Recurrence Decompositions in Finsler Space

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Abstract

Finsler geometry is a kind of differential geometry originated by P. Finsler. Indeed, Finsler geometry has several uses in a wide variety and it is playing an important role in differential geometry and applied mathematics of problems in physics relative, manual footprint. It is usually considered as a generalization of Riemannian geometry. In the present paper, we introduced some types of generalized W^h -birecurrent Finsler space, generalized W^h -birecurrent affinely connected space and we defined a Finsler space F_n for Weyl's projective curvature tensor W_{jkh}^i satisfies the generalized-birecurrence condition with respect to Cartan's connection parameters Γ_{kh}^{*i} , such that given by the condition (2.1), where $|m|n$ is h-covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^m and x^n , successively, λ_{mn} and μ_{mn} are non-null covariant vectors field and such space is called as a generalized W^h -birecurrent space and denoted briefly by GW^h -BRF $_n$. We have obtained some theorems of generalized W^h -birecurrent affinely connected space for the h-covariant derivative of the second order for Weyl's projective torsion tensor W_{kh}^i , Weyl's projective deviation tensor W_h^i in our space. We have obtained the necessary and sufficient condition for some tensors in our space.

Keywords: Generalized W^h -birecurrent affinely connected space, Generalized W^h -Birecurrent space, Weyl's projective curvature tensor W_{jkh}^i , Finsler space F_n .
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1. Introduction

The recurrent for different curvature tensors have been discussed by Al-Qashbari [1], Dikshit [2], Matsumoto [3], Pandey [4], Pandey, et al. [5], Qasem [6], Qasemet al. [7, 8] and others. Ahsanand Ali [9] who discussed a recurrent curvature tensor on some properties of W-curvature tensor of Weyl's projective curvature tensor W_{jkh}^i is skew-symmetric, it is indices k and h. The generalized birecurrent space was studied by Hadi [10], Qasem and Hadi [11]. Also, W-generalized birecurrent space studied by Qasem and Saleem [12] and others.

An n-dimensional Finsler space, Figure 1, equipped with the metric function f satisfies the requisite conditions [13]. Consider the components of the corresponding metric tensor

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g_{ij} , Cartan's connection parameters Γ_{jk}^{*i} and Berwald's connection parameters G_{jk}^i (the indices $i, j, k, \dots$ assume positive integral values from 1 to n). These are symmetric in their lower indices.

Figure 1: Finsler Space as a Locally Minkowskian Space

Figure 2: Metric Tensor g_{ij}

Figure 3: Metric Tensor g^{ij}

The vectors y_i and y^i satisfy the following relations [13]:

$$y_i = g_{ij} y^j, \quad (1.1)$$

$$y_i y^i = F^2, \quad (1.2)$$

$$\partial_i y_j = g_{ij}, \quad (1.3)$$

$$\partial_j y^i = \delta_j^i. \quad (1.4)$$

The metric tensor g_{ij} Figure 2, the metric tensor g^{ij} Figure 3 and the vector y^i are covariant constant with respect to the above process.

$$g_{ij|k} = 0, \quad (1.5)$$

$$y_{|k}^i = 0, \quad (1.6)$$

$$g_{|k}^{ij} = 0., \quad (1.7)$$

The h -covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j , successively, we get

$$X_{|k|j}^i = \partial_j (X_{|k}^i) - (X_{|r}^i) \Gamma_{kj}^{*r} + (X_{|k}^r) \Gamma_{rj}^{*i} - \partial_r (X_{|k}^i) \Gamma_{js}^{*r} y^s \quad (1.8)$$

Taking skew-symmetric part with respect to the indices k and j , we get the commutation formula for h -covariant differentiation as follows [13]:

$$X_{|k|j}^i - X_{|j|k}^i = X^r K_{rkj}^i - (\partial_r X^i) K_{skj}^r y^s, \quad (1.9)$$

where

$$K_{rkj}^i := \partial_j \Gamma_{kr}^{*i} + (\partial_l \Gamma_{rj}^{*i}) G_k^l + \Gamma_{mj}^{*i} \Gamma_{kr}^{*m} - \frac{j}{k}. \quad (1.10)$$

where the indices $i, j, k, \dots$ are positive integral values from 1 to n , and $-\frac{j}{k}$ means the subtraction from the former term by interchanging the indices k and j .

The tensor K_{rkj}^i as defined above is called Cartan's fourth curvature tensor.

The process of h -covariant differentiation, with respect to x^k , commute with partial differentiation with respect to y^j for arbitrary vector field X^i , according to [13],

$$\partial_j (X_{|k}^i) = (\partial_j X^i)_{|k} + X^r (\partial_j \Gamma_{rk}^{*i}) - (\partial_r X^i) P_{jk}^r \quad (1.11)$$

and

$$\partial_j \Gamma_{rk}^{*i} = \Gamma_{rkj}^{*i}. \quad (1.12)$$

The tensor W_{jkh}^i is known as projective curvature tensor (generalized Wely's projective curvature tensor), the tensor W_{jk}^i is known as projective torsion tensor (Wely's torsion tensor) and the tensor W_j^i is known as projective deviation tensor (Wely's deviation tensor) are defined by

$$\begin{aligned} W_{jkh}^i &= H_{jkh}^i + \frac{2\delta_j^i}{n+1} H_{hk} + \frac{2y^i}{n+1} \partial_j H_{kh} + \frac{\delta_k^i}{n^2-1} (nH_{jh} + H_{hj} + y^r \partial_j H_{hr}) \\ &\quad - \frac{\delta_h^i}{n^2-1} (nH_{jk} + H_{kj} + y^r \partial_j H_{kr}), \end{aligned} \quad (1.13)$$

$$W_{jk}^i = H_{jk}^i + \frac{y^i}{n+1} H_{jk} + \frac{2\delta_j^i}{n^2-1} (nH_{|k} - y^r H_{k|r}), \quad (1.14)$$

and

$$W_j^i = H_j^i - H \delta_j^i - \frac{1}{n+1} (\partial_r H_j^r - \partial_j H) y^i, \quad (1.15)$$

respectively.

The tensors W_{jkh}^i , W_{jk}^i and W_j^i are satisfying the following identities [13]

$$W_{jkh}^i y^j = W_{kh}^i, \quad (1.16)$$

$$W_{jk}^i y^j = W_k^i, \quad (1.17)$$

$$\partial_j W_{kh}^i = W_{jkh}^i, \quad (1.18)$$

$$\partial_k W_h^i = W_{kh}^i. \quad (1.19)$$

The projective curvature tensor W_{jkh}^i is skew-symmetric in its indices k and h see the definition in (1.13).

An affinely connected space has some properties as follows:

$$G_{jkh}^i = 0, \text{ and } C_{ijk|h} = 0 \quad (1.20)$$

Remark 1.1. An affinely connected space or Berwald space characterized by any one of the above two equivalent conditions. Also, we have the following properties.

The connection parameters Γ_{jk}^{*i} of Cartan and G_{jk}^i of Berwald coincide in an affinely connected space and they are independent of the directional arguments [13], i.e.

$$G_{jkh}^i = \dot{\partial}_j G_{kh}^i = 0, \quad (1.21)$$

$$\dot{\partial}_j \Gamma_{kh}^{*i} = 0, \quad (1.22)$$

$$y_r G_{ijk}^r = -2 C_{ijk|h} y^h = -2 P_{ijk} = 0. \quad (1.23)$$

2. On Necessary and Sufficient Condition of Generalized W^h -Birecurrent

A Finsler space F_n for which Weyl's projective curvature tensor W_{jkh}^i satisfies the recurrence property with respect to Cartan's coefficient connection parameters Γ_{jk}^{*i} which is characterized by the condition [13]

$$W_{jkh|m}^i = a_m W_{jkh}^i + b_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}), \quad W_{jkh}^i \neq 0, \quad (2.1)$$

where $|m$ is h -covariant derivative of first order (Cartan's second kind covariant differential operator) with respect to x^m , the quantities a_m and b_m are non-null covariant vectors field. It's space is called as a generalized W^h -recurrent space and is denoted briefly by GW^h - RF_n .

Taking the h -covariant derivative for (2.1) with respect to x^n and using (1.5), we get

$$W_{jkh|m|n}^i = a_{m|n} W_{jkh}^i + a_m W_{jkh|n}^i + b_{m|n} (\delta_h^i g_{jk} - \delta_k^i g_{jh})$$

where $g_{jk|n} = 0$.

In view of (2.1), the above equation yields

$$W_{jkh|m|n}^i = \lambda_{mn} W_{jkh}^i + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}), \quad (2.2)$$

where $\lambda_{mn} = a_{m|n} + a_m a_n$, $\mu_{mn} = a_m a_n + b_{m|n}$ and $|m|n$ is h -covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^m and x^n , successively, λ_{mn} and μ_{mn} are non-null covariant vectors field and such space is called as a generalized W^h -birecurrent space and denoted briefly by GW^h - $BIRF_n$.

Result Every generalized W^h -recurrent space is generalized W^h -birecurrent space.

Transvecting the condition (2.2) by y^j and by y^k , successively, using (1.6), (1.16), (1.17), (1.1) and (1.2), we get

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\delta_h^i y_k - \delta_k^i y_h), \quad (2.3)$$

$$W_{h|m|n}^i = \lambda_{mn} W_h^i + \mu_{mn} (\delta_h^i F^2 - y_h y^i), \quad (2.4)$$

Theorem 2.1. In GW^h - $BIRF_n$, the h -covariant derivative of the second order for Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (2.3) and (2.4), respectively.

Differentiating partially of (2.3) with respect to y^j , using (1.23) and (1.3), we get

$$\begin{aligned} \partial_r (W_{kh|m|n}^i) &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &\quad + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned}$$

Using the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $W_{kh|m}^i$, in the above equation, we get

$$\begin{aligned} &\left\{ \partial_r W_{kh|m}^i \right\}_{|n} + W_{kh|m}^r (\partial_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\partial_j \Gamma_{kn}^{*r}) \\ &\quad - W_{kr|m}^i (\partial_j \Gamma_{hn}^{*r}) - W_{kh|r}^i (\partial_j \Gamma_{mn}^{*r}) - (\partial_r W_{kh|m}^i) P_{jn}^r \\ &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &\quad + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned} \quad (2.5)$$

Again, applying the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor (W_{kh}^i) , in the equation (2.5) and using (1.18), we obtain

$$\begin{aligned} &\left\{ W_{jkh|m}^i + W_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - W_{rh}^i (\partial_j \Gamma_{km}^{*r}) - W_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - (\partial_r W_{kh}^i) P_{jm}^r \right\}_{|n} \\ &\quad + W_{kh|m}^r (\partial_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\partial_j \Gamma_{kn}^{*r}) - W_{kr|m}^i (\partial_j \Gamma_{hn}^{*r}) \\ &\quad - W_{kh|r}^i (\partial_j \Gamma_{mn}^{*r}) - (\partial_r W_{kh|m}^i) P_{jn}^r \\ &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &\quad + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}), \end{aligned}$$

which can be written as

$$\begin{aligned} &W_{jkh|m|n}^i + \left\{ W_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - W_{rh}^i (\partial_j \Gamma_{km}^{*r}) - W_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - (\partial_r W_{kh}^i) P_{jm}^r \right\}_{|n} \\ &\quad + W_{kh|m}^r (\partial_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\partial_j \Gamma_{kn}^{*r}) - W_{kr|m}^i (\partial_j \Gamma_{hn}^{*r}) \\ &\quad - W_{kh|r}^i (\partial_j \Gamma_{mn}^{*r}) - (\partial_r W_{kh|m}^i) P_{jn}^r \\ &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &\quad + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned} \quad (2.6)$$

This shows that

$$W_{jkh|m|n}^i = \lambda_{mn} W_{jkh}^i + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh})$$

if and only if

$$\begin{aligned}
& \{ W_{kh}^r (\partial_j \Gamma_{rm}^{*i}) - W_{rh}^i (\partial_j \Gamma_{km}^{*r}) - W_{kr}^i (\partial_j \Gamma_{hm}^{*r}) - (\partial_r W_{kh}^i) P_{jm}^r \}_{|n} \\
& + W_{kh|m}^r (\partial_j \Gamma_{rn}^{*i}) - W_{rh|m}^i (\partial_j \Gamma_{kn}^{*r}) - W_{kr|m}^i (\partial_j \Gamma_{hn}^{*r}) \\
& - W_{kh|r}^i (\partial_j \Gamma_{mn}^{*r}) - (\partial_r W_{kh|m}^i) P_{jn}^r - (\partial_j \lambda_{mn}) W_{kh}^i \\
& - (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) = 0.
\end{aligned} \tag{2.7}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 2.2. In GW^h - $BIRF_n$, Weyl's projective curvature tensor W_{jkh}^i is generalized birecurrent, if and only if the equation (2.7) is equal to zero.

Differentiating partially of (2.4) with respect to y^k , using (1.19) and (1.3), we obtain

$$\begin{aligned}
\partial_k (W_{h|m|n}^i) &= (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) \\
&+ \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h).
\end{aligned}$$

Using the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $W_{h|m}^i$, in the above equation, we have

$$\begin{aligned}
& (\partial_k W_{h|m}^i)_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*i}) - W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) \\
& - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\
& = (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) \\
& + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h).
\end{aligned} \tag{2.8}$$

Again, apply the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor (W_h^i) , in the equation (2.8) and using (1.19), we get

$$\begin{aligned}
& \{ W_{kh|m}^i + W_h^r (\partial_j \Gamma_{hm}^{*r}) - W_r^i (\partial_k \Gamma_{hm}^{*r}) - (\partial_r W_h^i) P_{km}^r \}_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) \\
& - W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\
& = (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i \\
& + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h),
\end{aligned}$$

which can be written as

$$\begin{aligned}
& W_{kh|m|n}^i + \{ W_h^r (\partial_j \Gamma_{hm}^{*r}) - W_r^i (\partial_k \Gamma_{hm}^{*r}) - (\partial_r W_h^i) P_{km}^r \}_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) \\
& - W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\
& = (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i \\
& + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h).
\end{aligned} \tag{2.9}$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h) \quad (2.10)$$

if and only if

$$\begin{aligned} & \{ W_h^r (\partial_j \Gamma_{hm}^{*r}) - W_r^i (\partial_k \Gamma_{hm}^{*r}) - (\partial_r W_h^i) P_{km}^r \}_{|n} + W_{h|m}^r (\partial_k \Gamma_{rn}^{*r}) \\ & - W_{r|m}^i (\partial_k \Gamma_{hn}^{*r}) - W_{h|r}^i (\partial_k \Gamma_{mn}^{*r}) - (\partial_r W_{h|m}^i) P_{kn}^r \\ & = (\partial_k \lambda_{mn}) W_h^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i). \end{aligned} \quad (2.11)$$

Therefore, it is concluded the following.

Theorem 2.3. In $GW^h - BIRF_n$, Weyl's projective torsion tensor W_{kh}^i is given by (2.10), if and only if the equation (2.11) holds good.

3. Affinely Connected Space on Generalized W^h -Birecurrent

Let us introduce definition of $GW^h - BIRF_n$ to be affinely connected space as follows:

Definition 3.1. The generalized W^h -birecurrent space which is an affinely connected space [satisfies any one of the conditions (1.21), (1.22) and (1.23)] will be called generalized W^h -birecurrentaffinely connected space and will denote it briefly by $GW^h - BIR$ -affinely connected space.

Remark 3.2. It will be sufficient to call the tensor which satisfies the condition of $GW^h - BIR$ -affinely connected space as generalized h -birecurrent tensor (briefly $Gh - BIR$).

By using the conditions (1.21), (1.22) and (1.23), the equation (2.6) reduce to

$$\begin{aligned} W_{jkh|m|n}^i &= (\partial_j \lambda_{mn}) W_{kh}^i + \lambda_{mn} W_{jkh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) \\ &+ \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}). \end{aligned} \quad (3.1)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{jkh}^i + \mu_{mn} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \quad (3.2)$$

if and only if

$$(\partial_j \lambda_{mn}) W_{kh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) = 0. \quad (3.3)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.3. In $GW^h - BIR$ - affinely connected space, Weyl's projective curvature tensor W_{jkh}^i is $Gh - BIR$ if and only if the equation (3.3) is equal to zero.

Transvecting the equation (3.1) by y^j , using (1.6), (1.16) and (1.1), we get

$$W_{kh|m|n}^i = (\partial_j \lambda_{mn}) W_{kh}^i y^j + \lambda_{mn} W_{kh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) y^j + \mu_{mn} (\delta_h^i y_k - \delta_k^i y_h). \quad (3.4)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\delta_h^i y_k - \delta_k^i y_h) \quad (3.5)$$

if and only if

$$(\partial_j \lambda_{mn}) W_{kh}^i + (\partial_j \mu_{mn}) (\delta_h^i y_k - \delta_k^i y_h) = 0, \text{ since } y^j \neq 0. \quad (3.6)$$

Transvecting the equation (3.4) by y^k , using (1.6), (1.16), (1.2) and (1.1), we obtain

$$W_{h|m|n}^i = (\partial_j \lambda_{mn}) W_h^i y^j + \lambda_{mn} W_h^i + (\partial_j \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^j + \mu_{mn} (\delta_h^i F^2 - y_h y^i).$$

This shows that

$$W_{h|m|n}^i = \lambda_{mn} W_h^i + \mu_{mn} (\delta_h^i F^2 - y_h y^i) \quad (3.7)$$

if and only if

$$(\partial_j \lambda_{mn}) W_h^i y^j + (\partial_j \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^j = 0. \quad (3.8)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.4. In GW^h - BIR -affinely connected space, Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (3.5) and (3.7), respectively, if and only if the equations (3.6) and (3.8), respectively are equals to zero.

By using the conditions (1.21), (1.22) and (1.23), the equation (2.9) reduce to

$$W_{kh|m|n}^i = (\partial_k \lambda_{mn}) W_h^i + \lambda_{mn} W_{kh}^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h). \quad (3.9)$$

This shows that

$$W_{kh|m|n}^i = \lambda_{mn} W_{kh}^i + \mu_{mn} (\partial_k \delta_h^i F^2 - \delta_k^i y_h) \quad (3.10)$$

if and only if

$$(\partial_k \lambda_{mn}) W_h^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) = 0. \quad (3.11)$$

Therefore, it is concluded the following.

Theorem 3.5. In GW^h - BIR - affinely connected space, Wely's projective torsion tensor W_{kh}^i is given by (3.10) if and only if the equation (3.11) hold good.

Transvecting the equation (3.9) by y^k , using (1.6), (1.17), (1.2) and (1.1), we get

$$W_{h|m|n}^i = (\partial_k \lambda_{mn}) W_h^i y^k + \lambda_{mn} W_h^i + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^k + \mu_{mn} (\delta_h^i F^2 - y_h y^i).$$

This shows that

$$W_{h|m|n}^i = \lambda_{mn} W_h^i + \mu_{mn} (\delta_h^i F^2 - y_h y^i) \quad (3.12)$$

if and only if

$$(\partial_k \lambda_{mn}) W_h^i y^k + (\partial_k \mu_{mn}) (\delta_h^i F^2 - y_h y^i) y^k = 0. \quad (3.13)$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.6. In GW^h - BIR -affinely connected space, Wely's projective deviation tensor W_h^i is given by (3.12) if and only if the equation (3.13) hold good.

4. Conclusions and Recommendations

A Finsler space is called generalized W^h -birecurrent space if it satisfies the condition (2.1). In GW^h - BIR, the h-covariant derivative of the second order for Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (2.3) and (2.4), respectively. In GW^h - BIR, the necessary and sufficient condition of Weyl's projective curvature tensor W_{jkh}^i is generalized birecurrent, if and only if the equation (2.7) is equal to zero and Wely's projective torsion tensor W_{kh}^i is given by the equation (2.10), if and only if the equation (2.11) is equal to zero.

In GW^h - BIR - affinely connected space, Weyl's projective curvature tensor W_{jkh}^i is Gh - BIR if and only if the equation (3.3) is equal to zero, GW^h - BIR - affinely connected space, Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by (3.5) and (3.7), respectively, if and only if the equations (3.6) and (3.8), respectively are equals to zero and in GW^h - BIR - affinely connected space, Wely's projective torsion tensor W_{kh}^i and Wely's projective deviation tensor W_h^i are given by equations (3.10) and (3.12), if and only if the equations (3.11) and (3.13) are hold good.

Authors recommend the need for continuing research and development in generalized W^h -birecurrent Finsler spaces and interlard it with the properties of special spaces for Finsler space.

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