



On fuzzy Henstock-Kurzweil-Stieltjes-Diamond-double integral on time scales

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Abstract

We introduce and study some properties of fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral on time scales. Also, we state and prove the uniform convergence theorem, monotone convergence theorem and dominated convergence theorem for the fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable functions on time scales.

Keywords: Fuzzy, Convergence theorems, Double integral, Henstock-Kurzweil integral, Time scales.

1. Introduction

In modern day analysis, Henstock [1] and Kurzweil [2] independently introduced the concepts of Henstock and Kurzweil integration. This later developed into Henstock-Kurzweil integration which is the generalization of the two integrals for real-valued functions. The generalization of this concept in the fuzzy setting is a rare case. Wu and Gong [3] introduced Henstock integral of fuzzy-number-valued functions, fuzzy sets and systems and presented some of its basic properties. Gong and Shao [4] gave the controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions, fuzzy sets and systems. For other interesting results involving fuzzy Henstock-Kurzweil integral, see e.g., the papers [3, 4, 5] and references cited therein.

In 1988, the theory of time scales was introduced by Hilger in his Ph.D. thesis [6]. The aim is to unify and generalize the concept of discrete and continuous dynamical systems. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [7], and Henstock-Kurzweil integrals on time scales were studied by Thomson [8]. Park et al. [9] studied the convergence results for the Henstock delta integral on time scales. It is clear that most of the properties of a time scale integral can be realized by

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using the techniques tailored to the time scale setting (see [6, 8, 9, 10, 11, 12, 13]) and references cited therein.

In this paper, we introduce fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral on time scales.

2. Preliminaries

A time scale \mathbb{T} is any closed non-empty subset of \mathbb{R} , with the topology inherited from the standard topology on the real numbers \mathbb{R} .

Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and a rectangle $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \rightarrow \mathbb{R}$ be two non-decreasing functions on $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$, respectively. Let $F : \mathbb{T}_1 \times \mathbb{T}_2 \rightarrow \mathbb{R}$ be bounded on \mathcal{R} . Let P_1 and P_2 be two partitions of $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, \dots, t_n\} \subset [a, b)_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, \dots, s_n\} \subset [c, d)_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, \dots, \xi_n\}$ denote an arbitrary selection of points from $[a, b)_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, \dots, n$. Similarly, let $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ denote an arbitrary selection of points from $[c, d)_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, \dots, k$.

Definition 2.1. ([5]) Let α be a real axis, a fuzzy subset of $\alpha : \mathbb{R} \rightarrow [0, 1]$ is called a fuzzy number if the following conditions occur:

- (i) α is normal. That is $x_0 \in \mathbb{R}$ exists with $\alpha(x_0) = 1$;
- (ii) α is fuzzy convex, i.e. $\alpha(kx_1 + (1 - k)x_2) \geq \min\{\alpha(x_1), \alpha(x_2)\}$ for all $x_1, x_2 \in \mathbb{R}$ and all $k \in (0, 1)$;
- (iii) α is upper semi-continuous;
- (iv) $[\alpha]^0 = \{x \in \mathbb{R} : \alpha(x) > 0\}$ is compact.

We shall denote the space of fuzzy numbers by $f_{\mathbb{R}}$ and define the λ -level set $[\alpha]^\lambda$ by

$$[\alpha]^\lambda = \{x \in \mathbb{R} : \alpha(x) \geq \lambda, \lambda \in (0, 1)\}.$$

By the conditions (i)-(iv) of Definition 2.1, denote $[\alpha]^\lambda$ by $[\alpha]^\lambda = [\underline{\alpha}^\lambda, \overline{\alpha}^\lambda]$ and for $\alpha_1, \alpha_2 \in f_{\mathbb{R}}$ and $k \in \mathbb{R}$, we define

$$[\alpha_1 + \alpha_2]^\lambda = [\alpha_1]^\lambda + [\alpha_2]^\lambda \text{ and } [k \odot \alpha_1]^\lambda = k[\alpha_1]^\lambda$$

for all $\lambda \in [0, 1]$.

Definition 2.2. ([14]) Let $(f_{\mathbb{R}}, D)$ be a complete metric space. The Hausdorff distance between α_1 and α_2 is defined by

$$D(\alpha_1, \alpha_2) = \sup_{\lambda \in [0, 1]} \max\{|\underline{\alpha}_1^\lambda - \underline{\alpha}_2^\lambda|, |\overline{\alpha}_1^\lambda - \overline{\alpha}_2^\lambda|\}.$$

We now introduce Henstock-Kurzweil-Stieltjes- \diamond -double integral over versions in $\mathbb{T}_1 \times \mathbb{T}_2$. Let $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ be a rectangle, and denote by $f_{\mathbb{R}}$ the space of fuzzy numbers on real line.

Definition 2.3. Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ be a bounded function on \mathcal{R} and let g_1 and g_2 be increasing functions defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ with partitions $P_1 = \{t_0, t_1, \dots, t_n\} \subset$

$[a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, 2, \dots, n$ and $P_2 = \{s_0, s_1, \dots, s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j = 1, 2, \dots, k$. Then

$$S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^n \sum_{j=1}^k F(\xi_i, \zeta_j) (g_1(t_i) - g_1(t_{i-1})) (g_2(s_j) - g_2(s_{j-1}))$$

is defined as fuzzy Henstock-Kurweil-Stieltjes- \diamond -double sum of F with respect to functions g_1 and g_2 .

Let $P = P_1 \times P_2$ and $\diamond g_{1_i} \diamond g_{2_j} = (g_1(t_i) - g_1(t_{i-1})) (g_2(s_j) - g_2(s_{j-1}))$, then the Henstock-Kurweil-Stieltjes- \diamond -double sum of F with respect to functions g_1 and g_2 is denoted by $S(P, F, g)$ is written as

$$S(P, F, g_1, g_2) = \sum_{i=1}^n \sum_{j=1}^k F(\xi_i, \zeta_j) \diamond g_{1_i} \diamond g_{2_j}, \quad (i = 1, \dots, n; j = 1, \dots, k).$$

3. Main Results

Definition 3.1. Let $F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ be a fuzzy function on $\mathcal{R} = [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} : t \in [a, b]_{\mathbb{T}_1}, s \in [c, d]_{\mathbb{T}_2}$. We say that F is fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable with respect to non-decreasing functions g_1, g_2 defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ if there is a number \tilde{L} , a member of \mathbb{R} such that for every $\varepsilon > 0$, there is a \diamond -gauge δ (or γ) such that

$$D(S(P, F, g_1, g_2), \tilde{L}) < \varepsilon$$

provided that $P_1 = \{t_0, t_1, \dots, t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, \dots, n$ and $P_2 = \{s_0, s_1, \dots, s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, \dots, k$ are δ -fine (or γ) partitions of $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$.

A positive function $\delta(t, s), \gamma(t, s) : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ such that $\delta(t, s) > 0$ for all t, s in $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ or ($\gamma(t, s) > 0$ for all t, s in $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$) is known as \diamond gauge on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$.

We say that \tilde{L} is the fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral of F with respect to g_1 and g_2 defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$, and write

$$\iint_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \tilde{L}.$$

The family of all fuzzy Henstock-Kurzweil-Stieltjes- \diamond -integrable functions on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is denoted by $\mathcal{FHKS}_{[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}}$.

Lemma 3.2. ([15]) Suppose that $\alpha \in f_{\mathbb{R}}$. Then,

(i) the interval $[\alpha]^\lambda$ is closed for $\lambda \in [0, 1]$;

(ii) $[\alpha]^{\lambda_1} \supset [\alpha]^{\lambda_2}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

(iii) for any sequence $\{\lambda_n\}$ satisfying $\lambda_n \leq \lambda_{n+1}$ and $\lambda_n \rightarrow \lambda \in (0, 1]$, we have

$$\bigcap_{n=1}^{\infty} [\alpha]^{\lambda_n} = [\alpha]^\lambda.$$

Now we state and provide proofs for our theorems.

Theorem 3.3. *If $F : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ is fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable with respect to two increasing functions g_1, g_2 on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$, then the fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral of F is unique.*

Proof. Suppose that \tilde{L}_1 and \tilde{L}_2 are both fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrals of F on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$. With the assumption that \tilde{L}_1 and \tilde{L}_2 are not unique, then F is said to be fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ if it satisfies the following point wise integrability criterion: for every $\varepsilon > 0$ there are \diamond -gauges δ_1 and δ_2 (or γ_1 and γ_2) defined on $(a, b]_{\mathbb{T}_1}$ and $(c, d]_{\mathbb{T}_2}$ respectively, such that for every $\varepsilon > 0$, there are \diamond -gauges δ_1^1 and δ_2^1 (or γ_1^1 and γ_2^1) for $[a, b]_{\mathbb{T}_1}$ and δ_1^2 and δ_2^2 (or γ_1^2 and γ_2^2) for $[c, d]_{\mathbb{T}_2}$ such that

$$D(S(P^1, F, g_1, g_2), \tilde{L}_1) < \frac{\varepsilon}{2} \text{ and } D(S(P^2, F, g_1, g_2), \tilde{L}_2) < \frac{\varepsilon}{2} \text{ for all pairs } P^1 = P_1^1 \times P_2^1 \text{ and } P^2 = P_1^2 \times P_2^2 \text{ of } \delta_1\text{-fine (or } \gamma_1\text{) and } \delta_2\text{-fine (or } \gamma_2\text{) partitions of } [a, b]_{\mathbb{T}_1} \text{ and } [c, d]_{\mathbb{T}_2} \text{ respectively.}$$

and for every $\varepsilon > 0$ and $i \in \{1, 2\}$, there are \diamond -gauges δ_1^i and δ_2^i (or γ_1^i and γ_2^i) for $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ respectively such that

$$D(S(P^i, F, g_1, g_2), \tilde{L}_i) < \frac{\varepsilon}{2}$$

provided that $P^i = P_1^i \times P_2^i$ is a pair of δ_1^i -fine (or γ_1^i) and δ_2^i -fine (or γ_2^i) partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ respectively.

Let $\delta_1 = \min\{\delta_1^1, \delta_1^2\}$ i.e. $(\delta_1)_L = \min\{(\delta_1^1)_L, (\delta_1^2)_L\}$ and $(\delta_1)_R = \min\{(\delta_1^1)_R, (\delta_1^2)_R\}$ and $\delta_2 = \min\{\delta_2^1, \delta_2^2\}$ i.e. $(\delta_2)_L = \min\{(\delta_2^1)_L, (\delta_2^2)_L\}$ and $(\delta_2)_R = \min\{(\delta_2^1)_R, (\delta_2^2)_R\}$, δ_1 and δ_2 are \diamond -gauges for $(a, b]_{\mathbb{T}_1}$ and $(c, d]_{\mathbb{T}_2}$ respectively, and given a pair $P = P_1 \times P_2$ of δ_1 -fine and δ_2 -fine partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, P_1 is a δ_1^1 -fine and δ_1^2 -fine partition of $(a, b]_{\mathbb{T}_1}$, P_2 is a δ_2^1 -fine and δ_2^2 -fine partition of $[c, d]_{\mathbb{T}_2}$, hence

$$\begin{aligned} D(\tilde{L}_1, \tilde{L}_2) &\leq D((\tilde{L}_1, S(P, F, g_1, g_2) + S(P, F, g_1, g_2), \tilde{L}_2)) \\ &\leq D(S(P, F, g_1, g_2), \tilde{L}_1) + D(S(P, F, g_1, g_2), \tilde{L}_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

since for all $\varepsilon > 0$, there are \diamond -gauges δ_1 and δ_2 (or γ_1 and γ_2), then it follows that $\tilde{L}_1 = \tilde{L}_2$.

Hence, the fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral of F on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is unique. \square

Theorem 3.4. *(Bolzano Cauchy Criterion). Let $F : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ be a fuzzy-valued function over a rectangle $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ with respect to g_1, g_2 on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$. Then, F is fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ if and only if for each $\varepsilon > 0$ there exists a positive function $\delta : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ such that $D(S(P_{\delta_1}, F, g_1, g_2), S(P_{\delta_2}, F, g_1, g_2), F, g)) < \varepsilon$ for all δ -fine tagged partitions P_1 and P_2 on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$.*

Proof. Suppose F is fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ with respect to g_1 and g_2 , and let

$$\tilde{L} = \int \int_{\mathcal{R}} F(t, s) \diamond_{g_1}(t) \diamond_{g_2}(s).$$

Let $\varepsilon > 0$. There are \diamond -gauges δ_1 and δ_2 for $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ respectively such that $D(S(P, F, g_1, g_2), \tilde{L}) < \frac{\varepsilon}{2}$ provided that $P = P_1 \times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a, b]_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c, d]_{\mathbb{T}_2}$. Therefore, if $P = P_1 \times P_2$ and $P = P'_1 \times P'_2$ are pairs of δ_1 (or γ_1) fine partition of $[a, b]_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c, d]_{\mathbb{T}_2}$, then

$$\begin{aligned} D(S(P, F, g_1, g_2), S(P^1, F, g_1, g_2)) &\leq D(S(P, F, g_1, g_2), \tilde{L}) \\ &\quad + D(\tilde{L}, S(P^1, F, g_1, g_2)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, suppose that for all $\varepsilon > 0$ there are \diamond -gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ respectively such that $D(S(P^1, F, g_1, g_2), S(P^2, F, g_1, g_2)) < \varepsilon$ for all pairs $P^1 = P^1_1 \times P^1_2$ and $P^2 = P^2_1 \times P^2_2$ of δ_1 (or γ_1)-fine partitions of $[a, b]_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c, d]_{\mathbb{T}_2}$.

Let $n \in \mathbb{N}$. Taking $\varepsilon = \frac{1}{n}$, there are \diamond -gauges $\delta_{1,n}$ and $\delta_{2,n}$ (or $\gamma_{1,n}$ and $\gamma_{2,n}$) for $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ respectively such that

$D(S(P', F, g_1, g_2), S(P^2, F, g_1, g_2)) < \varepsilon$ for all pairs $P^1 = P^1_1 \times P^1_2$ and $P^2 = P^2_1 \times P^2_2$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a, b]_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c, d]_{\mathbb{T}_2}$.

By replacing $\delta_{i,n}$ by $\min\{\delta_{i,1}, \delta_{i,2}, \dots, \delta_{i,n}\}$ with $i \in \{1, 2\}$, we may assume that $\delta_{i,n+1} \leq \delta_{i,n}$. Thus, for all $j > n$ $\delta_{i,j} \leq \delta_{i,n}$ so any pair $P^n = P^n_1 \times P^n_2$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a, b]_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c, d]_{\mathbb{T}_2}$ is also a pair of $\delta_{1,j}$ (or $\gamma_{1,j}$)-fine partitions of $[a, b]_{\mathbb{T}_1}$ and $\delta_{2,j}$ (or $\gamma_{2,j}$)-fine partitions of $[c, d]_{\mathbb{T}_2}$, hence

$$D(S(P^n, F, g_1, g_2), S(P^j, F, g_1, g_2)) < \frac{1}{j}.$$

This shows that $\{S(P^n, F, g_1, g_2)\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Let \tilde{L} be the limit of $\{S(P^n, F, g_1, g_2)\}_{n \in \mathbb{N}}$. For all $\varepsilon > 0$, choosing $N > \frac{2}{\varepsilon}$, for \diamond -gauges $\delta_{1,N}$ and $\delta_{2,N}$ (or $\gamma_{1,N}$ and $\gamma_{2,N}$) for $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ respectively,

$$\begin{aligned} D(S(P, F, g_1, g_2), \tilde{L}) &\leq D(S(P, F, g_1, g_2), S(P^N, F, g_1, g_2)) \\ &\quad + D(S(P^N, F, g_1, g_2), \tilde{L}) \\ &< \frac{1}{N} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for pair $P = P_1 \times P_2$ such that P_1 is a $\delta_{1,N}$ (or $\gamma_{1,N}$) fine partition of $[a, b]_{\mathbb{T}_1}$ and P_2 is a $\delta_{2,N}$ (or $\gamma_{2,N}$) fine partition of $[c, d]_{\mathbb{T}_2}$. \square

The following properties are obtained using the definition of fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral of F on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$. The proof of the next theorem is straightforward and therefore omitted.

Theorem 3.5. Let $F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ be a fuzzy-valued function and let g_1 and g_2 be non-decreasing functions respectively on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$. If F is fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable with respect to g_1 and g_2 on $\mathcal{R} = [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$. Then,

- i. $\int \int_{\mathcal{R}} \beta \diamond_1 g_1(t) \diamond_2 g_2(s) = \beta(g_1(b) - g_1(a))(g_2(d) - g_2(c))$, β is a constant;
- ii. $\int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = 0$ when g_1 or g_2 are constants;
- iii. $\int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = f(a, c)(g_1^{\tau_1}(a) - g_1(a))(g_2^{\tau_2}(c) - g_2(c))$
with $b = \tau_1(a)$ and $\tau_2(c)$;
- iv. $\int \int_{\mathcal{R}} \beta F(t, s) \mu[\diamond_1 g_1(t) \diamond_2 g_2(s)] = \int \int_{\mathcal{R}} \beta \mu f(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)$;
 β and μ are constants.

Theorem 3.6. Let F be a fuzzy-number-valued function, consider a sequence of fuzzy-number-valued function $F_n : (a, b)_{\mathbb{T}_1} \times (c, d)_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$, $n \in \mathbb{N}$ in $f_{\mathbb{R}}$ and increasing functions $g_1, g_2 : (a, b)_{\mathbb{T}_1} \times (c, d)_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$. Assume

- (i) $\lim_{n \rightarrow \infty} F_n(t, s) = F(t, s)$ holds \diamond a.e.;
- (ii) $G(t, s) \leq F_n(t, s) \leq H(t, s)$ holds \diamond a.e.;
- (iii) $F_n(t, s), G(t, s), H(t, s) \in \mathcal{FHKKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$.

Then $F(t, s) \in \mathcal{FHKKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$. Moreover,

$$\lim_{n \rightarrow \infty} \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s).$$

The proof of Theorem 3.6 is straightforward following the style of proof in [7].

Theorem 3.7. Let $F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ be a fuzzy-number-valued function and let g_1 and g_2 be non-decreasing functions respectively on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$. Function $F(t, s) \in \mathcal{FHKKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$ if and only if $\underline{F(t, s)}^\lambda, \overline{F(t, s)}^\lambda \in \mathcal{HKKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$ for all $\lambda \in [0, 1]$ uniformly.

Proof. For the necessary condition, let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, \dots, t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, \dots, s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\tilde{L} = \int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)$. Given $\varepsilon > 0$, there exists a $\delta(t, s)$ such that $D(S(P_\delta, F, g_1, g_2), \tilde{L}) < \varepsilon$ for any fine tag partition P_1 and P_2 . Then,

$$\begin{aligned} & \sup_{\lambda \in [0, 1]} \max \left\{ \left| [S(P_\delta, F, g_1, g_2)]^\lambda - \underline{\tilde{L}}^\lambda \right|, \left| [\overline{S(P_\delta, F, g_1, g_2)}]^\lambda - \overline{\tilde{L}}^\lambda \right| \right\} \\ &= \sup_{\lambda \in [0, 1]} \max \left\{ \left| [S(P_\delta, \underline{F}^\lambda, g_1^\lambda, g_2^\lambda)] - \underline{\tilde{L}}^\lambda \right|, \left| [S(P_\delta, \overline{F}^\lambda, g_1^\lambda, g_2^\lambda)] - \overline{\tilde{L}}^\lambda \right| \right\} < \varepsilon \end{aligned}$$

and

$$\left| [S(P_\delta, \underline{F}^\lambda, g_1^\lambda, g_2^\lambda)] - \underline{\tilde{L}}^\lambda \right| < \varepsilon, \quad \left| [S(P_\delta, \overline{F}^\lambda, g_1^\lambda, g_2^\lambda)] - \overline{\tilde{L}}^\lambda \right| < \varepsilon$$

for any $\lambda \in [0, 1]$ and for partitions $P = P_1 \cup P_2$. Thus, $\underline{F(t, s)^\lambda}, \overline{F(t, s)^\lambda} \in \mathcal{HK}_{[a, b]_{T_1} \times [c, d]_{T_2}}$ uniformly for any $\lambda \in [0, 1]$.

Now for sufficient condition, let $\varepsilon > 0$. By assumption, there exists a $\delta(t, s)$ such that

$$\left| [S(P_\delta, \underline{F}^\lambda, g_1^\lambda, g_2^\lambda)] - \underline{L}^\lambda \right| < \varepsilon, \quad \left| [S(P_\delta, \overline{F}^\lambda, g_1^\lambda, g_2^\lambda)] - \overline{L}^\lambda \right| < \varepsilon$$

for any $\lambda \in [0, 1]$ and for partitions $P = P_1 \cup P_2$ where

$$\underline{L}^\lambda = \int \int_{\mathcal{R}} \underline{F(t, s)^\lambda} \diamond_1 g_1(t) \diamond_2 g_2(s), \quad \overline{L}^\lambda = \int \int_{\mathcal{R}} \overline{F(t, s)^\lambda} \diamond_1 g_1(t) \diamond_2 g_2(s).$$

To prove that $\left\{ \left[\underline{L}^\lambda, \overline{L}^\lambda \right], \lambda \in [0, 1] \right\}$ represents a fuzzy number, check that $\left[\underline{L}^\lambda, \overline{L}^\lambda \right]$ satisfies the conditions (i)-(iii) of Lemma :

(i) for $\lambda \in [0, 1]$, if $\underline{F(t, s)^\lambda} \leq \overline{F(t, s)^\lambda}$, then $\underline{L}^\lambda \leq \overline{L}^\lambda$, i.e., the interval $\left[\underline{L}^\lambda, \overline{L}^\lambda \right]$ is closed.

(ii) $F(t, s)^\lambda$ and $\overline{F(t, s)^\lambda}$ nondecreasing and nonincreasing functions on $[0, 1]$ respectively. For any $0 \leq \lambda_1 \leq \lambda_2 \leq 1$

$$\begin{aligned} \int \int_{\mathcal{R}} \underline{F(t, s)^{\lambda_1}} \diamond_1 g_1(t) \diamond_2 g_2(s) &\leq \int \int_{\mathcal{R}} \underline{F(t, s)^{\lambda_2}} \diamond_1 g_1(t) \diamond_2 g_2(s) \\ &\leq \int \int_{\mathcal{R}} \overline{F(t, s)^{\lambda_2}} \diamond_1 g_1(t) \diamond_2 g_2(s) \\ &\leq \int \int_{\mathcal{R}} \overline{F(t, s)^{\lambda_1}} \diamond_1 g_1(t) \diamond_2 g_2(s). \end{aligned}$$

Thus, $\left[\underline{L}^{\lambda_1}, \overline{L}^{\lambda_1} \right] \supset \left[\underline{L}^{\lambda_2}, \overline{L}^{\lambda_2} \right]$.

(iii) Now, for any $\{\lambda_n\}$ satisfying $\lambda_n \leq \lambda_{n+1}$ and $\lambda_n \rightarrow \lambda \in (0, 1]$, we have

$$\bigcap_{n=1}^{\infty} [F(t, s)]^{\lambda_n} = [F(t, s)]^\lambda,$$

that is,

$$\bigcap_{n=1}^{\infty} \left[\underline{F(t, s)^{\lambda_n}}, \overline{F(t, s)^{\lambda_n}} \right] = \left[\underline{F(t, s)^\lambda}, \overline{F(t, s)^\lambda} \right],$$

$\lim_{n \rightarrow \infty} \underline{F(t, s)^{\lambda_n}} = \underline{F(t, s)^\lambda}$ and $\lim_{n \rightarrow \infty} \overline{F(t, s)^{\lambda_n}} = \overline{F(t, s)^\lambda}$. Moreover,

$$\underline{F(t, s)^0} \leq \underline{F(t, s)^{\lambda_n}} \leq \underline{F(t, s)^1}, \quad \overline{F(t, s)^1} \leq \overline{F(t, s)^{\lambda_n}} \leq \overline{F(t, s)^0}.$$

By Theorem 3.6, we have $\underline{F(t, s)^\lambda}, \overline{F(t, s)^\lambda} \in \mathcal{HK}_{[a, b]_{T_1} \times [c, d]_{T_2}}$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \int \int_{\mathcal{R}} \underline{F(t, s)^{\lambda_n}} \diamond_1 g_1(t) \diamond_2 g_2(s) &= \int \int_{\mathcal{R}} \underline{F(t, s)^\lambda} \diamond_1 g_1(t) \diamond_2 g_2(s), \\ \lim_{n \rightarrow \infty} \int \int_{\mathcal{R}} \overline{F(t, s)^{\lambda_n}} \diamond_1 g_1(t) \diamond_2 g_2(s) &= \int \int_{\mathcal{R}} \overline{F(t, s)^\lambda} \diamond_1 g_1(t) \diamond_2 g_2(s). \end{aligned}$$

Consequently,

$$\bigcap_{n=1}^{\infty} [\underline{L}^{\tilde{\lambda}_n}, \overline{L}^{\tilde{\lambda}_n}] = [\underline{L}^{\tilde{\lambda}}, \overline{L}^{\tilde{\lambda}}].$$

Now, $\tilde{L} = \{[\underline{L}^{\tilde{\lambda}}, \overline{L}^{\tilde{\lambda}}], \lambda \in [0, 1]\}$. Thus,

$$D(S(P, F, g_1, g_2), \tilde{L}) < \varepsilon$$

for each partitions $P = P_1 \cup P_2$ and for any n . This ends the proof. \square

Definition 3.8. A sequence $\{F_n(t, s)\}$ of Henstock-Kurzweil-Stieltjes integrable functions is called uniformly fuzzy Henstock-Kurzweil-Stieltjes- \diamond -integrable on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ with respect to increasing functions g_1 and g_2 if for each $\varepsilon > 0$ there exists a $\delta(t, s)$ such that

$$D\left(S(P, F_n(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) < \varepsilon$$

for any partition P and for any $n \in \mathbb{N}$.

Theorem 3.9. Let $F_n(t, s) \in \mathcal{FHKKS}_{[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}}$, $n = 1, 2, \dots$, satisfy:

(i) $\lim_{n \rightarrow \infty} F_n(t, s) = F(t, s)$ on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$;

(ii) $F_n(t, s)$ are uniformly fuzzy Henstock-Kurzweil-Stieltjes- \diamond -integrable on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$.

Then $F(t, s) \in \mathcal{FHKKS}_{[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}}$ and

$$\lim_{n \rightarrow \infty} \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s).$$

Proof. Let $\varepsilon > 0$. There exists a $\delta(t, s)$ such that

$$D\left(S(P, F_n(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) < \varepsilon$$

for any partition P and for every n . Fix a $P_0 \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$. From above statement of Theorem 3.6, there exists N such that $P_0 = P_1 \times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a, b]_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c, d]_{\mathbb{T}_2}$ and P_0 is a δ_0 fine partition of $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$. Then

$$D(S(P_{\delta_0}, F_n(t, s), g_1, g_2), S(P_{\delta_0}, F_m(t, s), g_1, g_2)) < \varepsilon$$

for arbitrary $n, m > N$. Then,

$$\begin{aligned} & D\left(\int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s), \int \int_{\mathcal{R}} F_m(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\ & \leq D\left(S(P_{\delta_0}, F_n(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\ & + D(S(P_{\delta_0}, F_n(t, s), g_1, g_2), S(P_{\delta_0}, F_m(t, s), g_1, g_2)) \\ & + D\left(S(P_{\delta_0}, F_m(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_m(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\ & < 3\varepsilon \end{aligned}$$

for any $n, m > N$ and, hence, $\{\int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\}$ is a Cauchy sequence. Let

$$\lim_{n \rightarrow \infty} \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \tilde{L}.$$

We now prove that

$$\int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \tilde{L}.$$

Let $\varepsilon > 0$. By hypothesis, there exists a $\delta(t, s)$ such that

$$D\left(S(P, F_n(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) < \varepsilon$$

for any partition P and for all n . Choose N satisfying

$$D\left(\int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s), \tilde{L}\right) < \varepsilon$$

for all $n > N$. Let there exists $N_0 > N$ satisfying

$$D(S(P_\delta, F_{N_0}(t, s), g_1, g_2), S(P_\delta, F(t, s), g_1, g_2)) < \varepsilon.$$

Therefore,

$$\begin{aligned} & D(S(P, F(t, s), g_1, g_2), \tilde{L}) \\ & \leq D(S(P_\delta, F(t, s), g_1, g_2), S(P_\delta, F_{N_0}(t, s), g_1, g_2)) \\ & + D\left(S(P_\delta, F_{N_0}(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_{N_0}(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\ & + D\left(\int \int_{\mathcal{R}} F_{N_0}(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s), \tilde{L}\right) \\ & < 3\varepsilon \end{aligned}$$

and the result follows. \square

Definition 3.10. ([12]) A function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is called absolutely continuous on $[a, b]_{\mathbb{T}}$, if for each $\varepsilon > 0$ there exists $\gamma > 0$ such that

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \varepsilon$$

whenever $\bigcup_{i=1}^n [x_{i-1}, x_i]_{\mathbb{T}} \subset [a, b]_{\mathbb{T}}$ and $\sum_{i=1}^n \Delta x_i < \gamma$

The following theorem is Dominated convergence theorem.

Theorem 3.11. Let $F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \rightarrow f_{\mathbb{R}}$ be a fuzzy-number-valued function and let g_1 and g_2 be non-decreasing functions respectively on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$. If $F_n(t, s) \in \mathcal{F}\mathcal{H}\mathcal{K}\mathcal{S}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$, $n = 1, 2, \dots$, satisfy:

(i) $\lim_{n \rightarrow \infty} F_n(t, s) = F(t, s)$ a.e.;

(ii) $G(t, s) \leq F_n(t, s) \leq H(t, s)$ a.e., and $G(t, s), H(t, s) \in \mathcal{FHKS}_{[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}}$; then sequence $\{F_n(t, s)\}$ is uniformly FHKS- \diamond -integrable. Thus, $F(t, s) \in \mathcal{FHKS}_{[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}}$ and

$$\lim_{n \rightarrow \infty} \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s).$$

Proof. Let $\varepsilon > 0$ and for arbitrary $p, q > N$, we have

$$\begin{aligned} D(F_p(t, s), F_q(t, s)) &= \sup_{\lambda \in [0,1]} \max \left\{ |\underline{F_p}(t, s)^\lambda - \underline{F_q}(t, s)^\lambda|, |\overline{F_p}(t, s)^\lambda - \overline{F_q}(t, s)^\lambda| \right\} \\ &\leq \sup_{\lambda \in [0,1]} \max \left\{ |\underline{H}(t, s)^\lambda - \underline{G}(t, s)^\lambda|, |\overline{H}(t, s)^\lambda - \overline{G}(t, s)^\lambda| \right\} \\ &= D(H(t, s), G(t, s)). \end{aligned}$$

Then, $D(H(t, s), G(t, s))$ is Lebesgue- \diamond -integrable. Let

$$D(t, s) = \int_{\mathcal{R}} D(H(x, y), G(x, y)) \diamond_1 g_1(x) \diamond_2 g_2(y).$$

From Definition 3.3, $D(t, s)$ is absolutely continuous on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$. Let $\varepsilon > 0$. Then there exists $\gamma > 0$ such that

$$\sum_{i=1}^n \sum_{j=1}^k |D(\xi_i, \zeta_j) - D(\xi_{i-1}, \zeta_{j-1})| < \frac{\varepsilon}{(b-a)(d-c)}$$

whenever $\bigcup_{i=1}^n [\xi_{i-1}, \xi_i]_{\mathbb{T}_1} \subset [a, b]_{\mathbb{T}_1}$ and $\bigcup_{j=1}^k [\zeta_{j-1}, \zeta_j]_{\mathbb{T}_2} \subset [c, d]_{\mathbb{T}_2}$ and $\sum_{i=1}^n \sum_{j=1}^k F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1})) < \gamma$. The limit

$$\lim_{n \rightarrow \infty} F_n(t, s) = F(t, s)$$

holds almost everywhere on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ and $\{D(F_n(t, s), F(t, s))\}$ is a sequence of \diamond -measurable functions. By Egorov's theorem, there is an open set Ω with $m(\Omega) < \delta$ such that $\lim_{n \rightarrow \infty} F_n(t, s) = F(t, s)$ uniformly for $t, s \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \setminus \Omega$. Thus, there exists N such that

$$D(F_p(t, s), F_q(t, s)) < \frac{\varepsilon}{(b-a)(d-c)}$$

for any $p, q > N$ and for any $t, s \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \setminus \Omega$.

Suppose that δ_1 is in the set of \diamond -gauge on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ such that

$$\left| S(D(H(t, s), G(t, s)), P_{\delta_1}, g_1, g_2) - \int_{\mathcal{R}} D(H(t, s), G(t, s)) \diamond_1 g_1(t) \diamond_2 g_2(s) \right| < \varepsilon$$

and

$$D\left(S(P_{\delta_1}, F_n(t, s), g_1, g_2), \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) < \varepsilon$$

for $1 \leq n \leq N$ and for any partition P . Define $\delta(\xi, \zeta)$ in the set of \diamond -gauge on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ as

$$\delta(\xi, \zeta) = \begin{cases} \delta_1(\xi, \zeta), & \text{if } \xi, \zeta \in [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \setminus \Omega \\ \min\{\delta_1(\xi, \zeta), \rho((\xi, \zeta), \Omega)\}, & \text{if } (\xi, \zeta) \in \Omega, \end{cases}$$

where $\rho((\xi, \zeta), \Omega) = \inf\{|\xi - \xi'| : (\xi', \zeta') \in \Omega\}$. Fix $n > N$. We have

$$\begin{aligned}
& D(S(P_\delta, F_n(t, s), g_1, g_2), S(P_\delta, F_N(t, s), g_1, g_2)) \\
&= D\left(\sum_{i=1}^n \sum_{j=1}^k F_n(\xi_i, \zeta_j) \diamond g_1(t) g_2(s), \sum_{i=1}^n \sum_{j=1}^k F_N(\xi_i, \zeta_j) \diamond g_1(t) g_2(s)\right) \\
&+ D\left(\sum_{\xi_i \in \Omega} \sum_{\zeta_j \in \Omega} F_n(\xi_i, \zeta_j) \diamond g_1(t) g_2(s), \sum_{\xi_i \in \Omega} \sum_{\zeta_j \in \Omega} F_N(\xi_i, \zeta_j) \diamond g_1(t) g_2(s)\right) \\
&\leq \varepsilon + \sum_{\xi_i \in \Omega} \sum_{\zeta_j \in \Omega} D(F_n(\xi_i, \zeta_j), F_N(\xi_i, \zeta_j)) \diamond g_1(t) g_2(s) \\
&\leq \varepsilon + \left| \sum_{\xi_i \in \Omega} \sum_{\zeta_j \in \Omega} D(H(\xi_i, \zeta_j), G(\xi_i, \zeta_j)) \diamond g_1(t) g_2(s) \right. \\
&\quad \left. - \int \int_{\Omega} D(H(t, s), G(t, s)) \diamond g_1(t) g_2(s) \right| \\
&+ \left| \int \int_{\Omega} D(H(t, s), G(t, s)) \diamond g_1(t) g_2(s) \right| \\
&\leq 3\varepsilon
\end{aligned}$$

for any partition P . Hence,

$$\begin{aligned}
& D\left(S(P, F_n(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\
&\leq D(S(P, F_n(t, s), g_1, g_2), S(P, F_N(t, s), g_1, g_2)) \\
&+ D\left(S(P, F_N(t, s), g_1, g_2), \int \int_{\mathcal{R}} F_N(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\
&+ D\left(\int \int_{\mathcal{R}} F_N(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s), \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s)\right) \\
&\leq 5\varepsilon,
\end{aligned}$$

which ends the proof of dominated convergence theorem for fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable functions on time scales. \square

We obtain the following monotone convergence theorem as a consequence of dominated convergence theorem. The monotone convergence theorem is stated as follows:

Theorem 3.12. *Let $F : [a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2} \rightarrow \mathbb{F}_{\mathbb{R}}$ be a fuzzy-number-valued function and let g_1 and g_2 be non-decreasing functions respectively on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$. If $F_n(t, s) \in \mathcal{FHKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$, $n = 1, 2, \dots$, satisfy:*

(i) $\lim_{n \rightarrow \infty} F_n(t, s) = F(t, s) \diamond a.e.$;

(ii) $\{F_n(t, s)\}$ is a monotone sequence and $F_n(t, s) \in \mathcal{FHKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$;

then sequence $\{F_n(t, s)\}$ is uniformly FHKS- \diamond -integrable. Consequently, $F(t, s) \in \mathcal{FHKS}_{[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}}$ and

$$\lim_{n \rightarrow \infty} \int \int_{\mathcal{R}} F_n(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s) = \int \int_{\mathcal{R}} F(t, s) \diamond_1 g_1(t) \diamond_2 g_2(s).$$

4. Conclusion

This paper exposed some of the properties of fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integral on time scales. Uniform convergence theorem and dominated convergence theorem are proved for the fuzzy Henstock-Kurzweil-Stieltjes- \diamond -double integrable functions on time scales. We also obtained the monotone convergence theorem as a consequence of dominated convergence theorem.

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