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Uniqueness and Stability Results on Non-local Stochastic Random Impulsive Integro-Differential Equations

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Abstract

The paper is concerned with stochastic random impulsive integro-differential equations with non-local conditions. The sufficient conditions guarantees uniqueness of mild solution derived using Banach fixed point theorem. Stability of the solution is derived by incorporating Banach fixed point theorem with certain inequality techniques.

Keywords: Uniqueness, Stability, Random impulse, Stochastic Integro-differential Equations.
2010 MSC: 34A12, 34K20, 58C30.

1. Introduction

Mathematical modelling in Engineering and Scientific fields results in integral, ordinary or partial differential equations, stochastic differential equations or integro-differential equations. Specifically, in the fluid dynamics, chemical kinetics and biological disciplines, systems in general are of integro-differential type refer [1, 2, 3, 4]. Evolution processes from fields of population dynamics, aeronautics, economics, telecommunications and physics experiences drastic change of state at certain moments of time between the intervals of continuous evolution. Comparatively, the duration of these changes are negligible to the total duration that acts instantaneously in the form of impulse. The theory of impulsive differential equations represents a more natural framework for mathematical modelling see [5, 6, 7] and the references therein.

Impulses exist at fixed time or random time. There are many articles featuring the qualitative properties of fixed impulsive type equations [8, 9]. Wu and Meng [10] initiated the study of random impulsive ordinary differential equations and investigated boundedness of solution by Liapunov's direct function. By regular fluctuations in the deterministic models due to noise which appears to be random, the researchers moved to the stochastic

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differential equations. There are several articles regarding the study the Stochastic Differential Equations (SDE) [11, 12, 13, 14, 15, 16, 17, 18]. Stochastic effects with the impulsive effect exist in evolution processes in the real world phenomena refer [19, 20]. Sakthivel and Luo [21] investigated the existence and asymptotical stability for mild solution of Impulsive Stochastic Differential Equations.

Pan and cao [22] solved the exponential stability of impulsive stochastic partial differential equations with delays. Cui and yan [23] investigated the existence results for fractional neutral stochastic integro-differential equations with infinite delay. Mao [24] established stability results of stochastic integro-differential equations. Li et.al [25] investigated the existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delay.

For instance, let us consider the classical stock price model [26]

$$\begin{aligned} d(N(t)) &= uN(t)dt + \sigma N(t)dB_t, \qquad t \geqslant 0, t \neq \delta_k, \\ N(\delta_k) &= a_k N(\delta_k^-), \qquad k = 1,2,3..., \\ N(0) &= N_0. \end{aligned} \tag{1.1}$$

The system 1.1 represents impulsive stochastic differential equations. Here B_t is a Brownian motion or Wiener process. $N(t)$ represents the price of stock at time t and δ_k represents the release time of an information relating to the stock $N(\delta_k^-) = \lim_{t \to \delta_k - 0} N(t)$ and $N_0 \in \mathbb{R}$. In reality, $\{\delta_k\}$ represents a sequence of random variable which satisfies $0 < \delta_1 < \delta_2 < ...$

El. Borai [27] studied the existence and uniqueness of stochastic fractional integro-differential equations. Ahmed and El. Borai [28] established the existence results of mild solutions of Hilfer fractional stochastic integro-differential equations with non-local conditions. Guo et.al [29] analysed the exponential stability of impulsive stochastic functional differential equations. Liu and Xu [30] studied the averaging results for impulsive fractional neutral stochastic differential equations.

Sayooj et.al [31] considered a non-local random impulsive integro-differential system and calculated the existence, uniqueness and stability results. There are several papers that includes the study of impulsive integro-differential equations involving random impulses [32, 33]. Therefore, the study of random impulsive stochastic differential equations has a room for improvement. Thus the main objective of this work is to present non-local random impulsive stochastic differential equations hoping that the results obtained would contribute to the area.

Let us consider a the non-local stochastic random impulsive integro-differential equation of the form

$$\begin{cases} dx(t) = \mathfrak{f}(t, x_t)dt + \int_0^t \mathfrak{g}(\eta, x(t+\eta))d\omega(\eta), & t \neq \xi_k, t \geqslant r, \\ x(\xi_k) = b_k(\delta_k)x(\xi_k^-), & k = 1,2,3..., \\ x_{t_0} + h(x) = x_0. \end{cases} \tag{1.2}$$

where δ_k is a random variable defined from Ω to $\mathfrak{D}_k \stackrel{def}{=} (0, d_k)$ for $k = 1, 2, ...$ where $0 < d_k < +\infty$. Moreover, assume that for $k = 1, 2, ...$ δ_k follows Erlang distribution, let δ_i and δ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, ...$ Here we suppose, $T \in (t_0, +\infty)$, $\mathfrak{f} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$, $\mathfrak{g} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^{d \times m}$, $\mathfrak{h} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$ and

$b_k : \mathcal{D}_k \to \mathbb{R}^{d\times d}$ and x_t is $\mathbb{R}^d$-valued stochastic process such that $x(t) \in \mathbb{R}^d, x(t) = \{x(t+\theta : -\delta \leqslant \theta \leqslant 0)\}$. The impulsive moments ξ_k form a strictly increasing sequence, i.e., $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots < \lim_{k\to\infty}$, and $x(\xi_k^-) = \lim_{t\to\xi-0} x(t)$. We assume that $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \delta_k$ for $k = 1, 2, 3...$ Obviously, ξ_k is a process with increment increments. We assume that $\{N(t), t \geqslant 0\}$ is a simple counting process generated by $\{\xi_k\}$ and $\{\omega(t) : t \geqslant 0\}$ is a given Weiner process.We denote $\mathfrak{F}^1$ the σ-algebra generated by $\{N(t), t \geqslant 0\}$, and denote $\mathfrak{F}^2$ the σ-algebra generated by $\{\omega(s), s \geqslant t\}$. We assume that $\mathfrak{F}^1_\infty, \mathfrak{F}^2_\infty$ and ξ_k are mutually independent.

The manuscript is summarized as follows. Section 2 presents certain preliminaries. Section 3 is devoted to the existence and uniqueness solution of random impulsive stochastic integro-differential equations with non-local condition. In Section 4, with the Lipschitz condition, the stability results are derived.

2. Preliminaries

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space with Expectation $\mathbb{E}$ and associate with the normal filtration $\{\mathfrak{F}_t\}$, $t \geqslant 0$ satisfying $\mathfrak{F}_t = \mathfrak{F}_t^{(1)} \vee \mathfrak{F}_t^{(2)}$. Let $\mathcal{L}^2(\Omega, \mathbb{R}^d)$ be the collection of all strongly measurable, $\mathfrak{F}_t$ measurable, $\mathbb{R}^d$-valued random variables x with norm $\|x\|_{\mathcal{L}_2} = \left(\mathbb{E}\|x\|^2\right)^{1/2}$, where $\mathbb{E} = \int_\Omega x d\mathbb{P}$. Let $\delta > 0$ denote the Banach space of all piecewise continuous $\mathbb{R}^d$-valued stochastic process $\{\xi(t), t \in [-\delta, 0]\}$ by $\mathfrak{C}\left([\delta, 0], \mathcal{L}^2\left(\Omega, \mathbb{R}^d\right)\right)$ equipped with the norm,

$$\|\beta\|_{\mathfrak{C}} = \sup_{\theta\in[-\delta,0]} \left(\mathbb{E}\|\beta(\theta)\|^2\right)^{1/2}.$$

The initial data,[25]

$$x_{t_0} = x_0 = \{\xi(\theta) : -\delta \leqslant \theta \leqslant 0\}, \tag{2.1}$$

is an $\mathfrak{F}_{t_0}$ measurable, $[-\delta, 0]$ to $\mathbb{R}^d$-valued random variable such that $\mathbb{E}\|\zeta\|^2 < \infty$.

Definition 2.1. *A $\mathbb{R}^d$ valued Stochastic process $x(t)$ on $t - \delta \leqslant t \leqslant T$ is called a mild solution to 2.1, if*
(i) For every $t_0 \leqslant t \leqslant T$, $x(t_0) = x_0, \{x_t\}_{t_0 \leqslant t \leqslant T}$ is $\mathfrak{F}_t$-adapted;
(ii)

$$\begin{aligned} x(t) \quad = \quad & \sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k} b_i(\delta_i)(x_0 - \mathfrak{h}(x)) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i} \mathfrak{f}(s, x(s))ds \right. \\ & + \int_{\xi_k}^{t} \mathfrak{f}(s, x(s))ds + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)ds \\ & \left. + \int_{\xi_k}^{t}\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)ds\right] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [\delta, T]. \end{aligned} \tag{2.2}$$

where $\prod_{j=p}^{q}(.) = 1$ *as* $p > q$,

$$\prod_{j=i}^{k} b_j(\delta_j) = b_k(\delta_k)b_{k-1}(\delta_{k-1})...b_i(\delta_i),$$

and $I_{(A)}(.)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Lemma 2.2. *For any* $r \geqslant 1$ *and for arbitrary* $\mathcal{L}_2^0$*- valued predictable process* $\Phi(.)$,

$$\sup_{s\in[0,t]} \mathbb{E} \left\| \int_0^s \Phi(u)dw(u) \right\|_{\mathbb{X}}^{2r} \leqslant (r(2r-1))^r \left(\int_0^t (\mathbb{E}\left\|\Phi(s)\right\|_{\mathcal{L}_2^0}^{2r})ds \right)^r.$$

The following hypotheses are considered to prove our results.

(H1) The function $\mathfrak{f} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$ satisfies the Lipschitz condition. For $x, y \in \mathbb{X}$ and $\delta \leqslant t \leqslant T$ there exist $\mathcal{L}_0, \mathcal{M}_0 \geqslant 0$ such that

$$\begin{aligned} \mathbb{E}\left\|\mathfrak{f}(t,x) - \mathfrak{f}(t,y)\right\|^2 &\leqslant \mathcal{L}_0\mathbb{E}\left\|x-y\right\|^2 \\ \mathbb{E}\left\|\mathfrak{f}(t,0)\right\|^2 &\leqslant \mathcal{M}_0. \end{aligned}$$

(H2) The function $\mathfrak{g} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^{d\times m}$ satisfies the Lipschitz condition. For $x, y \in \mathbb{X}$ and $\delta \leqslant t \leqslant T$ there exist $\mathcal{L}_1, \mathcal{M}_1 \geqslant 0$ such that

$$\begin{aligned} \mathbb{E}\left\| \int_0^T [\mathfrak{g}(\eta, x(t+\eta)) - \mathfrak{g}(\eta, y(t+\eta))]\, d\eta \right\|^2 &\leqslant \mathcal{L}_1\mathbb{E}\left\|x(t+\eta) - y(t+\eta)\right\|^2, \\ \mathbb{E}\left\|\mathfrak{g}(\eta,0)\right\|^2 &\leqslant \mathcal{M}_1. \end{aligned}$$

(H3) The condition $\max_{i,k} \left\{ \prod_{j=1}^{k} \left\|b_j(\delta_j)\right\| \right\}$ is uniformly bounded $\exists\, B > 0$ such that

$$\max_{i,k} \left\{ \prod_{j=1}^{k} \left\|b_j(\delta_j)\right\| \right\} \leqslant B,$$

for all $\delta_j \in D_j$, $j = 1, 2, 3....$

(H4) The function $\mathfrak{h} : [t_0, T] \times \mathfrak{C} \to \mathbb{R}^d$ satisfies the Lipschitz condition. For $x, y \in \mathbb{X}$ and $\delta \leqslant t \leqslant T$ there exist $\mathcal{L}_* \geqslant 0$ such that

$$E\left\|\mathfrak{h}(x) - \mathfrak{h}(y)\right\|^2 \leqslant \mathcal{L}_*\left\|x-y\right\|^2.$$

(H5)

$$\Gamma = B^2\max\{1, B^2\}(T-\delta)^2 \left[\mathcal{L}_0 + \mathcal{L}_1 + \frac{\mathcal{L}_*}{(T-\delta)^2} \right] < 1.$$

3. Uniqueness

This section is devoted to the study of uniqueness of mild solution of the system 1.2

Theorem 3.1. *Assume that the hypotheses (H1)-(H5) holds. Then the system 1.2 has a unique mild solution in* $\mathbb{B}$.

Proof. Let T be an arbitrary number $T \leqslant +\infty$. Initially from definition 2.1, we define a non-linear operator $\phi : \mathbb{B} \to \mathbb{B}$. Note that the problem 2.1 has a solution if and only if the operator ϕ has a fixed point.

$$\begin{aligned}\phi x(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\delta_i)(x_0 - \mathfrak{h}(x)) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i} \mathfrak{f}(s,x(s))ds \\ &+ \int_{\xi_k}^{t} \mathfrak{f}(s,x(s))ds + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)ds \\ &+ \int_{\xi_k}^{t}\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)ds\Bigg] \times I_{[\xi_k,\xi_{k+1})}(t),\end{aligned}$$

where $t \in [\delta, T]$. We need to show that $\mathbb{B}$ maps $\mathbb{B}$ under ϕ.
$\|\phi x(t)\|^2$

$$\begin{aligned} &\leqslant \sum_{k=0}^{+\infty}\Bigg[\left\|\prod_{i=1}^{k} b_i(\delta_i)\right\| \|(x_0 - \mathfrak{h}(x))\| + \sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i} \|\mathfrak{f}(s,x(s))\|\, ds \\ &+ \int_{\xi_k}^{t} \|\mathfrak{f}(s,x(s))\|\, ds + \sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i}\left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)\right\| ds \\ &+ \int_{\xi_k}^{t}\left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)\right\| ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [\delta, T] \\ &\leqslant 3\left[\sum_{k=0}^{+\infty}\left[\left\|\prod_{i=1}^{k} b_i(\delta_i)\right\|^2 \|(x_0 - \mathfrak{h}(x))\|^2 I_{[\xi_k,\xi_{k+1})}(t)\right] + \left[\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\delta_j)\right\|\right.\right.\right. \\ &\times \left.\left.\int_{\xi_{i-1}}^{\xi_i} \|\mathfrak{f}(s,x_s)\|\, ds + \int_{\xi_k}^{t} \|\mathfrak{f}(s,x(s))\|\, ds\right] I_{[\xi_k,\xi_{k+1})}(t)\right]^2 \\ &+ \left[\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\left\|\prod_{j=i}^{k} b_j(\delta_j)\right\| ds \times \int_{\xi_{i-1}}^{\xi_i}\left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)\right\|\right.\right. \\ &+ \left.\left.\int_{\xi_k}^{t}\left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta))d\omega(\eta)\right\| ds\right] I_{[\xi_k,\xi_{k+1})}(t)\right]^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant 3\max_{k}\left\{\left\|\prod_{i=1}^{k} b_i(\delta_i)\right\|^2\right\}\|x_0-\mathfrak{h}(x)\|^2+3\left[\max_{i,k}\left\{1,\left\|\prod_{j=i}^{k} b_j(\delta_j)\right\|^2\right\}\right]^2\\
&\times\left\{\int_{t_0}^{t}\|\mathfrak{f}(s,x(s))\|\,dsI_{[\xi_k,\xi_{k+1})}(t)\right\}^2+3\left[\max_{i,k}\left\{1,\left\|\prod_{j=i}^{k} b_j(\delta_j)\right\|^2\right\}\right]^2\\
&\times\left\{\int_{t_0}^{t}\left\|\int_0^T \mathfrak{g}(\eta,x(s+\eta))d\omega(\eta)\right\|dsI_{[\xi_k,\xi_{k+1})}(t)\right\}^2\\
&\leqslant 3B^2\|x_0-\mathfrak{h}(x)\|^2+3\max\{1,B^2\}(t-t_0)\int_{t_0}^{t}\|\mathfrak{f}(s,x(s))\|^2\,ds\\
&+\ 3\max\{1,B^2\}(t-t_0)\mathcal{C}_2\int_{t_0}^{t}\left[\left\|\int_0^T \mathfrak{g}(\eta,x(s+\eta))d\eta\right\|ds\right]^2.
\end{aligned}
$$

Thus we would obtain,
$\mathbb{E}\|\phi x(t)\|^2$

$$
\begin{aligned}
&\leqslant 3B^2\|x_0-\mathfrak{h}(x)\|^2+3\max\{1,B^2\}(t-t_0)\int_{t_0}^{t}\mathbb{E}\|\mathfrak{f}(s,x(s))\|^2\,ds+3\max\{1,B^2\}(t-t_0)\mathcal{C}_2\\
&\times\int_{t_0}^{t}\mathbb{E}\left[\left\|\int_0^T \mathfrak{g}(\eta,x(s+\eta))d\eta\right\|ds\right]^2\\
&\leqslant 3B^2\|x_0-\mathfrak{h}(x)\|^2+3\max\{1,B^2\}(T-\delta)\int_{t_0}^{t}\mathbb{E}\|\mathfrak{f}(s,x(s))\|^2\,ds\\
&+\ 3\max\{1,B^2\}(T-\delta)\mathcal{C}_2\int_{t_0}^{t}\mathbb{E}\left[\left\|\int_0^T \mathfrak{g}(\eta,x(s+\eta))d\eta\right\|ds\right]^2\\
&\leqslant 3B^2\|x_0-\mathfrak{h}(x)\|^2+6\max\{1,B^2\}(T-\delta)\mathcal{L}_0\int_{t_0}^{t}\mathbb{E}\|x(s)\|^2\,ds+6\max\{1,B^2\}(T-\delta)^2\mathcal{M}_0\\
&+\ 6\max\{1,B^2\}(T-\delta)\mathcal{L}_1\mathcal{C}_2\int_{t_0}^{t}\mathbb{E}\|x(s+\eta)\|^2\,ds+6\max\{1,B^2\}(T-\delta)^2\mathcal{C}_2\mathcal{M}_1
\end{aligned}
$$

Thus,
$\sup_{t\in[\delta,T]}\mathbb{E}\|\phi x(t)\|^2$

$$
\begin{aligned}
&\leqslant 3B^2\|x_0-\mathfrak{h}(x)\|^2+6\max\{1,B^2\}(T-\delta)\mathcal{L}_0\int_{t_0}^{t}\sup_{t\in[\delta,T]}\mathbb{E}\|x(s)\|^2\,ds+6\max\{1,B^2\}\\
&\times\ (T-\delta)^2\mathcal{M}_0+6\max\{1,B^2\}(T-\delta)\mathcal{L}_1\mathcal{C}_2\int_{t_0}^{t}\sup_{t\in[\delta,T]}\mathbb{E}\|x(s+\eta)\|^2\,ds\\
&+\ 6\max\{1,B^2\}(T-\delta)^2\mathcal{C}_2\mathcal{M}_1,\quad \delta\leqslant t\leqslant T.
\end{aligned}
$$

Hence ϕ maps $\mathbb{B}$ into itself. Now we have to show that ϕ is a contraction mapping,

$\|\phi x(t) - \phi y(t)\|^2$

$$
\begin{aligned}
&\leqslant \left[\sum_{k=0}^{+\infty} \left\|\prod_{i=1}^{k} b_i(\delta_i)\right\| \|\mathfrak{h}(x) - \mathfrak{h}(y)\| \, I_{[\xi_k,\xi_{k+1})}(t)\right]^2 + \left[\sum_{k=0}^{+\infty} \left[\sum_{i=1}^{k} \left\|\prod_{j=i}^{k} b_j(\delta_j)\right\|\right.\right. \\
&\times \left.\left. \int_{\xi_{i-1}}^{\xi_i} \|\mathfrak{f}(s,x(s)) - \mathfrak{f}(s,y(s))\| \, ds + \int_{\xi_k}^{t} \|\mathfrak{f}(s,x(s)) - \mathfrak{f}(s,y(s))\| \, ds\right] I_{[\xi_k,\xi_{k+1})}(t)\right]^2 \\
&+ \left[\sum_{k=0}^{+\infty} \left[\sum_{i=1}^{k} \left\|\prod_{j=i}^{k} b_j(\delta_j)\right\| ds \int_{\xi_{i-1}}^{\xi_i} \left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, y(s+\eta)) d\omega(\eta)\right\|\right.\right. \\
&+ \left.\left. \int_{\xi_k}^{t} \left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, y(s+\eta)) d\omega(\eta)\right\| ds\right] I_{[\xi_k,\xi_{k+1})}(t)\right]^2 \\
&\leqslant \max_k \left\{\prod_{i=1}^{k} \|b_i(\delta_i)\|^2\right\} \|\mathfrak{h}(x) - \mathfrak{h}(y)\|^2 + \left[\max_{i,k} \left\{1, \prod_{j=i}^{k} \|b_j(\delta_j)\|\right\}\right]^2 \\
&\times \left[\int_{t_0}^{t} \|\mathfrak{f}(s,x(s)) - \mathfrak{f}(s,y(s))\| \, ds I_{[\xi_k,\xi_{k+1})}(t)\right]^2 + \left[\max_{i,k} \left\{1, \prod_{j=i}^{k} \|b_j(\delta_j)\|\right\}\right]^2 \\
&\times \left[\int_{t_0}^{t} \left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, y(s+\eta)) d\omega(\eta)\right\| ds I_{[\xi_k,\xi_{k+1})}(t)\right]^2 \\
&\leqslant B^2 \|\mathfrak{h}(x) - \mathfrak{h}(y)\|^2 + \max\{1, B^2\} (t - t_0) \int_{t_0}^{t} \|\mathfrak{f}(s,x(s)) - \mathfrak{f}(s,y(s))\|^2 \, ds \\
&+ \max\{1, B^2\} (t - t_0) \mathcal{C}_2 \int_{t_0}^{t} \left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, y(s+\eta)) d\eta\right\|^2 ds.
\end{aligned}
$$

However,
$\mathbb{E}\|\phi x(t) - \phi y(t)\|^2$

$$
\begin{aligned}
&\leqslant B^2 \mathbb{E}\|\mathfrak{h}(x) - \mathfrak{h}(y)\|^2 + \max\{1, B^2\} (t - t_0) \int_{t_0}^{t} \mathbb{E}\|\mathfrak{f}(s,x(s)) - \mathfrak{f}(s,y(s))\|^2 \, ds \\
&+ \max\{1, B^2\} (t - t_0) \mathcal{C}_2 \int_{t_0}^{t} \mathbb{E}\left\|\int_0^T \mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, y(s+\eta)) d\eta\right\|^2 ds \\
&\leqslant B^2 \mathbb{E}\|\mathfrak{h}(x) - \mathfrak{h}(y)\|^2 + \max\{1, B^2\} (T - \delta) \mathcal{L}_0 \int_{t_0}^{t} \mathbb{E}\|x(s) - y(s)\|^2 \, ds \\
&+ \max\{1, B^2\} (T - \delta) \mathcal{C}_2 \mathcal{L}_1 \int_{t_0}^{t} \mathbb{E}\|x(s+\eta) - y(s+\eta)\|^2 \, ds.
\end{aligned}
$$

Taking supremum over t we would obtain,

$$\begin{aligned}\|\phi x-\phi y\|^2 &\leqslant B^2\mathcal{L}_*\|x-y\|^2+\max\{1,B^2\}(T-\delta)^2\mathcal{L}_0\|x-y\|^2\\ &+\max\{1,B^2\}(T-\delta)^2\mathcal{C}_2\mathcal{L}_1\|x-y\|^2\\ &\leqslant \left[B^2\mathcal{L}_*+\max\{1,B^2\}(T-\delta)\mathcal{L}_0+\max\{1,B^2\}(T-\delta)\mathcal{C}_2\mathcal{L}_1\right]\|x-y\|^2\\ &\leqslant \left[B^2\mathcal{L}_*+\max\{1,B^2\}(T-\delta)^2\{\mathcal{L}_0+\mathcal{C}_2\mathcal{L}_1\}\right]\|x-y\|^2\\ &\leqslant \Gamma\|x-y\|^2,\end{aligned}$$

where

$$\Gamma=B^2\mathcal{L}_*+\max\{1,B^2\}(T-\delta)^2\{\mathcal{L}_0+\mathcal{C}_2\mathcal{L}_1\}$$

By (H5) and $0<\Gamma<1$ we would obtain ϕ is a contraction mapping. By Banach fixed theorem ϕ has a unique fixed point on $\mathbb{B}$. Hence the system has a unique mild solution. □

Remark 3.2. Let $\mathfrak{f}:\mathbb{R}_\delta\times\mathbb{X}\to\mathbb{X}$, $\mathfrak{g}:\mathbb{R}_\delta\times\mathbb{X}\to\mathbb{X}$ and $\mathfrak{h}:\mathbb{X}\to\mathbb{X}$ satisfy the assumptions (H1)-(H5). Then there exists a unique, global, continuous solution x to the system 1.2 for any initial value (t_0,x_0) with $t_0\geqslant 0$ and $x_0\in\mathbb{B}$.

Remark 3.3. Assume that (H1)-(H5) holds. Then the mild solution without existence of non-local condition and the solution is

$$\begin{aligned}x(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\delta_i)x_0+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{f}(s,x(s))ds\\ &+\int_{\xi_k}^{t}\mathfrak{f}(s,x(s))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\delta_j)\\ &\times\int_{\xi_{i-1}}^{\xi_i}\int_0^T\mathfrak{g}(\eta,x(s+\eta))d\omega(\eta)ds+\int_{\xi_k}^{t}\int_0^T\mathfrak{g}(\eta,x(s+\eta))d\omega(\eta)ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t),\end{aligned}$$

where $t\in[\delta,T]$.

4. Stability

Theorem 4.1. *Let* $x(t)$ *and* $\widehat{x}(t)$ *be solution of the system 1.2 with initial value* $x_0-\mathfrak{h}(x)$ *and* $\widehat{x}_0-\mathfrak{h}(\widehat{x})$ *respectively. If the assumptions (H1)-(H4) of Theorem 3.1 are satisfied, then the system 1.2 is stable in the mean square.*

Proof. From the assumptions $x(t)$ and $\widehat{x}(t)$ are two solutions of the system 1.2 for every

$t \in [\delta, T]$. Then,

$$
\begin{aligned}
x(t) - \widehat{x}(t) &= \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} b_i(\delta_i)(x_0 - \widehat{x}_0) + (\mathfrak{h}(x) - \mathfrak{h}(\widehat{x})) \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\delta_j) \int_{\xi_{i-1}}^{\xi_i} [\mathfrak{f}(s, x(s)) - \mathfrak{f}(s, \widehat{x}(s))]\, ds \\
&+ \int_{\xi_k}^{t} [\mathfrak{f}(s, x(s)) - \mathfrak{f}(s, \widehat{x}(s))]\, ds + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\delta_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^T \Big[\mathfrak{g}(\eta, x(s+\eta)) \\
&- \mathfrak{g}(\eta, \widehat{x}(s+\eta)) \Big] d\omega(\eta) ds + \int_{\xi_k}^{t} \int_0^T [\mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, \widehat{x}(s+\eta))] \\
&\times d\omega(\eta) ds \Bigg] I_{[\xi_k, \xi_{k+1})}(t).
\end{aligned}
$$

Using the assumed hypotheses (H1)-(H4) we would obtain,

$$
\begin{aligned}
\|x(t) - \widehat{x}(t)\|^2 &\leqslant 4 \sum_{k=0}^{+\infty} \Bigg[\left\| \prod_{i=1}^{k} b_i(\delta_i) \right\| [\|x_0 - \widehat{x}_0\| + \|\mathfrak{h}(x) - \mathfrak{h}(\widehat{x}\|] \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \|b_j(\delta_j)\| \int_{\xi_{i-1}}^{\xi_i} \|\mathfrak{f}(s, x(s)) - \mathfrak{f}(s, \widehat{x}(s))\|\, ds \\
&+ \int_{\xi_k}^{t} \|\mathfrak{f}(s, x(s)) - \mathfrak{f}(s, \widehat{x}(s))\|\, ds + \sum_{i=1}^{k} \prod_{j=i}^{k} \|b_j(\delta_j)\| \\
&\times \int_{\xi_{i-1}}^{\xi_i} \left\| \int_0^T [\mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, \widehat{x}(s+\eta))]\, d\omega(\eta) \right\| ds \\
&+ \int_{\xi_k}^{t} \left\| \int_0^T [\mathfrak{g}(\eta, x(s+\eta)) - \mathfrak{g}(\eta, \widehat{x}(s+\eta))]\, d\omega(\eta) \right\| ds \Bigg]^2 \\
&\times I_{[\xi_k, \xi_{k+1})}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\, \|x(t) - \widehat{x}(t)\|^2 &\leqslant 4 \max_{k} \left\{ \left\| \prod_{i=1}^{k} b_i(\delta_i) \right\|^2 \right\} \left[\mathbb{E}\, \|x_0 - \widehat{x}_0\|^2 + \mathbb{E}\, \|\mathfrak{h}(x) - \mathfrak{h}(\widehat{x}\| \right] \\
&+ 4 \max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(\delta_j)\| \right\}^2
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \mathbb{E}\left\{\int_{t_0}^{t} \|\mathfrak{f}(s,x(s))-\mathfrak{f}(s,\widehat{x}(s))\|\, ds I_{[\xi_k,\xi_{k+1})}(t)\right\}^2 + 4\max_{i,k}\left\{1,\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|\right\}^2 \\
&\times \quad \mathbb{E}\left\{\int_{t_0}^{t}\left\|\int_0^T [\mathfrak{g}(\eta,x(s+\eta))-\mathfrak{g}(\eta,\widehat{x}(s+\eta))]\, d\omega(\eta)\right\| ds I_{[\xi_k,\xi_{k+1})}(t)\right\}^2 \\
\leqslant &\quad 4B^2\left[\mathbb{E}\|x_0-\widehat{x}_0\|^2+\mathbb{E}\|\mathfrak{h}(x)-\mathfrak{h}(\widehat{x}\|^2\right]+4\max\left\{1,B^2\right\}(t-t_0) \\
&\times \quad \int_{t_0}^{t}\|\mathfrak{f}(s,x(s))-\mathfrak{f}(s,\widehat{x}(s))\|^2\, ds + 4\max\left\{1,B^2\right\}(t-t_0)\mathcal{C}_2 \\
&\times \quad \int_{t_0}^{t}\left\|\int_0^T [\mathfrak{g}(\eta,x(s+\eta))-\mathfrak{g}(\eta,\widehat{x}(s+\eta))]\, d\eta\right\|^2 ds \\
\leqslant &\quad 4B^2\left[\mathbb{E}\|x_0-\widehat{x}_0\|^2+\mathbb{E}\|\mathfrak{h}(x)-\mathfrak{h}(\widehat{x}\|^2\right]+4\max\left\{1,B^2\right\}(t-t_0)\mathcal{L}_0 \\
&\times \quad \int_{t_0}^{t}\mathbb{E}\|x(t)-\widehat{x}(t)\|^2\, ds + 4\max\left\{1,B^2\right\}(t-t_0)\mathcal{L}_1\mathcal{C}_2 \\
&\times \quad \int_{t_0}^{t}\mathbb{E}\|x(t+\eta)-\widehat{x}(t+\eta)\|^2\, ds.
\end{aligned}
$$

Taking supremum over t,

$$
\begin{aligned}
&\sup_{t\in[\delta,T]}\mathbb{E}\|x(t)-\widehat{x}(t)\|^2 \\
\leqslant &\quad 4B^2\left[\mathbb{E}\|x_0-\widehat{x}_0\|^2+\sup_{t\in[\delta,T]}\mathbb{E}\|\mathfrak{h}(x)-\mathfrak{h}(\widehat{x}\|^2\right]+4\max\left\{1,B^2\right\} \\
\times &\quad (T-\delta)\mathcal{L}_0\int_{t_0}^{t}\sup_{t\in[\delta,T]}\mathbb{E}\|x(t)-\widehat{x}(t)\|^2\, ds + 4\max\left\{1,B^2\right\} \\
\times &\quad (T-\delta)\mathcal{L}_1\mathcal{C}_2\int_{t_0}^{t}\sup_{t\in[\delta,T]}\mathbb{E}\|x(t+\eta)-\widehat{x}(t+\eta)\|^2\, ds.
\end{aligned}
$$

By Grownwall inequality,

$$
\begin{aligned}
\sup_{t\in[\delta,T]}\mathbb{E}\|x(t)-\widehat{x}(t)\|^2 &\leqslant 4B^2\mathbb{E}\|x_0-\widehat{x}_0\|^2\exp\left[4\max\left\{1,B^2\right\}(T-\delta)^2\right]\mathcal{L} \\
&\leqslant \Gamma\mathbb{E}\|x_0-\widehat{x}_0\|^2.
\end{aligned}
$$

where, $\Gamma = 4B^2\exp\left[4\max\left\{1,B^2\right\}(T-\delta)^2\right]\mathcal{L}$ and $\mathcal{L} = \mathcal{L}_0+\mathcal{L}_1\mathcal{C}_2+\frac{\mathcal{L}_*}{(T-\delta)^2}$.
For given $\epsilon > 0$, we choose $\alpha = \frac{\epsilon}{\Gamma}$ such that $\mathbb{E}\|x_0-\widehat{x}_0\|^2 < \alpha$ then,

$$
\sup_{t\in[\delta,T]}\mathbb{E}\|x(t)-\widehat{x}(t)\|^2 \leqslant \epsilon
$$

. □

Remark 4.2. Random impulsive integro-differential equation with local initial condition is a special case of the system 1.2. So the random impulsive integro-differential equation with local initial condition is stable in the mean square.

5. conclusion

In this paper, the uniqueness and stability of random impulsive stochastic integro-differential system has been calculated using the initial data $x_{t_0} = x_0$. Contraction mapping principle is used to prove the existence and uniqueness. With inequality techniques and Contraction principle the stability is derived. In the future we can extend this work to stochastic partial differential equations.

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