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Positive solutions for generalized two-term fractional differential equations with integral boundary conditions

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Abstract

In this paper, we consider a class of boundary value problems for nonlinear two-term fractional differential equations with integral boundary conditions involving two ψ-Caputo fractional derivative. With the help of properties Green function, the fixed point theorems of Schauder and Banach, and the method of upper and lower solutions, we derive the existence and uniqueness of positive solution of proposed problem. Finally, an example is provided to illustrate the acquired results.

Keywords: fractional differential equations, ψ-Caputo fractional derivatives, Green function, positive solution, fixed point theorem.
2010 MSC: 34A08, 34B15, 34A12, 34B18, 47H10.

1. Introduction

Fractional calculus can be thought of as a generalization of calculus with integer order. Recently various definitions of derivatives and integrals of an arbitrary order have appeared. Despite the fact that inside the start, fractional calculus had an advancement as a simply purely mathematical idea, in current quite a while its utilization had moreover unfurl into numerous fields such as physics, mechanics, chemistry, biology, engineering, bioengineering and electrochemistry, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein. So in the literature, several studies handled comparable topics to various operators, as an instance, Riemann-Liouville [10, 11], Caputo [12, 13], Erdelyi–Kober [14, 15], generalized Caputo [16, 17], Hilfer [2], generalized Hilfer [18], Hadamard [19, 20], generalized Hadamard [21], Katugampola [22, 23], generalized Katugampola [24], Caputo-Fabrizio [25], Atangana-Baleanu [26], etc.

In this paper, we concentrate on the positivity of the solutions for the following nonlinear fractional differential equations (FDEs) with integral boundary conditions

$$\begin{cases} {}^{C}\mathfrak{D}_{0}^{\alpha,\psi}u(t) + f(t,u(t)) = {}^{C}\mathfrak{D}_{0}^{\beta,\psi}g(t,u(t)), & 0 < t < 1, \\ u(0) = 0, \ u(1) = \mathfrak{I}_{0}^{\alpha-\beta,\psi}g(1,u(1)), \end{cases} \tag{1.1}$$

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where $\mathfrak{D}_0^{\theta,\psi}$ is the generalized Caputo fractional derivative of order θ, $\theta \in \{\alpha, \beta : 1 < \alpha \leqslant 2, 0 < \beta \leqslant \alpha - 1\}$, $f, g : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are given continuous functions with $f(t,u)$ and $g(t,u)$ are not required any monotone assumption, $g(0,u(0)) = 0$, and

$$\mathfrak{I}_0^{\alpha-\beta,\psi} g(1,u(1)) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-\beta-1} g(s,u(s))ds.$$

In the literature, nonlinear one-term FDEs of the form

$$\mathfrak{D}_0^{\alpha} u(t) = f(t,x(t)) \text{ and } \mathfrak{D}_0^{\alpha,\psi} u(t) = f(t,x(t))$$

have been considered by many authors (see [27, 28, 29, 30, 31]). More generally, we can indicate to [32, 33, 34, 35, 36, 37] on the equations of kind

$$\mathfrak{D}_0^{\alpha} u(t) = f(t,x(t),\mathfrak{D}_0^{\alpha} u(t)) \text{ and } \mathfrak{D}_0^{\alpha,\psi} u(t) = f(t,x(t),\mathfrak{D}_0^{\alpha,\psi} u(t)).$$

Recently, the authors in [38] investigated the positivity results of the Caputo-type problem

$$\begin{cases} {}^C\mathfrak{D}_0^{\alpha} u(t) = f(t,u(t)) + {}^C\mathfrak{D}_0^{\alpha-1} g(t,u(t)), & 0 < t \leqslant T, \\ u(0) = \theta_1 > 0, \; u'(0) = \theta_2 > 0 \end{cases} \tag{1.2}$$

by using the method of upper and lower solutions and some fixed point theorems.

Very recently, Xu and Han in [39] studied the positivity results of the following nonlinear two-term FDEs

$$\begin{cases} \mathfrak{D}_0^{\alpha} u(t) + f(t,u(t)) = \mathfrak{D}_0^{\beta} g(t,u(t)), & 0 < t < 1, \\ u(0) = 0, \\ u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-\beta-1} g(s,u(s))ds, \end{cases}$$

in the Riemann–Liouville derivatives sense. Also, the positivity of solutions for the following nonlinear Hadamard-type FDEs

$$\begin{cases} \mathfrak{D}_1^{\alpha} u(t) + f(t,u(t)) = \mathfrak{D}_1^{\beta} g(t,u(t)), & 1 < t < e, \\ u(1) = 0, \\ u(e) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^e (\log \frac{e}{s})^{\alpha-\beta-1} g(s,u(s)) \frac{ds}{s}, \end{cases}$$

is another great study by Ardjouni in [40].

Over time, due to the operator's reliance on the integration kernel, many types of new fractional derivatives and integrals emerge to obtain a distinct kernel and this makes the range of definitions wide-ranging, due to the evolution of these operators, we refer here to some recent results that dealt with the existence of solution and positive solution to various problems of FDEs [41, 42, 43, 44, 45, 46].

To the best of our knowledge, no article has studied the existence of positive solutions for nonlinear FDEs with integral boundary conditions (1.1). This problem has two nonlinear terms and includes two generalized fractional derivatives. Compared to many two-term FDEs, the type of problem we considered is more general. To show the existence and uniqueness of the positive solution, we transform (1.1) into a fractional integral equation with the aid of the Green function, and then by the method of upper and lower solutions and use Schauder and Banach fixed point theorems we obtain our results.

The organization of this paper as follows: the representation of the problem with a brief survey for literature is presented in the introduction. In Section 2, we give the preliminary facts and some useful lemmas that will be used throughout the paper. In Section 3, we prove the existence and uniqueness of positive solutions to problem (1.1) via some fixed point theorems. An illustrative example is reported to justify our findings is presented in Section 4. Finally, the conclusions close the paper.

2. Preliminaries

Let $\Omega = [0,1]$ be a compact interval subset $\mathbb{R}$. By $X = C(\Omega, \mathbb{R})$ we indicate the Banach space of all continuous functions from Ω into $\mathbb{R}$ with the norm $\|u\| = \max_{t\in\Omega} |u(t)|$. Define the following space

$$\varepsilon = \{u \in X : u(t) \geqslant 0, \ \forall t \in \Omega\} \subset X.$$

By a positive solution $u \in X$, we mean a function $u(t) > 0$, for $t \in \Omega$.

Definition 2.1. Let $a, b \in \mathbb{R}^+$ such that $b > a$. For any $u \in [a, b]$, we define respectively the upper and lower contral functions as follows:

$$U(t,u) = \sup_{a\leqslant v\leqslant u} f(t,v), \ \text{and} \quad L(t,u) = \inf_{u\leqslant v\leqslant b} f(t,v)$$

$$U^*(t,u) = \sup_{a\leqslant v\leqslant u} g(t,v), \ \text{and} \quad L^*(t,u) = \inf_{u\leqslant v\leqslant b} g(t,v)$$

Certainly, the functions $U(t,u)$, $L(t,u)$, $U^*(t,u)$and $L^*(t,u)$ are monotonous nondecreasing with respect to u. Moreover, we have

$$L(t,u) \leqslant f(t,u) \leqslant U(t,u),$$

$$L^*(t,u) \leqslant g(t,u) \leqslant U^*(t,u).$$

We state some needful definitions and lemmas that will be used throughout this paper.

Definition 2.2. ([10]) Let $\alpha \in \mathbb{R}^+$, $\psi \in C^n[a,b]$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$, and $h : [a,b] \longrightarrow \mathbb{R}$ an integrable function. The left-sided ψ-Riemann-Liouville fractional integral and derivative of h of order α are given by

$$\mathfrak{I}_{a^+}^{\alpha,\psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s)ds,$$

and

$$\mathfrak{D}_{a^+}^{\alpha,\psi} h(t) = \mathfrak{D}^{n,\psi}\, \mathfrak{I}_{a^+}^{n-\alpha,\psi} h(t),$$

respectively, where $\mathfrak{D}^{n,\psi} = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n$, $n = [\alpha] + 1$, and $\Gamma(\cdot)$ is a gamma funcion.

Definition 2.3. [47]. Let $\alpha > 0$, $h, \psi \in C^n[a,b]$ two functions such that ψ is increasing function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$. The left-sided ψ-Caputo fractional derivative of h of order α is defined by

$$^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}h(t) = \mathfrak{I}_{a^+}^{n-\alpha,\psi}\mathfrak{D}^{n,\psi}h(t),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, and $n = \alpha$ for $\alpha \in \mathbb{N}$.

Lemma 2.4. *[47]. Let $\alpha > 0$. Then the following properties hold:*

1. *If $h \in C^1[a,b]$, then*
$$^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}\mathfrak{I}_{a^+}^{\alpha,\psi}h(t) = h(t).$$
2. *If $\psi, h \in C^n[a,b]$, then*
$$\mathfrak{I}_{a^+}^{\alpha,\psi}\ {}^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}h(t) = h(t) - \sum_{k=0}^{n-1}\frac{h_{\psi}^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k.$$
where $h_{\psi}^{[k]}(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k h(t)$ and $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$.
In particular, if $1 < \alpha < 2$, then
$$\mathfrak{I}_{a^+}^{\alpha,\psi}\ {}^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}h(t) = h(t) - h(a) - h'_{\psi}(a)(\psi(t)-\psi(a)),$$
where $h'_{\psi}(t) = \frac{h'(t)}{\psi'(t)}$.

Lemma 2.5. *[41]. Let $\alpha > 0$, $h \in C[a,b]$ and let $\psi \in C^1[a,b]$. Then for all $t \in [a,b]$*

(i) $\mathfrak{I}_{a^+}^{\alpha,\psi}(\cdot)$ *is bounded from $C[a,b]$ to $C[a,b]$.*

(ii) $\mathfrak{I}_{a^+}^{\alpha,\psi}h(a) = \lim_{t\to a^+}\mathfrak{I}_{a^+}^{\alpha,\psi}h(t) = 0.$

Lemma 2.6. *[16],[10]. Let $\alpha, \beta > 0$ and $h : [a,b] \longrightarrow \mathbb{R}$. Then*

1. $\mathfrak{I}_{a^+}^{\alpha,\psi}\mathfrak{I}_{a^+}^{\beta,\psi}h(t) = \mathfrak{I}_{a^+}^{\alpha+\beta,\psi}h(t)$.
2. $\mathfrak{I}_{a^+}^{\alpha,\psi}[\psi(t)-\psi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\psi(t)-\psi(a)]^{\alpha+\beta-1}$.
3. $^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}[\psi(t)-\psi(a)]^k = 0$, $\forall k \in \{0,1,...,n-1\}$, $n \in \mathbb{N}$.

Now, we state some fixed point theorems that enable us to demonstrate the existence and uniqueness of a positive solution of (1.1).

Definition 2.7. Let U be Banach space and $\phi : U \longrightarrow U$. The operator ϕ is a contraction operator if there is an $\lambda \in (0,1)$ such that $u, v \in U$ imply

$$\|\phi u - \phi v\| \leqslant \lambda\|u - v\| \tag{2.1}$$

Theorem 2.8. *Let K be a nonempty closed convex subset of a Banach space U and $\phi : K \longrightarrow K$ be a contraction operator. Then there is a unique $u \in K$ with $\phi u = u$.*

Theorem 2.9. *Let K be a nonempty bounded, closed and convex subset of a Banach space U and $\phi : K \longrightarrow K$ be a completely continuous operator. Then ϕ has a fixed point in K.*

3. Main results

In this section, we prove the existence and uniqueness results of (1.1) under Banach fixed point theorem and Schaefer fixed point theorem. Before starting the proof we will give the following fundamental lemma:

Lemma 3.1. *Let* $1 < \alpha \leqslant 2$, $u \in X$, $\psi, u'_\psi \in X^1$ *and* $f, g : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are continuous functions with* $g(0, u(0)) = 0$. *Then* u *is a solution of the boundary value problem (1.1) if and only if*

$$u(t) = \int_0^1 G_\psi(t,s)\psi'(s)f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1}g(s,u(s))ds. \tag{3.1}$$

where

$$G_\psi(t,s) = \frac{\Upsilon(t)}{\Gamma(\alpha)}\begin{cases} [\psi(1)-\psi(s)]^{\alpha-1} - \frac{1}{\Upsilon(t)}[\psi(t)-\psi(s)]^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ [\psi(1)-\psi(s)]^{\alpha-1}, \ 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{3.2}$$

with $\Upsilon(t) := \frac{\mathcal{N}(t)}{\mathcal{N}(1)}$, $\mathcal{N}(t) := [\psi(t)-\psi(0)]$ *and* $u'_\psi(t) = \frac{u'(t)}{\psi'(t)}$.

Proof. From Lemma 2.4, applying ψ-Reimann-Liouville fractional operator $\mathfrak{I}_0^{\alpha,\psi}$ on both sides of (1.1), it follows that

$$\begin{aligned} & u(t) - u(0) - u'_\psi(0)\,[\psi(t)-\psi(0)] \\ = \ & -\mathfrak{I}_0^{\alpha,\psi} f(t,u(t)) + \mathfrak{I}_0^{\alpha,\psi}\, {}^{C}\mathfrak{D}_0^{\beta,\psi} g(t,u(t))) \\ = \ & -\mathfrak{I}_0^{\alpha,\psi} f(t,u(t)) + \mathfrak{I}_0^{\alpha-\beta,\psi}(\mathfrak{I}_0^{\beta,\psi}\mathfrak{D}_0^{\beta,\psi} g(t,u(t))) \\ = \ & -\mathfrak{I}_0^{\alpha,\psi} f(t,u(t)) + \mathfrak{I}_0^{\alpha-\beta,\psi}(g(t,u(t)) - g(0,u(0))) \\ = \ & -\mathfrak{I}_0^{\alpha,\psi} f(t,u(t)) + \mathfrak{I}_0^{\alpha-\beta,\psi} g(t,u(t) - \frac{g(0,u(0))}{\Gamma(\alpha-\beta+1)}[\psi(t)-\psi(0)]^{\alpha-\beta}, \end{aligned}$$

where $u'_\psi(0) = \frac{u'(0)}{\psi'(0)}$. Then, by the initial condition $u(0) = 0$, and fact that $g(0, u(0) = 0$, we get

$$u(t) = u'_\psi(0)\,[\psi(t)-\psi(0)] - \mathfrak{I}_0^{\alpha,\psi} f(t,u(t)) + \mathfrak{I}_0^{\alpha-\beta,\psi} g(t,u(t). \tag{3.3}$$

By the boundary conditions $u(1) = \mathfrak{I}_0^{\alpha-\beta,\psi} g(1,u(1))$, we obtain

$$u'_\psi(0) = \frac{1}{[\psi(1)-\psi(0)]}\mathfrak{I}_0^{\alpha,\psi} f(1,u(1)). \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned} u(t) \ = \ & \frac{\Upsilon(t)}{\Gamma(\alpha)}\int_0^1 \psi'(s)\,[\psi(1)-\psi(s)]^{\alpha-1} f(s,u(s))ds \\ & -\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s))ds \\ & +\frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1} g(s,u(s))ds. \end{aligned} \tag{3.5}$$

By the Green function,

$$u(t) = \int_0^1 G_\psi(t,s)\psi'(s)f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1}g(s,u(s))ds.$$

For the converse, the equation (3.5) can be written as

$$u(t) = \Upsilon(t)\mathfrak{I}_0^{\alpha,\psi}f(1,u(1)) - \mathfrak{I}_0^{\alpha,\psi}f(t,u(t)) + \mathfrak{I}_0^{\alpha-\beta,\psi}g(t,u(t).$$

Applying ψ-Caputo fractional operator ${}^C\mathfrak{D}_0^{\alpha,\psi}(\cdot)$ on both sides of (3.5), and noting that

$${}^C\mathfrak{D}_0^{\alpha,\psi}\Upsilon(t) = {}^C\mathfrak{D}_0^{\alpha,\psi}\frac{\mathcal{N}(t)}{\mathcal{N}(1)} = \frac{1}{\mathcal{N}(1)}\,{}^C\mathfrak{D}_0^{\alpha,\psi}[\psi(t)-\psi(0)] = 0, \text{ for } 1<\alpha\leqslant 2,$$

we obtain

$$\begin{aligned} {}^C\mathfrak{D}_0^{\alpha,\psi}u(t) &= -\,{}^C\mathfrak{D}_0^{\alpha,\psi}\mathfrak{I}_0^{\alpha,\psi}f(t,u(t)) + {}^C\mathfrak{D}_0^{\alpha,\psi}\mathfrak{I}_0^{\alpha-\beta,\psi}g(t,u(t) \\ &= -f(t,u(t)) + {}^C\mathfrak{D}_0^{\beta,\psi}g(t,u(t). \end{aligned}$$

Taking the limits at $t\to 0$, and $t\to 1$ in equation (3.5) it follows that $u(0)=0$, and $u(1) = \mathfrak{I}_0^{\alpha-\beta,\psi}g(1,u(1))$. We proved that proplem (1.1) is equivalent to equation (3.1). □

Lemma 3.2. *The function* G_ψ *defined by (3.2) satisfies*

1. $G_\psi(t,s) > 0$ for $t,s\in(0,1)$.
2. $\Gamma(\alpha)\max_{0\leqslant t\leqslant 1} G_\psi(t,s) = [\psi(1)-\psi(s)]^{\alpha-1}$, $s\in(0,1)$.

Proof. The proof of part 1 was done, see [48]. To prove the part 2, we have $\mathcal{N}(t) = [\psi(t)-\psi(0)]$ and $\Upsilon(t) := \frac{\mathcal{N}(t)}{\mathcal{N}(1)}$. For $0\leqslant s\leqslant t\leqslant 1$, we get

$$\begin{aligned} G_\psi(t,s) &= \frac{\Upsilon(t)}{\Gamma(\alpha)}\left[[\psi(1)-\psi(s)]^{\alpha-1} - \frac{1}{\Upsilon(t)}[\psi(t)-\psi(s)]^{\alpha-1}\right] \\ &\leqslant \frac{\Upsilon(t)}{\Gamma(\alpha)}[\psi(1)-\psi(s)]^{\alpha-1} \\ &\leqslant \frac{[\psi(1)-\psi(s)]^{\alpha-1}}{\Gamma(\alpha)}, \end{aligned}$$

and for $0\leqslant t\leqslant s\leqslant 1$, we get

$$\begin{aligned} G_\psi(t,s) &= \frac{\Upsilon(t)}{\Gamma(\alpha)}[\psi(1)-\psi(s)]^{\alpha-1} \\ &\leqslant \frac{\Upsilon(s)}{\Gamma(\alpha)}[\psi(1)-\psi(s)]^{\alpha-1} \\ &\leqslant \frac{[\psi(1)-\psi(s)]^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

Therefore,

$$\max_{0\leqslant t\leqslant 1} G_\psi(t,s) = \frac{[\psi(1)-\psi(s)]^{\alpha-1}}{\Gamma(\alpha)}, \quad s\in(0,1).$$

□

Now we are able to prove more results there on existence and uniqueness of positive solution to the problem (1.1).

To use the fixed point theorem, according to Lemma 3.1, we consider the operator $\phi : X \longrightarrow X$ such that $\phi u = u$, where

$$(\phi u)(t) = \int_0^1 G_\psi(t,s)\psi'(s)f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1}g(s,u(s))ds. \tag{3.6}$$

We need the following assumptions to establish our reselts.

(H_1) Let $\overline{u}, \underline{u} \in \varepsilon$, such that $a \leqslant \underline{u} \leqslant \overline{u} \leqslant b$ and

$$\begin{cases} \mathfrak{D}_0^{\alpha,\psi}\overline{u}(t) + U(t,\overline{u}(t)) \geqslant \mathfrak{D}_0^{\alpha,\psi}U^*(t,\overline{u}(t)), \\ \mathfrak{D}_0^{\alpha,\psi}\overline{u}(t) + L(t,\underline{u}(t)) \leqslant \mathfrak{D}_0^{\alpha,\psi}L^*(t,\underline{u}(t)), \end{cases}$$

for any $t \in \Omega$, where $\overline{u}$ and $\underline{u}$ are the upper and lower solutions for (1.1) respectively.

Theorem 3.3. *Assum that (H_1) is satisfied, then the FDE (1.1) has at least one positive solution $u \in X$ satisfying $\underline{u} \leqslant u \leqslant \overline{u}$, $t \in \Omega$.*

Proof. Let $P = \{u \in X : \underline{u}(t) \leqslant u(t) \leqslant \overline{u}(t), t \in \Omega\}$ with the norm $\|u\| = \max_{0\leqslant t\leqslant 1}|u(t)|$, then we have $\|u\| \leqslant b$. Hence, P is a convex, bounded, and closed subset of the Banach space X. Moreover, the continuity of g and f implies the continuity of the operator ϕ defined by (3.6) on P. Now, if $u \in P$, there exist positive constants p_f and p_g such that

$$\max\{f(t,u(t)) : t \in \Omega, u(t) \leqslant b\} < p_f,$$

and

$$\max\{g(t,u(t)) : t \in \Omega, u(t) \leqslant b\} < p_g.$$

Then

$$\begin{aligned}
(\phi u)(t) &\leqslant \int_0^1 G_\psi(t,s)\psi'(s)f(s,u(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1}g(s,u(s))ds \\
&\leqslant \int_0^1 \max_{0\leqslant t\leqslant 1} G_\psi(t,s)\psi'(s)f(s,u(s))ds + \mathfrak{I}_0^{\alpha-\beta,\psi}g(t,u(t)) \\
&\leqslant \frac{p_f}{\Gamma(\alpha)}\int_0^1 [\psi(1)-\psi(s)]^{\alpha-1}\psi'(s)ds + \frac{p_g\,[\psi(t)-\psi(0)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\leqslant \frac{p_f}{\Gamma(\alpha+1)}[\psi(1)-\psi(0)]^{\alpha} + \frac{p_g}{\Gamma(\alpha-\beta+1)}[\psi(1)-\psi(0)]^{\alpha-\beta}.
\end{aligned}$$

Thus,

$$\|\phi u\| \leqslant \left(\frac{p_f}{\Gamma(\alpha+1)} + \frac{p_g}{\Gamma(\alpha-\beta+1)}\right)[\psi(1)-\psi(0)]^{\alpha}.$$

Hence, $\phi(P)$ is uniformly bounded. Next, we prove the equicontinuity of $\phi(P)$. Let $u \in P$, then for any $t_1, t_2 \in \Omega$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|(\phi u)(t_2) - (\phi u)(t_1)| \\
= &\left| \int_0^1 \left(G_\psi(t_2, s) - G_\psi(t_1, s) \right) \psi'(s) f(s, u(s)) ds \right. \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-\beta-1} g(s, u(s)) ds \\
&\left. - \frac{1}{\Gamma(\alpha-\beta)} \int_0^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-\beta-1} g(s, u(s)) ds \right| \\
\leqslant &\int_0^1 \left| G_\psi(t_2, s) - G_\psi(t_1, s) \right| \psi'(s) \left| f(s, u(s)) \right| ds \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^{t_1} \left([\psi(t_2) - \psi(s)]^{\alpha-\beta-1} - [\psi(t_1) - \psi(s)]^{\alpha-\beta-1} \right) \psi'(s) \left| f(s, u(s)) \right| ds \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} \left([\psi(t_2) - \psi(s)]^{\alpha-\beta-1} \right) \psi'(s) \left| f(s, u(s)) \right| ds.
\end{aligned}$$

We have

$$\begin{aligned}
\left| G_\psi(t_2, s) - G_\psi(t_1, s) \right| &= \left| \frac{\Upsilon(t_2)}{\Gamma(\alpha)} \left[[\psi(1) - \psi(s)]^{\alpha-1} - \frac{1}{\Upsilon(t_2)} [\psi(t_2) - \psi(s)]^{\alpha-1} \right] \right. \\
&\quad \left. - \frac{\Upsilon(t_1)}{\Gamma(\alpha)} \left[[\psi(1) - \psi(s)]^{\alpha-1} - \frac{1}{\Upsilon(t_1)} [\psi(t_1) - \psi(s)]^{\alpha-1} \right] \right| \\
&\leqslant \frac{\Upsilon(t_2) - \Upsilon(t_1)}{\Gamma(\alpha)} [\psi(1) - \psi(s)]^{\alpha-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
|(\phi u)(t_2) - (\phi u)(t_1)| &\leqslant p_f \frac{\Upsilon(t_2) - \Upsilon(t_1)}{\Gamma(\alpha+1)} [\psi(1) - \psi(s)]^{\alpha} \\
&\quad + \frac{p_g}{\Gamma(\alpha-\beta+1)} \left([\psi(t_2) - \psi(0)]^{\alpha-\beta} - [\psi(t_1) - \psi(0)]^{\alpha-\beta} \right).
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the previous inequality is independent of u and tends to zero. Therefore, (ϕu) is equicontinuous. The Arzela-Ascoli theorem shows that $\phi : X \longrightarrow X$ is compact. To apply Theorem 2.9 it remains to prove that $\phi P \subseteq P$. Let $u \in P$. Then by assumption (H_1) and Definition 2.1, we have

$$\begin{aligned}(\phi u)(t) &= \int_0^1 G_\psi(t,s)\psi'(s)f(s,u(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} g(s,u(s))ds \\ &\leqslant \int_0^1 G_\psi(t,s)\psi'(s)U(s,u(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} U^*(s,u(s))ds \\ &\leqslant \int_0^1 G_\psi(t,s)\psi'(s)U(s,\overline{u}(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} U^*(s,\overline{u}(s))ds \\ &\leqslant \overline{u}(t),\end{aligned}$$

and

$$\begin{aligned}(\phi u)(t) &= \int_0^1 G_\psi(t,s)\psi'(s)f(s,u(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} g(s,u(s))ds \\ &\geqslant \int_0^1 G_\psi(t,s)\psi'(s)L(s,u(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} L^*(s,u(s))ds \\ &\geqslant \int_0^1 G_\psi(t,s)\psi'(s)L(s,\underline{u}(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} L^*(s,\underline{u}(s))ds \\ &\geqslant \underline{u}(t).\end{aligned}$$

Hence, $\underline{u} \leqslant (\phi u)(t) \leqslant \overline{u}$, $t \in \Omega$, that is, $\phi(P) \subseteq P$. According to Theorem 2.9, the operator ϕ has at least one fixed point $u \in P$. Therefore, the problem (1.1) has at least one positive solution $u \in X$ and $\underline{u} \leqslant u \leqslant \overline{u}$, $t \in \Omega$. □

Next, we give further special cases of the preceding theorem.

Corollary 3.4. *Suppose that there exist positive constants k_1, k_2, k_3 and k_4 such that*

$$0 < k_1 \leqslant f(t,u(t)) \leqslant k_2 < \infty, \ (t,u) \in \Omega \times \mathbb{R}^+, \tag{3.7}$$

and

$$0 < k_3 \leqslant g(t,u(t)) \leqslant k_4 < \infty, \ (t,u) \in \Omega \times \mathbb{R}^+, \tag{3.8}$$

Then the problem (1.1) has at least one positive solution $u \in P$. *Moreover,*

$$u(t) \geqslant k_1 \int_0^1 G_\psi(t,s)\psi'(s)ds + \frac{k_3}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} ds, \tag{3.9}$$

$$u(t) \leqslant k_2 \int_0^1 G_\psi(t,s)\psi'(s)ds + \frac{k_4}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} ds. \tag{3.10}$$

Proof. Consider the following problems

$$\begin{cases} {}^C\mathfrak{D}_0^{\alpha,\psi}\overline{u}(t) + k_2 = {}^C\mathfrak{D}_0^{\beta,\psi}k_4, & 0<t<1, \\ \overline{u}(0)=0,\ \overline{u}(1) = \left(\mathfrak{J}_0^{\alpha-\beta,\psi}k_4\right)(1), \end{cases} \tag{3.11}$$

$$\begin{cases} {}^C\mathfrak{D}_0^{\alpha,\psi}\underline{u}(t) + k_1 = {}^C\mathfrak{D}_0^{\beta,\psi}k_3, & 0<t<1, \\ \underline{u}(0)=0,\ \underline{u}(1) = \left(\mathfrak{J}_0^{\alpha-\beta,\psi}k_3\right)(1), \end{cases} \tag{3.12}$$

In view of Lemma 3.1, the problems (3.11) and (3.12) are equivalent to

$$\overline{u}(t) = k_2 \int_0^1 G_\psi(t,s)\psi'(s)ds + \frac{k_4}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} ds, \tag{3.13}$$

$$\underline{u}(t) = k_1 \int_0^1 G_\psi(t,s)\psi'(s)ds + \frac{k_3}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} ds. \tag{3.14}$$

By the given assumption (3.8) and the definition of control function, we have

$$k_1 \leqslant L(t,y) \leqslant U(t,y) \leqslant k_2 < \infty, (t,y) \in \Omega \times [a,b],$$

$$k_3 \leqslant L^*(t,y) \leqslant U^*(t,y) \leqslant k_4 < \infty, (t,y) \in \Omega \times [a,b],$$

where a, b are the minimum and maximum of y on Ω. It follows that

$$y(t) \leqslant \int_0^1 G_\psi(t,s)\psi'(s)L(s,y)ds + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} L^*(s,y)ds,$$

$$z(t) \geqslant \int_0^1 G_\psi(t,s)\psi'(s)U(s,z)ds + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1} U^*(s,z)ds.$$

Obviously, (3.13) and (3.14) are the upper and lower solutions of the problem (1.1). An application of Theorem 3.3 shows that (1.1) has at least one solution $u \in P$ and satisfies $z(t) \leqslant u(t) \leqslant y(t)$. □

Corollary 3.5. *Suppose that*

$$\sigma_f < f(t,u(t)) < \gamma_f u(t) + \eta_f < \infty \text{ for } t \in \Omega,$$

$$\sigma_g < g(t,u(t)) < \gamma_g u(t) + \eta_g < \infty \text{ fort} \in \Omega,$$

where $\sigma_f, \sigma_f, \gamma_f, \gamma_g, \eta_f, \eta_g$ *are positive constants with*

$$\Phi := \left(\frac{\gamma_f}{\Gamma(\alpha+1)} + \frac{\gamma_g}{\Gamma(\alpha-\beta+1)}\right)[\psi(1)-\psi(0)]^\alpha < 1. \tag{3.15}$$

. *Then the problem (1.1) has at least a positive solution* $u \in X$.

Proof. Consider the following problem

$$\begin{cases} \mathfrak{D}_0^{\alpha,\psi}u(t) + (\gamma_f u(t) + \eta_f) = \mathfrak{D}_0^{\beta,\psi}(\gamma_g u(t) + \eta_g), \quad 0 < t < 1, \\ u(0) = 0, u(1) = \mathfrak{I}_0^{\alpha-\beta,\psi}((\gamma_g u(1) + \eta_g)). \end{cases} \tag{3.16}$$

Problem (3.16) is equivalent to fractional integral equation

$$\begin{aligned} u(t) &= \int_0^1 G_\psi(t,s)\psi'(s)(\gamma_f u(s) + \eta_f)\,ds \\ &+ \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)[(\psi(t)-\psi(s)]^{\alpha-\beta-1}(\gamma_g u(s) + \eta_g)\,ds. \end{aligned}$$

Let ϖ be a positive real number such that □

$$\varpi > (1-\Phi)^{-1}\left(\frac{\eta_f}{\Gamma(\alpha+1)} + \frac{\eta_g}{\Gamma(\alpha-\beta+1)}\right)[\psi(1)-\psi(0)]^\alpha. \tag{3.17}$$

Then, the set $\mathbb{B}_\varpi = \{u \in X : \|u\| \leqslant \varpi\}$ is convex, closed, and bounded subset of X. The operator $F : \mathbb{B}_\varpi \longrightarrow \mathbb{B}_\varpi$ defined by

$$\begin{aligned} (Fu)(t) &= \int_0^1 G_\psi(t,s)\psi'(s)(\gamma_f u(s) + \eta_f)\,ds \\ &+ \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)[(\psi(t)-\psi(s)]^{\alpha-\beta-1}(\gamma_g u(s) + \eta_g)\,ds \end{aligned}$$

is completely continuous in X as in the proof of Theorem 3.3. Moreover,

$$(Fu)(t) \leqslant \int_0^1 \max_{0\leqslant t\leqslant 1} G_\psi(t,s)\psi'(s)(\gamma_f u(s) + \eta_f)\,ds + \mathfrak{I}_0^{\alpha-\beta,\psi}(\gamma_g u(t) + \eta_g),$$

which gives

$$\begin{aligned} |(Fu)(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 [\psi(1)-\psi(0)]^{\alpha-1}\psi'(s)(\gamma_f |u(s)| + \eta_f)\,ds \\ &+ \mathfrak{I}_0^{\alpha-\beta,\psi}(\gamma_g |u(t)| + \eta_g) \\ &\leqslant \frac{\gamma_f \|u\|}{\Gamma(\alpha+1)}[\psi(1)-\psi(0)]^\alpha + \frac{\eta_f}{\Gamma(\alpha+1)}[\psi(1)-\psi(0)]^\alpha \\ &+ \frac{\gamma_g \|u\|}{\Gamma(\alpha-\beta+1)}[\psi(1)-\psi(0)]^{\alpha-\beta} \\ &+ \frac{\eta_g}{\Gamma(\alpha-\beta+1)}[\psi(1)-\psi(0)]^{\alpha-\beta} \\ &\leqslant \left(\frac{\gamma_f}{\Gamma(\alpha+1)} + \frac{\gamma_g}{\Gamma(\alpha-\beta+1)}\right)\|u\|[\psi(1)-\psi(0)]^\alpha \\ &+ \left(\frac{\eta_f}{\Gamma(\alpha+1)} + \frac{\eta_g}{\Gamma(\alpha-\beta+1)}\right)[\psi(1)-\psi(0)]^\alpha \end{aligned}$$

If $u \in \mathbb{B}_{\varpi}$, then it follows from (3.15) and (3.17) that

$$\begin{aligned}
|(Fu)(t)| &\leqslant \left(\frac{\gamma_f}{\Gamma(\alpha+1)} + \frac{\gamma_g}{\Gamma(\alpha-\beta+1)}\right)[\psi(1)-\psi(0)]^{\alpha}\varpi \\
&\quad + \left(\frac{\eta_f}{\Gamma(\alpha+1)} + \frac{\eta_g}{\Gamma(\alpha-\beta+1)}\right)[\psi(1)-\psi(0)]^{\alpha} \\
&\leqslant \Phi\varpi + (1-\Phi)\varpi = \varpi.
\end{aligned}$$

This shows that $F : \mathbb{B}_{\varpi} \to \mathbb{B}_{\varpi}$ is a compact operator. Hence, the Theorem 2.9 ensures that F has at least one fixed point in $\mathbb{B}_{\varpi}$, and then problem (3.16) has at least one positive solution $\overline{u}(t)$, where $0 < t < 1$. Therefore, if $t \in \Omega$ one can asserts that

$$\begin{aligned}
\overline{u}(t) &= \int_0^1 G_{\psi}(t,s)\psi'(s)\left(\gamma_f\overline{u}(s)+\eta_f\right)ds \\
&\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1}\left(\gamma_g\overline{u}(s)+\eta_g\right)ds \\
&= \gamma_f\int_0^1 G_{\psi}(t,s)\psi'(s)\overline{u}(s)ds + \frac{\eta_f}{\Gamma(\alpha+1)}\left[\left(\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}\right)^{1-\alpha} - 1\right][\psi(t)-\psi(0)]^{\alpha} \\
&\quad + \frac{\gamma_g}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1}\overline{u}(s)ds \\
&\quad + \frac{\eta_g}{\Gamma(\alpha-\beta+1)}[\psi(1)-\psi(0)]^{\alpha-\beta}
\end{aligned}$$

By the Definition 2.1, we obtain

$$\overline{u}(t) \geqslant \int_0^1 G_{\psi}(t,s)\psi'(s)U(s,\overline{u}(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1}U^*(s,\overline{u}(s))ds.$$

Then $\overline{u}$ is an upper positive solution of the problem (1.1). Similarly,

$$\begin{aligned}
\underline{u}(t) &= \int_0^1 G_{\psi}(t,s)\psi'(s)\sigma_f ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1}\sigma_g ds \\
&= \sigma_f\int_0^1 G_{\psi}(t,s)\psi'(s)ds + \frac{\sigma_g}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1}ds \\
&= \frac{\sigma_f}{\Gamma(\alpha+1)}\left[\left(\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}\right)^{1-\alpha} - 1\right][\psi(t)-\psi(0)]^{\alpha} \\
&\quad + \frac{\sigma_g}{\Gamma(\alpha-\beta+1)}[\psi(1)-\psi(0)]^{\alpha-\beta},
\end{aligned}$$

and by the Definition 2.1, we get

$$\underline{u}(t) \leqslant \int_0^1 G_{\psi}(t,s)\psi'(s)L(s,\underline{u}(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)\left[(\psi(t)-\psi(s)\right]^{\alpha-\beta-1}L^*(s,\underline{u}(s))ds.$$

Thus, $\overline{u}$ is a lower positive solution of problem (1.1). By Theorem 3.3, the problem (1.1) has at least one positive solution $u \in X$, where $\underline{u}(t) \leqslant u(t) \leqslant \overline{u}(t)$.

Our final result discusses the uniqueness of positive solution to (1.1) using Theorem 2.8.

Theorem 3.6. *Suppose that* $f, g : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are continuous functions, and there exist two constants* $M_1, M_2 > 0$ *such that*

$$\begin{cases} |f(t,u) - f(t,v)| \leqslant M_1 |u - v|, \\ |g(t,u) - g(t,v)| \leqslant M_2 |u - v|, \end{cases}$$

for $t \in \Omega$ and $u, v \in \mathbb{R}^+$. Then, if

$$\mathfrak{R} := \left(\frac{M_1 [\psi(1) - \psi(0)]^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_2 [\psi(1) - \psi(0)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) < 1. \tag{3.18}$$

then the problem (1.1) has a unique positive solution $u \in P$.

Proof. In view of Theorem 3.3, the problem (1.1) has at least one positive solution in P. Hence, we just prove that the operator defined by (3.6) is a contraction on P. Obviously, if $u \in P$, then $\phi u \in P$. Indeed, for any $t \in \Omega$ and $u, v \in \mathbb{R}^+$ we have

$$\begin{aligned}
\|\phi u - \phi v\| &= \max_{t\in\Omega} |(\phi u)(t) - (\phi v)(t)| \\
&\leqslant \max_{t\in\Omega} \left(\int_0^1 G_{\psi}(t,s)\psi'(s) |f(s,u(s)) - f(s,v(s))| \, ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-\beta-1} |g(s,u(s)) - g(s,v(s))| \, ds \right) \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^1 [\psi(1) - \psi(0)]^{\alpha-1} \psi'(s) M_1 \|u - v\| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-\beta-1} M_2 \|u - v\| \, ds \\
&\leqslant \left(\frac{M_1 [\psi(1) - \psi(0)]^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_2 [\psi(1) - \psi(0)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \|u - v\| \\
&= \mathfrak{R} \|u - v\| .
\end{aligned}$$

As $\mathfrak{R} < 1$, the operator ϕ is a contraction mapping due to (3.18). So,Theorem 2.8 shows that the problem (1.1) has a unique positiv solution $u \in P$. □

4. An example

Consider the Boundary fractional differential equation

$$ {}^{C}\mathfrak{D}_{0^+}^{\frac{3}{2},\psi} u(t) + \frac{1}{4+t} \left(4 + \frac{tu(t)}{(3+u(t))} \right) = {}^{C}\mathfrak{D}_{0^+}^{\frac{1}{4},\psi} \left(\frac{u(t)}{5+u(t)} \right), \quad t \in (0,1), \tag{4.1}$$

$$u(0) = 0, \; u(1) = \mathfrak{I}_0^{\frac{7}{4},\psi} \left(\frac{u(1)}{5+u(1)} \right), \tag{4.2}$$

By comparing with problem (1.1), we have: $\alpha = \frac{3}{2}$, $\beta = \frac{1}{4}$, $\alpha - \beta = \frac{5}{4}$,

$$f(t,u) = \frac{1}{4+t} \left(4 + \frac{tu}{(3+u)} \right),$$

$$g(t,u) = \frac{u}{5+u}.$$

Then, $g(0,u(0)) = 0$ and for any $u, v \in \mathbb{R}^+$ and $t \in (0,1)$, we obtain

$$|f(t,u) - f(t,v)| = \frac{1}{4+t}\left|\frac{tu}{3+u} - \frac{tv}{3+v}\right| \leqslant \frac{1}{12}|u-v| = M_1|u-v|, \text{ and}$$

$$|g(t,u) - g(t,v)| = \left|\frac{u}{5+u} - \frac{v}{5+v}\right| \leqslant \frac{1}{5}|u-v| = M_2|u-v|.$$

Take $\psi(t) = e^{\frac{t}{3}}$, for all $t \in [0,1]$. Since

$$\mathfrak{R} = \frac{\left[\sqrt[3]{e}-1\right]^{\frac{3}{2}}}{9\sqrt{\pi}} + \frac{\left[\sqrt[3]{e}-1\right]^{\frac{5}{4}}}{5\Gamma(\frac{9}{4})} \approx 0.07 < 1.$$

Thus by Theorem 3.6, the problem (4.1)-(4.2) has a unique positive solution.

Moreover, since $f(t,u)$ and $g(t,u)$ are nondecreasing on u,

$$\lim_{u\to\infty} g(t,u) = 1, \ \lim_{u\to\infty} f(t,u) = 1,$$

and

$$\frac{4}{5} \leqslant f(t,u) \leqslant 1, \ \frac{1}{5} \leqslant g(t,u) \leqslant 1,$$

for $t \in [1,0]$, and $u \in \mathbb{R}^+$. Therefore, Corollary 3.4 holds with $k_1 = \frac{4}{5}$, $k_2 = 1$, $k_3 = \frac{1}{5}$ and $k_4 = 1$. Hence, the problem (4.1)-(4.2) has a positive solution which verifies $\underline{u}(t) \leqslant u(t) \leqslant \overline{u}(t)$ where

$$\begin{aligned}
\overline{u}(t) &= \frac{k_2}{\Gamma(\alpha+1)}\left[\left(\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}\right)^{1-\alpha} - 1\right][\psi(t)-\psi(0)]^{\alpha} \\
&\quad + \frac{k_4}{\Gamma(\alpha-\beta+1)}[\psi(t)-\psi(0)]^{\alpha-\beta} \\
&= \frac{4}{3\sqrt{\pi}}\left[\sqrt{\frac{e^{\frac{1}{3}}-1}{e^{\frac{t}{3}}-1}} - 1\right]\left[e^{\frac{t}{3}}-1\right]^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{9}{4})}\left[e^{\frac{t}{3}}-1\right]^{\frac{5}{4}}
\end{aligned}$$

and

$$\begin{aligned}
\underline{u}(t) &= \frac{k_1}{\Gamma(\alpha+1)}\left[\left(\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}\right)^{1-\alpha} - 1\right][\psi(t)-\psi(0)]^{\alpha} \\
&\quad + \frac{k_3}{\Gamma(\alpha-\beta+1)}[\psi(t)-\psi(0)]^{\alpha-\beta} \\
&= \frac{16}{15\sqrt{\pi}}\left[\sqrt{\frac{e^{\frac{1}{3}}-1}{e^{\frac{t}{3}}-1}} - 1\right]\left[e^{\frac{t}{3}}-1\right]^{\frac{3}{2}} + \frac{1}{5\Gamma(\frac{9}{4})}\left[e^{\frac{t}{3}}-1\right]^{\frac{5}{4}}.
\end{aligned}$$

5. Conclusions

In this paper, we have considered a class of boundary value problems for nonlinear two-term fractional differential equations with integral boundary conditions involving two ψ-Caputo fractional derivative. The studied problem has two nonlinear terms and includes two generalized fractional derivatives. Compared to many two-term FDEs, the type of problem we considered is more general. With the aid of the properties Green function, known fixed point theorems, and the method of upper and lower solutions, we have established the existence and uniqueness of positive solutions for a proposed problem. Finally, the main results are well illustrated with the help of an example. Many results of problems that contain classical fractional operators are obtained as special cases of (1.1). The reported results in this paper are novel and an important contribution to the existing literature on the topic.

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