrent tensor (briefly $Gh - BIR$) .

By using the conditions (1.21), (1.22) and (1.23), the equation (2.6) reduce to

$$\begin{aligned} W_{jkh|m|n}^i &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &\quad + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned} \quad (3.1)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{jkh}^i + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \quad (3.2)$$

if and only if

$$(\partial_j \lambda_{mn}) W_{kh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) = 0. \quad (3.3)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.3. In $GW^h - BIR$ - affinely connected space, Weyl's projective curvature tensor W_{jkh}^i is $Gh - BIR$ if and only if the equation (3.3) is equal to zero.

Transvecting the equation (3.1) by y^j , using (1.6), (1.16) and (1.1), we get

$$\begin{aligned} W_{kh|m|n}^i &= (\partial_j \lambda_{mn}) W_{kh}^i y^j + \lambda_{mn} W_{kh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) y^j \\ &\quad + \mu_{mn} (\delta_h^i y_k - \delta_k^i y_h). \end{aligned} \quad (3.4)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\delta_h^i y_k - \delta_k^i y_h) \quad (3.5)$$

if and only if

$$(\partial_j \lambda_{mn}) W_{kh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) = 0, \text{ since } y^j \neq 0. \quad (3.6)$$

Transvecting the equation (3.4) by y^k , using (1.6), (1.16), (1.2) and (1.1), we obtain

$$\begin{aligned} W_{h|m|n}^i &= (\partial_j \lambda_{mn}) W_h^i y^j + \lambda_{mn} W_h^i + (\partial_j \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^j \\ &\quad + \mu_{mn} (\delta_h^i F^2 - y_h y^i). \end{aligned}$$

This shows that

$$W_{h|m|n}^i = \lambda_{mn} W_h^i + \mu_{mn} (\delta_h^i F^2 - y_h y^i) \quad (3.7)$$

if and only if

$$(\partial_j \lambda_{mn}) W_h^i y^j + (\partial_j \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^j = 0. \quad (3.8)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.4. *In GW^h - BIR -affinely connected space, Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (3.5) and (3.7), respectively, if and only if the equations (3.6) and (3.8), respectively are equals to zero.*

By using the conditions (1.21), (1.22) and (1.23), the equation (2.9) reduce to

$$\begin{aligned} W_{kh|m|n}^i &= (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) \\ &\quad + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h). \end{aligned} \quad (3.9)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h) \quad (3.10)$$

if and only if

$$(\partial_k \lambda_{mn}) W_h^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) = 0. \quad (3.11)$$

Therefore, it is concluded the following.

Theorem 3.5. *In GW^h - BIR - affinely connected space, Wely's projective torsion tensor W_{kh}^i is given by (3.10) if and only if the equation (3.11) hold good.*

Transvecting the equation (3.9) by y^k , using (1.6), (1.17), (1.2) and (1.1), we get

$$\begin{aligned} W_{h|m|n}^i &= (\partial_k \lambda_{mn}) W_h^i y^k + \lambda_{mn} W_h^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^k \\ &\quad + \mu_{mn} (\delta_h^i F^2 - y_h y^i). \end{aligned}$$

This shows that

$$W_{h|m|n}^i = \lambda_{mn} W_h^i + \mu_{mn} (\delta_h^i F^2 - y_h y^i) \quad (3.12)$$

if and only if

$$(\partial_k \lambda_{mn}) W_h^i y^k + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^k = 0. \quad (3.13)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.6. *In GW^h - BIR -affinely connected space, Wely's projective deviation tensor W_h^i is given by (3.12) if and only if the equation (3.13) hold good.*

4. Conclusions and Recommendations

A Finsler space is called generalized W^h -birecurrent space if it satisfies the condition (2.1). In GW^h - BIR , the h-covariant derivative of the second order for Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (2.3) and (2.4), respectively. In GW^h - BIR , the necessary and sufficient condition of Weyl's projective curvature tensor W_{jkh}^i is generalized birecurrent, if and only if the equation (2.7) is equal to zero and Wely's projective torsion tensor W_{kh}^i is given by the equation (2.10), if and only if the equation (2.11) is equal to zero.

In GW^h - BIR - affinely connected space, Weyl's projective curvature tensor W_{jkh}^i is Gh - BIR if and only if the equation (3.3) is equal to zero, GW^h - BIR - affinely connected space, Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (3.5) and (3.7), respectively, if and only if the equations (3.6) and (3.8), respectively are equals to zero and in GW^h - BIR - affinely connected space, Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by equations (3.10) and (3.12), if and only if the equations (3.11) and (3.13) are hold good.

Authors recommend the need for continuing research and development in generalized W^h -birecurrent Finsler spaces and interlard it with the properties of special spaces for Finsler space.

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