Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2021)2(2) : 30-37
doi:10.48185/jmam.v2i2.276

On B-covariant derivative of first order for some tensors in different spaces

ALAA A. ABDALLAH [a,*], A. A. NAVLEKAR [b], KIRTIWANT P. GHADLE [c]

[a,c] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, 431001, India

[b] Department of Mathematics Pratishitan Mahavidyalaya, Paithan (M.S.) India

• *Received: 06 May 2021* • *Accepted: 24 June 2021* • *Published Online: 30 June 2021*

Abstract

In this paper, we study the relationship between Cartan's second curvature tensor P^i_{jkh} and $(h)hv-$torsion tensor C^i_{jk} in sense of Berwald. Moreover, we discuss the necessary and sufficient condition for some tensors which satisfy a recurrence property in BC-RF_n, P2-Like-BC-RF_n, P^*-BC-RF_n and P-reducible-$BC-RF_n$.

Keywords: BC-Recurrent Finsler space, Cartan's second curvature tensor, $(h)hv-$torsion tensor, recurrence property.

1. Introduction

The concept of C-recurrent space in sense of Cartan and Berwald has been studying by Matsumoto [9] and Sarangi and Goswami [8], respectively. Mishra and Lodhi [2] discussed the properties of C^h-recurrent and C^v-recurrent Finsler spaces, Mohammed [1] introduced P^h-recurrent space and studied the properties of P2-like space and P^*-space in P^h-recurrent space, Pandey and Dikshit [12] discussed P^*-and P-reducible Finsler space of recurrent curvature.

Let F_n be an n-dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [4]. The (h) hv-torsion tensor C^i_{jk} is the associate tensor of the tensor C_{ijk} which are defined by

$$\begin{cases} a)\ C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0, & b) C^i_{jk} y^j = C^i_{kj}\, y^j = 0, \\ c) C^h_{ik} = g^{hj} C_{ijk}, & d) C_{ijk} = g_{hi} C^h_{jk}, \\ e) C^i_{ji} = C_j \text{ and} & f) C_k\, y^k = C, \end{cases} \tag{1.1}$$

*Corresponding author: ala733.ala00@gmail.com where $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\, \dot{\partial}_k\, F^2$.

Berwald covariant derivative $B_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k given by

$$B_k T^i_j := \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

Berwald covariant derivative of the vector y^i vanish identically, i.e.

$$B_k y^i = 0. \tag{1.2}$$

But, in general, Berwald covariant derivative of the metric tensor g_{ij} does not vanish and given by

$$B_k g_{ij} = -2C_{ijk|h} y^h = -2y^h B_h C_{ijk}\,. \tag{1.3}$$

The tensor P^i_{jkh} called $hv-$ curvature tensor (Cartan's second curvature tensor) is positively homogeneous of degree -1 in y^i and defined by [4]

$$P^i_{jkh} := C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^i{}_{rh} - P^r_{jh} C^i{}_{rk}, \tag{1.4}$$

which satisfies the relations

$$a) P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r \quad \text{and} \quad b)\ P_{ijkh} = g_{ir}\, P^r_{jkh}, \tag{1.5}$$

where P_{ijkh} called *associate tensor of* $hv-$ *curvature tensor* and P^i_{kh} called $v\ (hv)-$ *torsion tensor* which satisfies

$$a)\ P^i_{kh} y^k\ =\ 0, \qquad b)\ P_{rkh} = g_{ir} P^i_{kh} \quad \text{and} \quad c)\ P^i_{kh} = P_{rkh} g^{ir}, \tag{1.6}$$

where P_{rkh} called *associative tensor for* $v\,(hv)$*-torsion tensor*.

$P-$ Ricci tensor P_{jk} , curvature vector P_k and curvature scalar P of Cartan's second curvature tensor given by

$$a) P_{jk} = P^i_{jki}, \qquad b)\ P_k = P^i_{ki} \quad \text{and} \quad c)\ P = P_k\, y^k, \tag{1.7}$$

respectively.

Recently, Alaa et al. [5, 6, 7] discussed the necessary and sufficient condition for some tensors which satisfy the generalized recurrence property in $G(BP) - RF_n$ and studied some properties of P2-like space, P^*-space and P-reducible space in it.

2. On BC-Recurrent Space

Matsumoto [9] introduced a Finsler space which the *(h)hv*-torsion tensor C^i_{jk} satisfies the recurrence property in $h-$ covariant derivative (Cartan's second kind covariant differentiation) and called it C^h-recurrent space. This space characterized by the conditions

$$a) C^i_{kh|m} = \lambda_m C^i_{kh} \qquad \text{and} \qquad b) C_{jkh|m} = \lambda_m C_{jkh}, \tag{2.1}$$

where C_{ijk} is associate tensor of C^i_{jk} .

Sarangi and Goswami [8] introduced a Finsler space for which the (h) $h\nu$-torsion tensor C^i_{jk} satisfies the recurrence property in sense of Berwald and called it C-recurrent space. Let us denote this space briefly by a $BC\text{-}RF_n$.

This space characterized by the conditions

$$a) B_m C^i_{kh} = \lambda_m C^i_{kh} \qquad \text{and} \qquad b) B_m C_{jkh} = \lambda_m C_{jkh}. \tag{2.2}$$

Let consider a $BC - RF_n$.

Using (2.1)(a) and (2.1)(b) in (1.4), we get

$$P^i_{jkh} = \lambda_j C^i_{kh} - \lambda^i C_{jkh} + C^r_{jk} P^i_{rh} - C^i_{rk} P^r_{jh}, \tag{2.3}$$

where $\lambda^i = \lambda_r g^{ir}$.

Taking B-covariant derivative for eq. (2.3) with respect to x^m , using (2.2)(a) and (2.2)(b) in the resulting equation, we get

$$\begin{aligned} B_m P^i_{jkh} &= \lambda_m \left(\lambda_j C^i_{kh} - \lambda^i C_{jkh} + C^r_{jk} P^i_{rh} - C^i_{rk} P^r_{jh}\right) + \left(B_m \lambda_j\right) C^i_{kh} - \left(B_m \lambda^i\right) C_{jkh} \\ &+ C^r_{jk} B_m P^i_{rh} - C^i_{rk} B_m P^r_{jh}. \end{aligned}$$

Using eq. (2.3) in above equation, we get

$$B_m P^i_{jkh} = \lambda_m P^i_{jkh} + \left(B_m \lambda_j\right) C^i_{kh} - \left(B_m \lambda^i\right) C_{jkh} + C^r_{jk} B_m P^i_{rh} - C^i_{rk} B_m P^r_{jh}. \tag{2.4}$$

This shows that $B_m P^i_{jkh} = \lambda_m P^i_{jkh}$ if and only if

$$\left(B_m \lambda_j\right) C^i_{kh} - \left(B_m \lambda^i\right) C_{jkh} + C^r_{jk} B_m P^i_{rh} - C^i_{rk} B_m P^r_{jh} = 0. \tag{2.5}$$

Transvecting eq. (2.4) by g_{ir}, then using (1.3) and (1.5)(b), we get

$$\begin{aligned} B_m P_{rjkh} &= \lambda_m P_{rjkh} + g_{ir}\{\left(B_m \lambda_j\right) C^i_{kh} - \left(B_m \lambda^i\right) C_{jkh} + C^r_{jk} B_m P^i_{rh} \\ &- C^i_{rk} B_m P^r_{jh}\} - 2 P^i_{jkh} y^s B_s C_{irm}. \end{aligned} \tag{2.6}$$

This shows that $B_m P_{rjkh} = \lambda_m P_{rjkh}$ if and only if

$$g_{ir}\{\left(B_m \lambda_j\right) C^i_{kh} - \left(B_m \lambda^i\right) C_{jkh} + C^r_{jk} B_m P^i_{rh} - C^i_{rk} B_m P^r_{jh}\} - 2 P^i_{jkh} y^s B_s C_{irm} = 0. \tag{2.7}$$

Transvecting eq. (2.4) by y^j , using (1.5)(a), (1.2), (1.1)(a), (1.1)(b) and (1.6)(a), we get

$$B_m P^i_{kh} = \lambda_m \ P^i_{kh} + \left(B_m \lambda_j\right) C^i_{kh} \ y^j. \tag{2.8}$$

This shows that $B_m P^i_{kh} = \lambda_m P^i_{kh}$ if and only if

$$\left(B_m \lambda_j\right) C^i_{kh} \ y^j = 0. \tag{2.9}$$

Contracting the indices i and h in eq. (2.4), using (1.7)(a), (1.1)(e) and (1.7)(b), we get

$$B_m P_{jk} = \lambda_m P_{jk} + \left(B_m \lambda_j\right) C_k - \left(B_m \lambda^i\right) C_{jki} + C^r_{jk} B_m P_r - C^i_{rk} B_m P^r_{ji}. \tag{2.10}$$

This shows that $B_m P_{jk} = \lambda_m P_{jk}$ if and only if

$$\left(B_m \lambda_j\right) C_k - \left(B_m \lambda^i\right) C_{jki} + C^r_{jk} B_m P_r - C^i_{rk} B_m P^r_{ji} = 0. \tag{2.11}$$

Contracting the indices i and h in eq. (2.8), using (1.7)(b) and (1.1)(e), we get

$$B_m P_k = \lambda_m P_k + \left(B_m \lambda_j\right) C_k\, y^j. \tag{2.12}$$

This shows that $B_m P_k = \lambda_m P_k$ if and only if

$$\left(B_m \lambda_j\right) C_k\, y^j = 0. \tag{2.13}$$

Transvecting eq. (2.12) by y^k , using (1.7)(c), (1.2) and (1.1)(f), we get

$$B_m P = \lambda_m P + \left(B_m \lambda_j\right) C y^j. \tag{2.14}$$

This shows that $B_m P = \lambda_m P$ if and only if

$$\left(B_m \lambda_j\right) C\, y^j = 0. \tag{2.15}$$

Consequently, from previous equations, we proved that, the behavior of P^i_{jkh} , P_{rjkh} , P^i_{kh} , P_{jk} , P_k and P as recurrent if and only if eqs. (2.5), (2.7), (2.9), (2.11), (2.13) and (2.15), respectively hold. Thus, we conclude:

Theorem 2.1. *In $BC\text{-}RF_n$, Cartan's second curvature tensor P^i_{jkh}, the associate curvature tensor P_{rjkh} , the torsion tensor P^i_{kh}, the P-Ricci tensor P_{jk}, the curvature vector P_k and the curvature scalar P satisfy the recurrence property if and only if eqs. (2.5), (2.7), (2.9), (2.11), (2.13) and (2.15) respectively hold.*

3. A P2-LikeBC-Recurrent Space

A P2-Like space is characterized by the condition[10]

$$P^i_{jkh} = \varphi_j C^i_{kh} - \varphi^i C_{jkh}, \tag{3.1}$$

where φ_j and φ^i are non-zero covariant and contravariant vectors field, respectively.

Definition 3.1. The BC-recurrent space which is P2-Like space [satisfies the condition (3.1)], will be called a P2-Like BC-recurrent space and we will denote it briefly by a P2-Like-BC-RF_n.

Let consider a P2-Like-BC-RF_n. Taking B-covariant derivative for the condition (3.1) with respect to x^m , using (2.2)(a) and (2.2)(b) in the resulting equation, we get

$$B_m P^i_{jkh} = \lambda_m \left(\varphi_j C^i_{kh} - \varphi^i C_{jkh}\right) + \left(B_m \varphi_j\right) C^i_{kh} - \left(B_m \varphi^i\right) C_{jkh}.$$

Using the condition (3.1) in above equation, we get

$$B_m P^i_{jkh} = \lambda_m P^i_{jkh} + \left(B_m \varphi_j\right) C^i_{kh} - \left(B_m \varphi^i\right) C_{jkh}. \tag{3.2}$$

This shows that $B_m P^i_{jkh} = \lambda_m P^i_{jkh}$ if and only if

$$\left(B_m \varphi_j\right) C^i_{kh} - \left(B_m \varphi^i\right) C_{jkh} = 0. \tag{3.3}$$

Transvecting eq. (3.2) by g_{ir} , using (1.5)(b) and (1.3), we get

$$B_m P_{rjkh} = \lambda_m P_{rjkh} + g_{ir}\{\left(B_m \varphi_j\right) C^i_{kh} - \left(B_m \varphi^i\right) C_{jkh}\} - 2P^i_{jkh} y^s B_s C_{irm}. \tag{3.4}$$

This shows that $B_m P_{rjkh} = \lambda_m P_{rjkh}$ if and only if

$$g_{ir}\{\left(B_m \varphi_j\right) C^i_{kh} - \left(B_m \varphi^i\right) C_{jkh}\} - 2P^i_{jkh} y^s B_s C_{irm} = 0. \tag{3.5}$$

Contracting the indices i and h in eq. (3.2), using (1.7)(a) and (1.1)(e), we get

$$B_m P_{jk} = \lambda_m P_{jk} + \left(B_m \varphi_j\right) C_k - \left(B_m \varphi^i\right) C_{jki}. \tag{3.6}$$

This shows that $B_m P_{jk} = \lambda_m P_{jk}$ if and only if

$$\left(B_m \varphi_j\right) C_k - \left(B_m \varphi^i\right) C_{jki} = 0. \tag{3.7}$$

Consequently, from previous equations, we proved that, the behavior of P^i_{jkh} , P_{rjkh} and P_{jk} as recurrent if and only if eqs. (3.3), (3.5) and (3.7), respectively hold. Thus, we conclude

Theorem 3.2. *In P2-Like-BC-RF$_n$, Cartan's second curvature tensor P^i_{jkh}, the associate curvature tensor P_{rjkh} and the P-Ricci tensor P_{jk} satisfy the recurrence property if and only if eqs. (3.3), (3.5) and (3.7), respectively hold.*

4. A P*- BC-Recurrent Space

A P^*-Finsler space is characterized by the condition [3]

$$P^i_{kh} = \varphi C^i_{kh}. \tag{4.1}$$

Definition 4.1. The BC-recurrent space which is P^*-space [satisfies the condition (4.1)], will be called a P^*-BC-recurrent space and we will denote it briefly by a P^*-BC-RF$_n$.

Let consider a P^*-BC-RF$_n$. Taking B-covariant derivative for the condition (4.1) with respect to x^m , using (2.2)(a) in the resulting equation, we get

$$B_m P^i_{kh} = \left(B_m \varphi\right) C^i_{kh} + \varphi \lambda_m C^i_{kh}.$$

Using the condition (4.1) in above equation, we get

$$B_m P^i_{kh} = \lambda_m P^i_{kh} + \left(B_m \varphi\right) C^i_{kh}. \tag{4.2}$$

This shows that $B_m P^i_{kh} = \lambda_m P^i_{kh}$ if and only if

$$\left(B_m \varphi\right) C^i_{kh} = 0. \tag{4.3}$$

Transvecting eq. (4.2) by g_{ij} , using (1.6)(b), (1.3) and (1.1)(d), we get

$$B_m P_{jkh} = \lambda_m P_{jkh} + (B_m \varphi)\, C_{jkh} + 2P^i_{kh} y^s B_s C_{ijm}. \tag{4.4}$$

This shows that $B_m P_{jkh} = \lambda_m P_{jkh}$ if and only if

$$(B_m \varphi)\, C_{jkh} + 2P^i_{kh} y^s B_s C_{ijm} \; = 0. \tag{4.5}$$

Contracting the indices i and h in eq. (4.2), using (1.7)(b) and (1.1)(e), we get

$$B_m P_k = \lambda_m P_k + (B_m \varphi)\, C_k. \tag{4.6}$$

This shows that $B_m P_k = \lambda_m P_k$ if and only if

$$(B_m \varphi)\, C_k = 0. \tag{4.7}$$

Transvecting eq. (4.6) by y^k , using (1.7)(c) and (1.1)(f), we get

$$B_m P = \lambda_m P + (B_m \varphi)\, C. \tag{4.8}$$

This shows that $B_m P = \lambda_m P$ if and only if

$$(B_m \varphi)\, C = 0. \tag{4.9}$$

Consequently, from previous equations, we proved that, the behavior of P^i_{kh}, P_{jkh}, P_k and P as recurrent if and only if eqs. (4.3), (4.5), (4.7) and (4.9), respectively hold. Thus, we conclude:

Theorem 4.2. *In P^*-BC-RF$_n$, the torsion tensor P^i_{kh}, the associate curvature tensor P_{jkh}, the curvature vector P_k and the curvature scalar P satisfy the recurrence property if and only if eqs. (4.3), (4.5), (4.7) and (4.9), respectively hold.*

5. A P-Reducible-BC-Recurrent Space

A P− reducible space is characterized by the condition [11, 13]

$$P_{jkh} = \lambda C_{jkh} + \varphi \left(h_{jk} C_h + h_{kh} C_j + h_{hj} C_k \right), \tag{5.1}$$

where λ and φ are scalar vectors positively homogeneous of degree one in y^j and h_{jk} is the angular metric tensor.

Definition 5.1. The BC-recurrent space which is P -reducible space [satisfies the condition (5.1)], will be calleda P-reducible-BC-recurrent space and we will denote it briefly by a P-reducible-BC-RF$_n$.

Transvecting the condition (5.1) by g^{ij} using (1.6)(c) and (1.1)(c), we get

$$P^i_{kh} = \lambda C^i_{kh} + \varphi \left(h^i_k C_h + h_{kh} C^i + h^i_h C_k \right), \tag{5.2}$$

where $h^i_k = g^{ij} h_{jk}$ and $C^i = g^{ij} C_j$. Let consider a P-reducible-BC-RF$_n$.

Taking B-covariant derivative for the condition (5.2) with respect to x^m, using (2.2)(a) in the resulting equation, we get

$$B_m P^i_{kh} = \lambda\lambda_m C^i_{kh} + (B_m\lambda)\, C^i_{kh} + B_m\left[\varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right)\right].$$

Using the condition (5.2) in above equation, we get

$$\begin{aligned} B_m P^i_{kh} &= \lambda_m P^i_{kh} - \lambda_m \varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right) + (B_m\lambda)\, C^i_{kh} \\ &\quad + B_m\left[\varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right)\right]. \end{aligned}$$

This shows that

$$B_m\left[\varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right)\right] = \lambda_m \varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right) - (B_m\lambda)\, C^i_{kh} \quad (5.3)$$

if and only if $B_m P^i_{kh} = \lambda_m P^i_{kh}$. Taking B− covariant derivative for the condition (5.1) with respect to x^m , using (2.2)(b) in the resulting equation, we get

$$B_m P_{jkh} = \lambda\lambda_m C_{jkh} + (B_m\lambda)\, C_{jkh} + B_m\left[\varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right)\right].$$

Using the condition (5.1) in above equation, we get

$$\begin{aligned} B_m P_{jkh} &= \lambda_m P_{jkh} - \lambda_m \varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right) + (B_m\lambda)\, C_{jkh} \\ &\quad - B_m\left[\varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right)\right]. \end{aligned}$$

This shows that

$$B_m\left[\varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right)\right] = \lambda_m \varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right) - (B_m\lambda)\, C_{jkh} \quad (5.4)$$

if and only if $B_m P_{jkh} = \lambda_m P_{jkh}$.

Consequently, we proved that, Berwald's covariant derivative of the first order for the tensors $\varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right)$ and $\varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right)$ satisfy eqs. (5.3) and (5.4) if and only if the torsion tensor P^i_{kh} and the associate torsion tensor P_{jkh} behave as recurrent, respectively hold. Thus, we conclude the following theorem:

Theorem 5.2. *In P-reducible-BC-RF$_n$, Berwald's covariant derivative of the first order for the tensors $\varphi\left(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\right)$ and $\varphi\left(h^i_k C_h + h_{kh}C^i + h^i_h C_k\right)$ are given by eqs. (5.3) and (5.4) if and only if the torsion tensor P^i_{kh} and the associate torsion tensor P_{jkh} satisfy the recurrence property, respectively hold.*

6. Conclusion

We obtained the necessary and sufficient condition for Cartan's second curvature tensor P^i_{jkh}, associate curvature tensor P_{ijkh} , torsion tensor P^i_{kh}, P-Ricci tensor P_{jk}, curvature vector P_k and scalar curvature P which satisfy the recurrence property in BC-RF$_n$, P2-Like -BC-RF$_n$, P*-BC-RF$_n$ and P-reducible -BC-RF$_n$, we got the relationship between Cartan's second curvature tensor P^i_{jkh} and (h) hv-torsion tensor C^i_{jk} in sense of Berwald.

References

[1] Mohammed AH (2016). *On study of* P^h*recurrent curvature tensors in different types of Finsler space*. M.Sc. Thesis, University of Aden, (Aden) (Yemen).

[2] Mishra CK, Lodhi G (2008). *On* C^h*-recurrent and* C^v- *recurrent Finsler spaces of second order*. Int. Jour. Contemp. Math. Sci., **3**(17): 801-810.

[3] Izumi H (1976). *On* P^**-Finsler spaces I*. Memo. Defence Acad. (Japan), **16**: 133-138.

[4] Rund H (1981). *The differential geometry of Finsler spaces*. Springer-Verlag, Berlin Göttingen, (1959); 2nd Edit. (in Russian), Nauka, (Moscow).

[5] Alaa AA, Navlekar AA, Ghadle KP (2019). *The necessary and sufficient condition for some tensors which satisfy a generalized* BP*-recurrent Finsler space*. International Journal of Scientific and Engineering Research, **10**(11): 135-140.

[6] Alaa AA, Navlekar AA, Ghadle KP (2019). *Special types of generalized* BP*-recurrent spaces*. Journal of Computer and Mathematical Sciences, **10**(5): 972-979.

[7] Alaa AA, Navlekar AA, Ghadle KP (2019). *On* P^**-and* P*-reducible of Cartan's second curvature tensor*. Journal of Applied Science and computation, **VI**(XI): 13-24.

[8] Sarangi KC, Goswami A (2006). *On C-recurrent Finsler space*. Journal of International Academy of physical Science, **10**: 17-24.

[9] Matsumoto M (1971). *On* h*-isotropic and* C^h*-recurrent Finsler*. J. Math. Kyoto Univ. **11**: 1-9.

[10] Matsumoto M (1971). *On Finsler spaces with curvature tensor of some special forms*. Tensor N.S. **22**: 201-204.

[11] Dwivedi PK (2011). P*-reducible Finsler spaces and applications*. int. journal of Math. Analysis, **5**(5): 223-229.

[12] Pandey PN, Dikshit S (1998). *On* P^**- and* P*-ruducible Finsler spaces of recurrent curvature*. Journal of International Academy of Physical Science, **2**: 11-17.

[13] Saxena PS (2013). *A study of* P^**-ruducible Finsler space with Douglas tensor*. Journal of International Academy of physical Science, **17**(3): 277-285.