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Some fixed point results of $\mathbb{F}$-Contraction mapping in $\mathbb{D}$-metric spaces by Samet's method

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Abstract

The aim of this paper is to study the $\mathbb{F}$-contraction mapping introduced by Wardowski to obtain fixed point results by method of Samet in generalized complete metric spaces. Our findings extend the results announced by Samet methods and some other works in generalized metric spaces.

Keywords: $\mathbb{F}$-contraction, $\mathbb{D}$-metric spaces, Fixed point.
2010 MSC: 74H10, 54H25.

1. introduction

Dhage [11], presented the extended metric or $\mathbb{D}$-metric spaces and obtained some results about it. Many researchers have taken these results for granted and applied them in studying fixed point results in $\mathbb{D}$-metric spaces. Rhoades [21], extended Dhages contractive condition by increasing a number of factors and studied the existence and uniqueness of fixed point of some mappings in $\mathbb{D}$-metric space. Wardowski introduced the concepts of F-contraction and F-weak contraction to generalize the Banach's contraction in many ways (see [28],[29]). Sedghi et al. [26] have introduced the concept of S-metric space and investigated that this is a generalization of a G-metric space and a $\mathbb{D}^*$ -metric space. Also, they have studied properties of S-metric spaces and some fixed point results for a self-map on an S-metric space. In the following, some authors extended this work (see [2, 20, 24, 25]). Samet et al. [23, 22] proved that $\alpha - \phi$ contractions unify large classes of contractive type operators, whose fixed points can be obtained by means of the Picard iteration. Afterward, these results expanded by many mathematicians (see, for example [1, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 17]).
Here, we investigate the result of Wardowski and Samet in generalized applying the result obtained by E. Karapinar in [16] we prove new fixed point theorems in generalized metric spaces which have many applications in solving integral equations ([14, 15, 27]).

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2. Preliminaries

Let $\mathcal{M}$ be a nonempty set. A generalized $\mathbb{D}$-metric on $\mathcal{M}$ is a function,
$\mathbb{D} : \mathcal{M}^3 \to \mathbb{R}^+$ that satisfies the following
(D1) $\mathbb{D}(\zeta, \eta, \zeta) \geqslant 0$,
(D2) $\mathbb{D}(\zeta, \eta, \zeta) = 0$ if and only if $\zeta = \eta = \zeta$,
(D3) $\mathbb{D}(\zeta, \eta, \zeta) = \mathbb{D}(p\{\zeta, \eta, \zeta\})$, (symmetric) where p is a permutation function,
(D4) $\mathbb{D}(\zeta, \eta, \zeta) \leqslant \mathbb{D}(\zeta, a, a) + \mathbb{D}(a, \eta, \zeta)$,
where $\zeta, \eta, \zeta, a \in \mathcal{M}$. $\mathbb{D}$ is called a generalized $\mathbb{D}$-metric and the pair $(\mathcal{M}, \mathbb{D})$ is called a generalized $\mathbb{D}$-metric space.
Let

$$d_{\mathbb{D}}(\zeta, \eta) = \mathbb{D}(\zeta, \eta, \eta) + \mathbb{D}(\eta, \zeta, \zeta), \quad \forall \zeta, \eta \in \mathcal{M}. \tag{2.1}$$

Obviously that $d_{\mathbb{D}}$ is a metric.

Remark 2.1. In a $\mathbb{D}$-metric space, we have:
(i) $\mathbb{D}(\zeta, \zeta, \eta) \leqslant \mathbb{D}(\zeta, \zeta, \zeta) + \mathbb{D}(\zeta, \eta, \eta) = \mathbb{D}(\zeta, \eta, \eta)$,
(ii) $\mathbb{D}(\eta, \eta, \zeta) \leqslant \mathbb{D}(\eta, \eta, \eta) + \mathbb{D}(\eta, \zeta, \zeta) = \mathbb{D}(\eta, \zeta, \zeta)$,
(iii) $\mathbb{D}(\zeta, \zeta, \eta) = \mathbb{D}(\zeta, \eta, \eta)$.

Definition 2.2. [19] Let $\{\zeta_n\}$ be a sequence of $(\mathcal{M}, \mathbb{D})$. $\{\zeta_n\}$ is $\mathbb{D}$-convergent to $\zeta \in \mathcal{M}$ if

$$\lim_{n,m \to +\infty} \mathbb{D}(\zeta, \zeta_n, \zeta_m) = 0.$$

that is, for $\varepsilon > 0$, there exists $N \in \mathbf{N}$ with $\mathbb{D}(\zeta, \zeta_n, \zeta_m) < \varepsilon$, for $n, m \geqslant N$.

Proposition 2.3. *[19] Let $(\mathcal{M}, \mathbb{D})$ be a $\mathbb{D}$-metric space. The following are equivalent*

(i) $\{\zeta_n\}$ *is* $\mathbb{D}$*-convergent to* $\mathcal{M}$,

(ii) $\mathbb{D}(\zeta_n, \zeta_n, \zeta) \to 0$ *as* $n \to \infty$,

(iii) $\mathbb{D}(\zeta_n, \zeta, \zeta) \to 0$ *as* $n \to \infty$,

(vi) $\mathbb{D}(\zeta_n, \zeta_m, \zeta) \to 0$ *as* $n, m \to \infty$.

Definition 2.4. [19] Let $(\mathcal{M}, \mathbb{D})$ be a $\mathbb{D}$-metric space. $\{\zeta_n\}$ is called a $\mathbb{D}$-Cauchy if for any $\varepsilon > 0$, there exists $N \in \mathbf{N}$ with $\mathbb{D}(\zeta_n, \zeta_m, \zeta_l) < \varepsilon$ for all $m, n, l \geqslant N$, that is, $\mathbb{D}(\zeta_n, \zeta_m, \zeta_l) \to 0$ as $n, m, l \to \infty$.

Definition 2.5. [28] Let $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying,
(F_1) F is strictly increasing, i.e. for $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta, F(\alpha) < F(\beta)$;
(F_2) for $\{\alpha_n\}_{n \in \mathbf{N}}$ of positive real numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F_3) $\exists\, k \in (0, 1)$ with $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

We say that $T : \mathcal{M} \to \mathcal{M}$ is F-contraction if there exists $\tau > 0$ such that

$$\forall \ \zeta, \eta \in \mathcal{M}, \ \ d(T\zeta, T\eta) > 0 \Rightarrow \tau + F(d(T\zeta, T\eta)) \leqslant F(d(\zeta, \eta)).$$

Example 2.6. Let $f_i : \mathcal{R}^+ \to \mathcal{R}$ $(i = 1, 2, 3, 4, 5, 6)$, defined by
$e_1)$ $f_1(t) = -\frac{1}{\sqrt{\alpha}}, \alpha > 0$,
$e_2)$ $f_2(t) = \ln(\alpha), \alpha > 0$,
$e_3)$ $f_3(t) = \alpha + \ln(\alpha), \alpha > 0$,
$e_4)$ $f_4(t) = \ln(\alpha^2 + \alpha), \alpha > 0$,
$e_5)$ $f_5(t) = F(\alpha) = \tan(\alpha + \frac{\pi}{2})$,
$e_6)$ $f_6(t) = F(\alpha) = -\frac{1}{\alpha^2}$, $\alpha > 0$. Then f_1, f_2, f_3, f_4, f_5 and f_6 satisfy (F_1)-(F_3) and so are F-contractions.

Definition 2.7. Suppose $T : \mathcal{M} \to \mathcal{M}$ and $\alpha : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathfrak{R}^+$. T is α-admissible if

$$\zeta, \eta, \zeta \in \mathcal{M}, \qquad \alpha(\zeta, \eta, \zeta) \geqslant 1 \implies \alpha(T\zeta, T\eta, T\zeta) \geqslant 1.$$

Denote with Φ the family of nondecreasing functions $\phi : [0, +\infty) \to [0, +\infty)$ continuous in $t = 0$ such that
(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t + s) \leqslant \phi(t) + \phi(s)$,
and $\sum_{n=1}^{+\infty} \phi^n(t) < +\infty$ for $t > 0$, where ϕ^n is the n-th iterate of ϕ.

Lemma 2.8. *For $\phi : \mathfrak{R}^+ \to \mathfrak{R}^+$ the following holds:*
if ϕ is nondecreasing then for $t > 0$, $\lim_{n\to+\infty} \phi^n(t) = 0$ implies $\phi(t) < t$.

Theorem 2.9. *Let $\{\mathfrak{A}_j\}_{j=1}^m$ be a family of nonempty $\mathbb{D}$-closed subsets of complete space $(\mathcal{M}, \mathbb{D})$. Let $\eta = \cup_{j=1}^m \mathfrak{A}_j$ and $T : Y \to Y$ be a $\alpha-$admissible satisfying*

$$T(\mathfrak{A}_j) \subseteq \mathfrak{A}_{j+1}, \quad j = 1, ..., m, \quad \text{where} \quad \mathfrak{A}_{m+1} = \mathfrak{A}_1.$$

If there exist $\alpha : Y \times Y \times Y \to \mathfrak{R}^+$ and $\phi \in \Phi$ such that

$$\alpha(\zeta, \eta, Tz)\mathbb{D}(T\zeta, T\eta, T\zeta) \leqslant \phi(\mathbb{D}(\zeta, \eta, \zeta)), \tag{2.2}$$

holds for all $\zeta \in \mathfrak{A}_j$ and $\eta, \zeta \in \mathfrak{A}_{j+1}$, $j = 1, ..., m$, and there exist $\zeta_0 \in Y$ such that $\alpha(\zeta_0, T\zeta_0, T^2\zeta_0) \geqslant 1$, then T has a unique fixed point in $\cap_{j=1}^m \mathfrak{A}_j$.

Proof. Let $\zeta_0 \in Y$ with $\alpha(\zeta_0, T\zeta_0, T^2\zeta_0) \geqslant 1$. Suppose $\zeta_0 \in \mathfrak{A}_1$. Define the sequence $\{\zeta_n\}$ in Y as follows

$$\zeta_n = T\zeta_{n-1} \quad \text{for all} \quad n \in \mathbf{N}.$$

Since T is cyclic, $\zeta_0 \in \mathfrak{A}_1$, $\zeta_1 = T(\zeta_0) \in \mathfrak{A}_2, ...$ and so, if $\zeta_{n_0+1} = \zeta_{n_0}$ for some $n_0 \in \mathbf{N}$, clearly, the fixed point of T is ζ_{n_0}. Let $\zeta_n \neq \zeta_{n+1}$ for $n \in \mathbf{N}$. Since T is $\alpha-$admissible, we have

$$\alpha(\zeta_0, \zeta_1, \zeta_2) = \alpha(\zeta_0, T\zeta_0, T^2\zeta_0) \geqslant 1 \Rightarrow \alpha(T\zeta_0, T\zeta_1, T\zeta_2) = \alpha(\zeta_1, \zeta_2, \zeta_3) \geqslant 1.$$

By induction, we get

$$\alpha(\zeta_{n-1}, \zeta_n, \zeta_{n+1}) \geqslant 1, \quad \text{for all} \quad n \in \mathbf{N}. \tag{2.3}$$

Applying (2.9) with $\zeta = \zeta_{n-1}$ and $\eta = \zeta = \zeta_n$, and utilizing (2.10), we deduce

$$\begin{aligned} 0 \leqslant \mathbb{D}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) &= \mathbb{D}(T\zeta_{n-1}, T\zeta_n, T\zeta_n) \\ &\leqslant \alpha(\zeta_{n-1}, \zeta_n, T\zeta_n)\mathbb{D}(T\zeta_{n-1}, T\zeta_n, T\zeta_n) \leqslant \phi(\mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n)). \end{aligned}$$

Therefore,

$$\mathbb{D}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leqslant \phi^n(\mathbb{D}(\zeta_0, \zeta_1, \zeta_1)), \quad \text{for all } n \in \mathbf{N}. \tag{2.4}$$

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbf{N}$ such that $\sum_{n \geqslant n(\varepsilon)} \phi^n(\mathbb{D}(\zeta_0, \zeta_1, \zeta_1)) < \varepsilon$.

By (D3) and (D4), we have

$$\mathbb{D}(\zeta, \eta, \eta) = \mathbb{D}(\eta, \eta, \zeta) \leqslant \mathbb{D}(\eta, \zeta, \zeta) + \mathbb{D}(\zeta, \eta, \zeta) = 2\mathbb{D}(\eta, \zeta, \zeta). \tag{2.5}$$

The inequality (2.5) with $\zeta = \zeta_n$ and $\eta = \zeta_{n-1}$ becomes

$$\mathbb{D}(\zeta_n, \zeta_{n-1}, \zeta_{n-1}) \leqslant 2\mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n). \tag{2.6}$$

Letting $n \to \infty$ in (2.6), we get

$$\lim_{n \to \infty} \mathbb{D}(\zeta_n, \zeta_{n-1}, \zeta_{n-1}) = 0,$$

we show $\{\zeta_n\}$ is Cauchy $(\mathcal{M}, d_{\mathbb{D}})$. Let $n, l \in \mathbf{N}$ with $n > l > n(\varepsilon)$ we obtain

$$\begin{aligned} d_{\mathbb{D}}(\zeta_n, \zeta_l) &\leqslant d_{\mathbb{D}}(\zeta_n, \zeta_{n-1}) + d_{\mathbb{D}}(\zeta_{n-1}, \zeta_{n-2}) + \dots + d_{\mathbb{D}}(\zeta_{l+1}, \zeta_l) \\ &= \mathbb{D}(\zeta_n, \zeta_{n-1}, \zeta_{n-1}) + \mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n) \\ &\quad + \mathbb{D}(\zeta_{n-1}, \zeta_{n-2}, \zeta_{n-2}) + \mathbb{D}(\zeta_{n-2}, \zeta_{n-1}, \zeta_{n-1}) + \dots \\ &\quad + \mathbb{D}(\zeta_{l+1}, \zeta_l, \zeta_l) + \mathbb{D}(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \\ &= \sum_{i=l+1}^{n} [\mathbb{D}(\zeta_i, \zeta_{i-1}, \zeta_{i-1}) + \mathbb{D}(\zeta_{i-1}, \zeta_i, \zeta_i)], \end{aligned} \tag{2.7}$$

By using of (2.12) and (2.6) we obtain

$$\begin{aligned} 0 \leqslant d_{\mathbb{D}}(\zeta_n, \zeta_l) &\leqslant \sum_{i=l+1}^{n} [2\mathbb{D}(\zeta_{i-1}, \zeta_i, \zeta_i) + \mathbb{D}(\zeta_{i-1}, \zeta_i, \zeta_i)] \\ &\leqslant \sum_{i=l+1}^{n} 3\phi^{i-1}(\mathbb{D}(\zeta_0, \zeta_1, \zeta_1)) \\ &\leqslant \sum_{l > n(\varepsilon)} 3\phi^{l}(\mathbb{D}(\zeta_0, \zeta_1, \zeta_1)) < \varepsilon. \end{aligned}$$

Thus $\{\zeta_n\}$ is Cauchy in $(\mathcal{M}, d_D)$. $(\mathcal{M}, \mathbb{D})$ is $\mathbb{D}$-complete, hence $(\mathcal{M}, d_{\mathbb{D}})$ is complete and then, $\{\zeta_n\}$ converges to $u \in \mathcal{M}$. Furthermore, $\{\zeta_n\}$ is $\mathbb{D}$-Cauchy in $(\mathcal{M}, \mathbb{D})$. Now we show that $u \in \cap_{j=1}^{m} \mathfrak{A}_j$. If $\zeta_0 \in \mathfrak{A}_1$, then, $\{\zeta_{m(n-1)}\}_{n=1}^{\infty} \in \mathfrak{A}_1$, $\{\zeta_{m(n-1)+1}\}_{n=1}^{\infty} \in \mathfrak{A}_2$, by continuing, $\{\zeta_{mn-1}\}_{n=1}^{\infty} \in \mathfrak{A}_m$. All the m subsequences are $\mathbb{D}$-convergent so converge to the same limit u. Moreover, the sets $\mathfrak{A}_j$ are D-closed, thus the limit $u \in \cap_{j=1}^{m} \mathfrak{A}_j$. In fact $u \in \mathcal{M}$ is a fixed point of T, considering (2.9) and setting $\zeta = \zeta_n$, $\eta = \zeta = Tu$ with assuming that $u \neq Tu$ or $d_{\mathbb{D}}(u, Tu) > 0$, we get,

$$\begin{aligned} 0 \leqslant d_{\mathbb{D}}(\zeta_n, Tu) &= \mathbb{D}(\zeta_n, Tu, Tu) + \mathbb{D}(Tu, \zeta_n, \zeta_n) \\ &= \mathbb{D}(T\zeta_{n-1}, Tu, Tu) + \mathbb{D}(Tu, T\zeta_{n-1}, T\zeta_{n-1}) \\ &\leqslant \mathbb{D}(T\zeta_{n-1}, Tu, Tu) + 2\mathbb{D}(T\zeta_{n-1}, Tu, Tu) \\ &\leqslant 3\alpha(\zeta_{n-1}, u, u)\mathbb{D}(T\zeta_{n-1}, Tu, Tu) \\ &\leqslant 3\phi(\mathbb{D}(\zeta_{n-1}, u, u)). \end{aligned} \tag{2.8}$$

Tending $n \to \infty$, we end up with $0 \leqslant d_{\mathbb{D}}(u, Tu) \leqslant 0$ which contradicts the assumption $d_{\mathbb{D}}(u, Tu) > 0$, hence $u = Tu$ and then $u \in \mathcal{M}$ is a fixed point of T. To prove the uniqueness, suppose $\nu \in \mathcal{M}$ is another fixed point of T such that $\nu \neq u$. Both u and ν lie in $\cap_{j=1}^{m}\mathfrak{A}j$, thus we can substitute $\zeta = u$ and $\eta = \zeta = \nu$ in (2.9). So

$$\mathbb{D}(Tu, T\nu, T\nu) \leqslant \alpha(u, \nu, \nu)\mathbb{D}(Tu, T\nu, T\nu) \leqslant \phi(\mathbb{D}(u, \nu, \nu)).$$

From lemma 2.8 and $\nu = T\nu$ we have

$$\mathbb{D}(u, \nu, \nu) \leqslant \alpha(u, \nu, \nu)\mathbb{D}(Tu, T\nu, T\nu) < \mathbb{D}(u, \nu, \nu),$$

which is contradiction, Thus $u = \nu$, and the fixed point of T is unique. □

Theorem 2.10. *Let $(\mathcal{M}, \mathbb{D})$ be a $\mathbb{D}-$complete $\mathbb{D}$-metric space and $\{\mathfrak{A}_j\}_{j=1}^{m}$ be a family of nonempty $\mathbb{D}$-closed subsets of $\mathcal{M}$. Let $\eta = \cup_{j=1}^{m}\mathfrak{A}_j$ and $T : Y \to Y$ be a $\alpha-$admissible satisfying*

$$T(\mathfrak{A}_j) \subseteq \mathfrak{A}_{j+1}, \quad j = 1, ..., m, \quad \textit{where} \quad \mathfrak{A}_{m+1} = \mathfrak{A}_1.$$

If there exist two functions $\alpha : Y \times Y \times Y \to \mathfrak{R}^{+}$ and $\phi \in \Phi$ such that

$$\forall \ \zeta, \eta \in \mathcal{M}, \ \ (\mathbb{D}(T\zeta, T\eta, T\zeta) > 0 \Rightarrow \tau + \alpha(\zeta, \eta, T\zeta)F(\mathbb{D}(T\zeta, T\eta, T\zeta)) \leqslant F(\phi(\mathbb{D}(\zeta, \eta, \zeta)))). \tag{2.9}$$

holds for all $\zeta \in \mathfrak{A}_j$ and $\eta, \zeta \in \mathfrak{A}_{j+1}$, $j = 1, ..., m$, and there exists $\zeta_0 \in Y$ such that $\alpha(\zeta_0, T\zeta_0, T^2\zeta_0) \geqslant 1$, then T has a unique fixed point in $\cap_{j=1}^{m}\mathfrak{A}_j$.

Proof. Suppose $\zeta_0 \in Y$ with $\alpha(\zeta_0, T\zeta_0, T^2\zeta_0) \geqslant 1$, assume that $\zeta_0 \in \mathfrak{A}_1$. Define

$$\zeta_n = T\zeta_{n-1} \quad \text{for all} \ \ n \in \mathbf{N}.$$

Since T is cyclic, $\zeta_0 \in \mathfrak{A}_1$, $\zeta_1 = T(\zeta_0) \in \mathfrak{A}_2, ...$ and so on. If $\zeta_{n_0+1} = \zeta_{n_0}$ for some $n_0 \in \mathbf{N}$, then, the fixed point of T is ζ_{n_0}. If $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbf{N}$, since T is $\alpha-$admissible, we have

$$\alpha(\zeta_0, \zeta_1, \zeta_2) = \alpha(\zeta_0, T\zeta_0, T^2\zeta_0) \geqslant 1 \Rightarrow \alpha(T\zeta_0, T\zeta_1, T\zeta_2) = \alpha(\zeta_1, \zeta_2, \zeta_3) \geqslant 1.$$

By induction, we get

$$\alpha(\zeta_{n-1}, \zeta_n, \zeta_{n+1}) \geqslant 1, \quad \text{for all} \ \ n \in \mathbf{N}. \tag{2.10}$$

Applying the inequality (2.9) with $\zeta = \zeta_{n-1}$ and $\eta = \zeta = \zeta_n$, and using (2.10), we obtain

$$\begin{aligned} 0 \leqslant F(\mathbb{D}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) &= F(\mathbb{D}(T\zeta_{n-1}, T\zeta_n, T\zeta_n)) \\ &\leqslant \alpha(\zeta_{n-1}, \zeta_n, T\zeta_n)F(\mathbb{D}(T\zeta_{n-1}, T\zeta_n, T\zeta_n)) \\ &\leqslant F(\phi(\mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n))) - \tau \\ &< F((\mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n))) - \tau. \end{aligned} \tag{2.11}$$

So,

$$F(\mathbb{D}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) \leqslant F(\mathbb{D}(\zeta_0, \zeta_1, \zeta_1)) - n\tau, \quad \text{for all} \ \ n \in \mathbf{N}. \tag{2.12}$$

tending $n \to \infty$ we have

$$\lim_{n\to\infty} F(\mathbb{D}(\zeta_n, \zeta_{n+1}, \zeta_{n+1})) = -\infty.$$

Thus,

$$\lim_{n\to\infty} \mathbb{D}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0.$$

Hence

$$\lim_{n\to\infty} \mathbb{D}(\zeta_n, \zeta_{n-1}, \zeta_{n-1}) = 0.$$

Also if put $\gamma_n = \mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n)$, then by using (2.11) we obtain,

$$(\gamma_n)^k F(\gamma_n) \quad \leqslant (\gamma_n)^k F(\gamma_0) - (\gamma_n)^k n\tau. \tag{2.13}$$

Thus

$$(\gamma_n)^k F(\gamma_n) - (\gamma_n)^k F(\gamma_0) \leqslant (\gamma_n)^k (F(\gamma_0) - n\gamma) - (\gamma_n)^k F(\gamma_0) = -(\gamma_n)^k n\tau \leqslant 0$$

By attention to, $\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$ and by $\lim_{n\to\infty} \gamma_n = 0$ and Letting $n \to \infty$ in (2.13), we get

$$\lim_{n\to\infty} (\gamma_n)^k n = 0. \tag{2.14}$$

From (2.14) there exists $n_1 \in \mathbf{N}$ with $(\gamma_n)^k n \leqslant 1$ for all $n \geqslant n_1$, hen we have

$$\gamma_n \leqslant \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \geqslant n_1.$$

We show $\{\zeta_n\}$ is cauchy $(\mathcal{M}, d_{\mathbb{D}})$ where $d_{\mathbb{D}}$ is given in (2.1). Let $n, l \in \mathbf{N}$ with $n > l > n_1$ we obtain

$$\begin{aligned} d_{\mathbb{D}}(\zeta_n, \zeta_l) &\leqslant d_{\mathbb{D}}(\zeta_n, \zeta_{n-1}) + d_{\mathbb{D}}(\zeta_{n-1}, \zeta_{n-2}) + ... + d_{\mathbb{D}}(\zeta_{l+1}, \zeta_l) \\ &= \mathbb{D}(\zeta_n, \zeta_{n-1}, \zeta_{n-1}) + \mathbb{D}(\zeta_{n-1}, \zeta_n, \zeta_n) \\ &\quad + \mathbb{D}(\zeta_{n-1}, \zeta_{n-2}, \zeta_{n-2}) + \mathbb{D}(\zeta_{n-2}, \zeta_{n-1}, \zeta_{n-1}) + ... \\ &\quad + \mathbb{D}(\zeta_{l+1}, \zeta_l, \zeta_l) + \mathbb{D}(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \\ &= \sum_{i=l+1}^{n} [\mathbb{D}(\zeta_i, \zeta_{i-1}, \zeta_{i-1}) + \mathbb{D}(\zeta_{i-1}, \zeta_i, \zeta_i)]. \end{aligned} \tag{2.15}$$

Tending $n, l \to \infty$ we obtain that $\{\zeta_n\}$ is Cauchy in the $(\zeta, d_{\mathbb{D}})$. $(\mathcal{M}, \mathbb{D})$ is $\mathbb{D}$-complete, hence, $\{\zeta_n\}$ converges $u \in \mathcal{M}$. Furthermore, $\{\zeta_n\}$ is $\mathbb{D}$-Cauchy in $(\mathcal{M}, \mathbb{D})$. Now $u \in \cap_{j=1}^m \mathfrak{A}_j$. if $\zeta_0 \in \mathfrak{A}_1$, then $\{\zeta_{m(n-1)}\}_{n=1}^\infty \in \mathfrak{A}_1$, $\{\zeta_{m(n-1)+1}\}_{n=1}^\infty \in \mathfrak{A}_2$, by continuing, $\{\zeta_{mn-1}\}_{n=1}^\infty \in \mathfrak{A}_m$. All the m subsequences are $\mathbb{D}$-convergent and hence, they all converge to the same limit u. In addition, the sets $\mathfrak{A}_j$ are $\mathbb{D}$-closed, thus the limit $u \in \cap_{j=1}^m \mathfrak{A}_j$. $u \in \mathcal{M}$ is a fixed point of T, because by (2.1) and (2.9) with $\zeta = \zeta_n$, $\eta = \zeta = Tu$ if $u \neq Tu$ or $d_{\mathbb{D}}(u, Tu) > 0$, then,

$$\begin{aligned} 0 \leqslant d_{\mathbb{D}}(\zeta_n, Tu) &= \mathbb{D}(\zeta_n, Tu, Tu) + \mathbb{D}(Tu, \zeta_n, \zeta_n) \\ &= \mathbb{D}(T\zeta_{n-1}, Tu, Tu) + \mathbb{D}(Tu, T\zeta_{n-1}, T\zeta_{n-1}) \\ &\leqslant \mathbb{D}(T\zeta_{n-1}, Tu, Tu) + 2\mathbb{D}(T\zeta_{n-1}, Tu, Tu) \\ &= 3\mathbb{D}(T\zeta_{n-1}, Tu, Tu) \\ &= 3(\mathbb{D}(\zeta_n, u, u)). \end{aligned} \tag{2.16}$$

Tending $n \to \infty$, $0 \leqslant d_{\mathbb{D}}(u, Tu) \leqslant 0$ which contradicts the assumption $d_{\mathbb{D}}(u, Tu) > 0$, hence $u = Tu$, so $u \in \mathcal{M}$ is a fixed point of T. Suppose $v \in \mathcal{M}$ is another fixed point of T such that $v \neq u$. Both u and v lie in $\cap_{j=1}^{m} \mathbb{D}j$, thus we can substitute $\zeta = u$ and $\eta = \zeta = v$ in (2.9). This yields

$$F(\mathbb{D}(Tu, Tv, Tv)) + \tau \leqslant \alpha(u, v, v)F(\mathbb{D}(Tu, Tv, Tv)) + \tau \leqslant F(\varphi(\mathbb{D}(u, v, v))),$$

therefore

$$F(\mathbb{D}(Tu, Tv, Tv)) \leqslant F(\varphi(\mathbb{D}(u, v, v))).$$

F is strictly increasing, by 2.8 we obtain

$$\mathbb{D}(Tu, Tv, Tv) \leqslant \varphi(\mathbb{D}(u, v, v)) < \mathbb{D}(u, v, v),$$

this is a contradiction, thus $u = v$, and the fixed point of T is unique. □

3. Conclusion

In the current study, we used the F-contraction mapping introduced by Wardowski to obtain fixed point results by method of Samet in generalized complete metric spaces. Over the last decade authors proved some fixed point results for F-contraction mappings in metric spaces. We showed that this results hold in $\mathbb{D}$-metric spaces under some conditions.

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