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A Hybrid Kharrat-Toma Transform with Homotopy Perturbation Method for Solving Integro-Differential Equations

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Abstract

This article presents an effective hybrid semi-analytical method for dealing with the integro-differential equations. This new technique is based on the combining of the Kharrat-Toma integral transform with the homotopy perturbation method to find the exact or approximate solutions of both linear and nonlinear models. Several examples are illustrated to test the power and efficiency of the proposed method. Further, the obtained results of the hybrid approach are implemented in Maple software.

Keywords: Kharrat-Toma integral transform method, Homotopy perturbation method, Integro-differential equations.

1. Introduction

Integro—differential equations play an important role in many fields of engineering and applied sciences. The linear and nonlinear integro—differential equations are used to model many phenomena in chemistry, biology, astrophysics and physical science. Therefore, the researchers are interested in these types of problems, which are sometimes difficult to solve by traditional methods.

In the past, different numerical, analytical and semi—analytical methods were employed to solve the integro—differential equations such as the moving least square method [1], finite difference method [3], spectral homotopy analysis method [4], Taylor expansion method [5], variational iteration method and Adomian decomposition method [6] and a hybrid natural transform homotopy perturbation method [7]. Although there are several classical techniques to solve the integro—differential equations, these methods have their own difficulties like the initial approximation and slow convergence. Therefore, many hybrid techniques have been provided to overcome these disadvantages.

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Many research works have been introduced to solve these integro–differential equations such as differential transform method for both integro–differential and integral equation systems [8].Pinheiro et al. developed a new approach that relies on eigenfunctions expansions for solving integro–differential equations in conductive–radiative heat transfer [9]. The authors in [10] employed a computational Haar wavelet collocation technique for the solution of linear delay integral equations that include delay Fredholm, Volterra and Volterra–Fredholm integral equations. In [11], Al–Mdallal presented an effective algorithm which is based on a monotone method for the solution of a class of nonlinear integro–differential equations of the second order. Ahmad and Saeed [12] solved Multi–higher nonlinear Fredholm integro–differential equations of the fractional order using He's homotopy perturbation method with modification. Amin et al. [13] improved the Haar wavelet collocation technique for the numerical solution of Volterra and Volterra–Fredholm fractional integro–differential equations, where the derived system was solved by Gauss elimination method. A theoretical and numerical study of the boundary–value problems for nonlinear fractional integro–differential equation was introduced in [14].

Moreover, Zulkarnain et al. [15] solved a linear Fredholm–Volterra integro–differential equation of the third kind using a modified homotopy perturbation method to obtain the approximate solutions. Asif et al. [16] suggested a new collocation scheme for numerical solution of Fredholm, Volterra and mixed Volterra–Fredholm integral equations of the second kind and the authors developed a numerical integration formula based on the basis of linear Legendre multi–wavelets.
In [17], a combination of the homotopy analysis method and Laplace transform–Adomian method was proposed to find the analytical solution of nonlinear integro–differential equations.

In addition, a new integral transform called Tarig transform was introduced in [18] to solve integro–differential equations with constant coefficients. Babolian and Shamloo [19] used operational matrices of piecewise constant orthogonal functions for solving integro–differential equation numerically. Ziyaee and Tari [20] developed a new differential transform scheme for solving the two–dimensional integro–differential equations of the second kind. Also, a hybrid mechanism for improving the differential transform method which combines the differential transform method, Laplace transform and the Pade‘ approximate, was employed by the authors in [21]. Ahmed et al. demonstrated an efficient double transform named the Laplace–Sumudu transform method to solve integral differential equations [22]. Hendi and Qarni proposed and applied a coupling of the variational iteration method with homotopy perturbation method for nonlinear mixed integro–differential equations [23]. The authors in [24] conducted a comparative study among He's homotopy perturbation method and three traditional techniques for an analytic and approximate treatment of nonlinear integral and integro–differential equations. In [25], the numerical solutions of a system of two nonlinear integro–differential equations, arising in biological species living applications,were obtained via the well–known homotopy perturbation method.

The main contributions of this paper are summarized as follows:

- Suggest a new hybrid scheme for the solutions of integro–differential equations.

- Produce an exact or approximate solution when the classical methods lead to poor results.
- Provide an alternative mechanism to numerical and analytical methods.
- Extend the application of Kharrat–Toma integral transform.

The proposed hybrid method has the following benefits:

- It reduces the differential equation toan algebraic equation.
- The proposed method does not require the choice of an initial approximation as in the homotopy perturbation method.
- The converge to the exact solution is rapid.

Based on these advantages, the proposed hybrid technique is easy to implement versus the existing methods.

The rest of this paper is organized as follows: Section 2 illustrates the Kharrat–Toma integral transform method.Section 3 describes the proposed hybrid methodology for solving linear and nonlinear integro–differential equations. Section 4 shows the numerical examples and results. Finally, the Section 5 draws the conclusion.

2. Kharrat-Toma Transform Method

In 2020, Kharrat and Toma proposed a new integral transform, namely Kharrat–Toma integral transform [26] to solve the initial and boundary value problems represented as ordinary differential equations with initial and boundary conditions.

Definition 2.1. [26] The Kharrat–Toma integral transform and the inverse Kharrat–Toma transform are expressed as follows:

$$B\left[f(x)\right] = G\left(S\right) = s^3 \int_0^\infty f(x)\ e^{\frac{-x}{s^2}}\,dx, \quad x \geqslant 0,$$

$$f(x) = B^{-1}\left[G\left(S\right)\right] = B^{-1}\left[s^3 \int_0^\infty f(x)\ e^{\frac{-x}{s^2}}\,dx\right].$$

The B integral transform states that if $f(x)$is piecewise continuous on $[0,+\infty)$and has exponential order, the B^{-1} will be the inverse of the B integral transform, where

$$B\left[f^{(n)}(x)\right] = \frac{1}{s^{2n}}\ G(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5}\, f^{(k)}(0) \quad ; \ n \geqslant 1$$

The Kharrat–Toma transform of some functions is as follows:

(1)$f(x) = 1 \underset{B^{-1}}{\overset{B}{\longleftrightarrow}} G(s) = s^5$

(2)$f(x) = x^n \underset{B^{-1}}{\overset{B}{\longleftrightarrow}} G(s) = s^{2n+5}.\ n!$

(3)$f(x) = \sin(kx) \underset{B^{-1}}{\overset{B}{\longleftrightarrow}} G(s) = \frac{k\, s^7}{1+k^2\, s^4}$

(4)$f(x) = \cos(kx) \underset{B^{-1}}{\overset{B}{\longleftrightarrow}} G(s) = \frac{s^5}{1+k^2\, s^4}$

(5)$f(x) = sh(kx) \underset{B^{-1}}{\overset{B}{\longleftrightarrow}} G(s) = \frac{k\, s^7}{1-k^2\, s^4}$

(6)$f(x) = \cosh(kx) \underset{B^{-1}}{\overset{B}{\longleftrightarrow}} G(s) = \frac{s^5}{1-k^2\, s^4}$

3. Methodology and Formulation

To demonstrate the basic idea of the new hybrid approach for the solutions of linear and nonlinear integro–differential equations, we consider the following integro–differential equation:

$$v^{(n)}(x) + \psi\left(x, y, y', ..., y^{(n-1)}\right) = f(x) + \int_0^x K(x,t)\phi\, [v(t)]\, dt. \tag{3.1}$$

With initial conditions

$$v^{(k)}(0) = \alpha_k, \quad \alpha_k = const.\ \ k = 0, 1, ..., n-1,$$

where $v(x)$ is an unknown function, $\varphi\, [v(x)]$ is a linear or nonlinear function, $f(x)$ is a known analytical functions on $[0, a]$ where is a real constant and the kernel is $K(x,t)$.

Taking the Kharrat–Toma transform on (3.1), yields

$$\frac{1}{s^{2n}}\, B\,(v) - \sum_{k=0}^{n-1} s^{-2n+2k+5}\, v^{(k)}(0) = B\left[f(x) - \psi\left(x, v, v', ..., v^{(n-1)}\right) + \int_0^x K(x,t)\phi\, [v(t)]\, dt\right]. \tag{3.2}$$

Then

$$B\,(v) = \sum_{k=0}^{n-1} s^{2k+5}\, v^{(k)}(0) + s^{2n} B\,[f(x)] + s^{2n} B\left[-\psi\left(x, v, v', ..., v^{(n-1)}\right) + \int_0^x K(x,t)\, \phi\, [v(t)]\, dt\right]. \tag{3.3}$$

A homotopy of (3.3) which can be constructed as follows:

$$B\,(v) = \sum_{k=0}^{n-1} s^{2k+5}\, v^{(k)}(0) + s^{2n} B\,[f(x)] + p\, s^{2n} B\left[-\psi\left(x, v, v', ..., v^{(n-1)}\right) + \int_0^x K(x,t)\, \phi\, [v(t)]\, dt\right], \tag{3.4}$$

where $p \in [0, 1]$ is an embedding parameter.

According to HPM the solution of (3.4) can be written as a power series in p

$$v=\sum_{i=0}^{\infty} p^i v_i \tag{3.5}$$

The substitution of (3.5) into (3.4), yields

$$\begin{aligned} B\left(\sum_{i=0}^{\infty} p^i v_i\right) &= \sum_{k=0}^{n-1} s^{2k+5} v^{(k)}(0) + s^{2n} B\left[f(x)\right] + p\, s^{2n} B \\ &\times \left[-\psi\left(x, \sum_{i=0}^{\infty} p^i v_i, \sum_{i=0}^{\infty} p^i v'_i, \ldots, \sum_{i=0}^{\infty} p^i v_i^{(n-1)}\right)\right. \\ &\left. + \int_0^x K(x,t)\, \phi\left[\sum_{i=0}^{\infty} p^i v_i(t)\right] dt\right] \end{aligned} \tag{3.6}$$

Comparing the coefficients of terms with identical powers of *p* in (3.6), leads to

$$p^0 : B\left[v_0\right] = \sum_{k=0}^{n-1} s^{2k+5} v^{(k)}(0) + s^{2n} B\left[f(x)\right] \tag{3.7}$$

$$p^{i+1} : B\left[v_{i+1}\right] = s^{2n} B\left[-\psi\left(x, \sum_{i=0}^{\infty} p^i v_i, \sum_{i=0}^{\infty} p^i v'_i, \ldots, \sum_{i=0}^{\infty} p^i v_i^{(n-1)}\right) + \int_0^x K(x,t)\, \phi\left[\sum_{i=0}^{\infty} p^i v_i(t)\right] dt\right], \tag{3.8}$$

where $i=0,1,2,\ldots$.
Applying the inverse Kharrat—Toma integral transform, gives $v_i \quad ; \quad i=0,1,2,\ldots$.
Setting, then the approximate solution of (3.1) can be written as follows:

$$v(x)=\sum_{i=0}^{\infty} v_i(x). \tag{3.9}$$

The convergence of series (3.9) was proved by J. He [26].

4. Illustrative Examples

In this section, we consider four examples to illustrate the performance and efficiency of the present hybrid method.

Example 4.1. Consider the following linear integro-differential equation

$$v'(x)=1+\int_0^x v(t)dt, \quad 0 \leqslant t, x \leqslant 1. \tag{4.1}$$

With initial condition

$$v(0)=1$$

Taking the Kharrat-Toma transform on (4.1), finds

$$\frac{1}{s^2}B\left(v\right)-s^3v(0)=s^5+B\left[\int_0^x v(t)dt\right]. \tag{4.2}$$

Then we have

$$B\left(v\right)=s^5+s^7+s^2B\left[\int_0^x v(t)\ dt\right]. \tag{4.3}$$

Now, constructing the homotopy on (4.3) as follows

$$B\left(v\right)=s^5+s^7+p\ s^2B\left[\int_0^x v(t)\ dt\right]. \tag{4.4}$$

Substituting (3.5) into (4.4), we get

$$B\left(\sum_{i=0}^{\infty}p^i v_i\right)=s^5+s^7+p\ s^2B\left[\int_0^x \sum_{i=0}^{\infty}p^i v_i(t)\ dt\right]. \tag{4.5}$$

Comparing coefficients of terms with identical powers of p in (4.5), leads to

$$p^0: B\left[v_0\right]=s^5+s^7. \tag{4.6}$$

$$p^1: B\left[v_1\right]=s^2B\left[\int_0^x v_0(t)\ dt\right]. \tag{4.7}$$

$$p^2: B\left[v_2\right]=s^2B\left[\int_0^x v_1(t)\ dt\right]. \tag{4.8}$$

Taking the inverse Kharrat–Toma transform of Equations (4.6), (4.7) and (4.8), yields

$$\begin{aligned} v_0 &= 1+x \\ v_1 &= \tfrac{x^2}{2!}+\tfrac{x^3}{3!} \\ v_2 &= \tfrac{x^4}{4!}+\tfrac{x^5}{5!} \end{aligned}$$

Then we get the exact solution

$$v(x)=v_0+v_1+v_2+...=1+x+\tfrac{x^2}{2!}+\tfrac{x^3}{3!}+\tfrac{x^4}{4!}+\tfrac{x^5}{5!}+...=e^x.$$

Example 4.2. Consider the following linear integro–differential equation

$$v^{(3)}(x)-xv''(x)=\frac{4}{7}x^9-\frac{8}{5}x^7-x^6+6x^2-6+4\int_0^x x^2t^3v(t)dt,\ \ 0\leqslant t,x\leqslant 1, \tag{4.9}$$

with initial conditions

$$v(0)=1, v'(0)=2, v''(0)=0.$$

Taking the Kharrat–Toma transform on (4.9), yields

$$\begin{aligned}&\frac{1}{s^6}B\,(v)-\frac{1}{s}v(0)-s\,v'(0)-s^3\,v''(0)\\&=B\left[\frac{4}{7}x^9-\frac{8}{5}x^7-x^6+6x^2-6\right]+B\left[x\,v''+4\int_0^x x^2\,t^3\,v(t)\,dt\right].\end{aligned}\tag{4.10}$$

Then we have

$$B\,(v)=s^5+2s^7+s^6\,B\left[\frac{4}{7}x^9-\frac{8}{5}x^7-x^6+6x^2-6\right]+s^6\,B\left[x\,v''+4\int_0^x x^2\,t^3\,v(t)\,dt\right].\tag{4.11}$$

Now, constructing the homotopy on (4.11) as follows

$$B\,(v)=s^5+2s^7+s^6\,B\left[\frac{4}{7}x^9-\frac{8}{5}x^7-x^6+6x^2-6\right]+p\,s^6\,B\left[x\,v''+4\int_0^x x^2\,t^3\,v(t)\,dt\right].\tag{4.12}$$

Substituting (3.5) into (4.12), we get

$$\begin{aligned}B\left(\sum_{i=0}^{\infty}p^i\,v_i\right)&=s^5+2s^7+s^6\,B\left[\frac{4}{7}x^9-\frac{8}{5}x^7-x^6+6x^2-6\right]\\&+p\,s^6\,B\left[x\sum_{i=0}^{\infty}p^i\,v''_i+4\int_0^x x^2\,t^3\sum_{i=0}^{\infty}p^i\,v_i(t)\,dt\right].\end{aligned}\tag{4.13}$$

Comparing the coefficients of terms with identical powers of p in (4.13), leads to

$$p^0:B\,[v_0]=s^5+2s^7+s^6\,B\left[\frac{4}{7}x^9-\frac{8}{5}x^7-x^6+6x^2-6\right],\tag{4.14}$$

$$p^1:B\,[v_1]=s^6\,B\left[x\,v''_0+4\int_0^x x^2\,t^3\,v_0(t)\,dt\right],\tag{4.15}$$

$$p^2:B\,[v_2]=s^6\,B\left[x\,v''_1+4\int_0^x x^2\,t^3\,v_1(t)\,dt\right].\tag{4.16}$$

Taking the inverse Kharrat–Toma transform of equations (4.14), (4.15) and (4.16), yields

$$v_0=1+2x-x^3+\frac{x^5}{10}-\frac{x^9}{504}-\frac{x^{10}}{450}+\frac{x^{12}}{2310}$$

x	**Error ofproposed hybrid method(n=2)** $v_0 + v_1$	**Error ofproposed hybrid method (n=3)** $v_0 + v_1 + v_2$
0	0	0
0.1	9.52586 e −10	9.99999 e −13
0.2	1.21900 e −07	4.06991 e −10
0.3	2.08253 e −06	1.56181 e −08
0.4	1.55953 e −05	2.07965 e −07
0.5	7.43002 e −05	1.54847 e −06
0.6	2.65789 e −04	7.97994 e −06
0.7	7.79691 e −04	3.18868 e −05
0.8	1.97652 e −03	1.05714 e −04
0.9	4.47774 e −03	3.03686 e −04
1.0	9.27429 e −03	7.78774 e −04

$$\begin{aligned} v_1 &= -\frac{x^5}{10}+\frac{x^7}{105}+\frac{x^9}{504}+\frac{x^{10}}{450}-\frac{x^{11}}{6930}-\frac{9x^{12}}{15400} \\ &\quad +\frac{4x^{14}}{85995}-\frac{x^{18}}{8019648}-\frac{x^{19}}{9157050}+\frac{x^{21}}{73735200} \end{aligned}$$

$$\begin{aligned} v_2 &= -\frac{x^7}{105}+\frac{x^9}{1260}+\frac{x^{11}}{6930}+\frac{x^{12}}{6600}-\frac{x^{13}}{108108}-\frac{383x^{14}}{6879600} \\ &\quad +\frac{31x^{16}}{8731800}+\frac{x^{18}}{8019648}+\frac{x^{19}}{9157050}-\frac{1657x^{20}}{147891744000} \\ &\quad -\frac{13x^{21}}{565488000}+\frac{20747x^{23}}{13750604627760}-\frac{x^{27}}{774096523200} \\ &\quad -\frac{x^{28}}{1034948105100}+\frac{x^{30}}{11226184200000}. \end{aligned}$$

Table 1 shows the absolute errors at different values of *x* for two terms $v_0 + v_1$and three terms $v_0 + v_1 + v_2$of the proposed hybrid method.

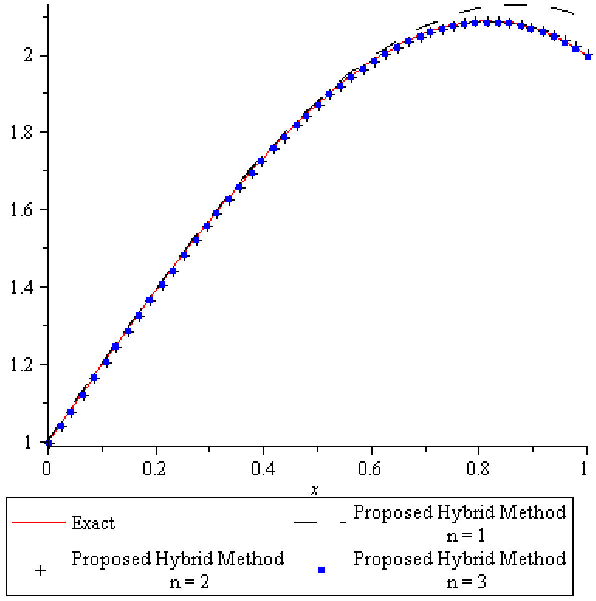
Where the exact solution of Eq. (4.9) is $v(x) = 1 + 2x - x^3$.

Table 1: Comparison of the absolute errors for Example 2

Based on the Table 1, it can be seen that the hybrid technique, when two terms are truncated from the solution series, is more accurate than the three terms truncated.

Figure 1 displays the comparison of approximate solutions and the exact solution.

Figure 1: Comparison of exact and approximate solutions for Example (2)



Example 4.3. Consider the nonlinear integro—differential equation

$$v^{(4)}(x) = e^{-3x} + e^{-x} - 1 + 3\int_0^x v^3(t)dt, \;\; 0 \leqslant t, x \leqslant 1, \tag{4.17}$$

with initial conditions

$$v(0) = v''(0) = 1 \;\; , \;\; v'(0) = v^{(3)}(0) = -1.$$

By the same manipulation as in Example 4.1 and Example 4.2, we obtain

$$\begin{aligned} v_0 &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \frac{x^6}{72} - \frac{x^7}{180} + \frac{41x^8}{20160} - \frac{61x^9}{90720} \\ &\quad + \frac{73x^{10}}{362880} - \frac{547x^{11}}{9979200} + \frac{3281x^{12}}{239500800} - \frac{703x^{13}}{222393600} \end{aligned}$$

$$\begin{aligned} v_1 &= \frac{x^5}{40} - \frac{x^6}{80} + \frac{3x^7}{560} - \frac{9x^8}{4480} + \frac{3x^9}{4480} - \frac{x^{10}}{4800} + \frac{x^{11}}{15400} - \frac{3x^{12}}{140800} \\ &\quad - \frac{421x^{13}}{57657600} - \frac{449x^{14}}{179379200} + \frac{27x^{15}}{32032000} - \frac{7243x^{16}}{25830604800} \\ &\quad + \frac{8497x^{17}}{91483392000} - \frac{2693x^{18}}{87824056320} + \frac{151381x^{19}}{15084957888000} - \frac{1963361x^{20}}{603398315520000} \\ &\quad + \frac{855851x^{21}}{8212924151680000} - \frac{60472771x^{22}}{183660249636864000} + \frac{15492259x^{23}}{1586377648742 4000} + \ldots \end{aligned}$$

Table 2 presents the comparison of absolute errors at different values of *x* for one term$v_0$and two terms $v_0 + v_1$ of the proposed hybrid method, where the exact solution of (4.17) is $v(x) = e^{-x}$.

X	Error ofproposed hybrid method(n=1) v_0	Error ofproposed hybrid method(n=2) $v_0 + v_1$
0	0	0
0.1	2.37980 e −07	6.66337 e −11
0.2	7.26377 e −06	6.70026 e −11
0.3	5.26894 e −05	1.00109 e −10
0.4	2.12417 e −04	4.42250 e −10
0.5	6.21078 e −04	3.86042 e −09
0.6	1.48273 e −03	2.10312 e −08
0.7	3.07878 e −03	8.84641 e −08
0.8	5.77388 e −03	3.09395 e −07
0.9	1.00203 e −02	9.46010 e −07
1.0	1.63612 e −02	2.63392 e −06

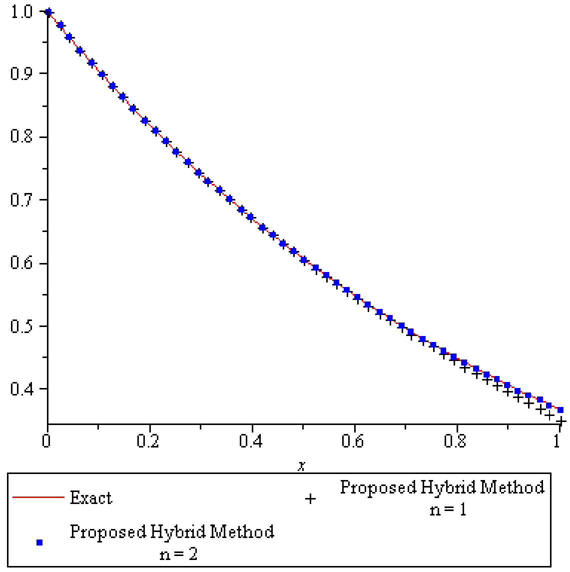


Table 2: The comparison of absolute errors for Example 3

According to the Table 2, it can be notice that the hybrid technique, when one term is truncated from the solution series, is more accurate than two terms truncated.

The following Figure 2 explains the convergence to the exact solution.

Figure 2: Comparison of exact and approximate solutions for Example (3)

x	**Error of proposed hybrid method (n=1)** v_0	**Error of proposed hybrid method (n=2)** $v_0 + v_1$
0	0	0
0.1	8.61168 e −08	2.57917 e −10
0.2	7.28545 e −06	4.20461 e −10
0.3	2.24630 e −05	2.42168 e −09
0.4	9.81543 e −05	2.21042 e −08
0.5	3.10891 e −04	1.37126 e −07
0.6	8.03653 e −04	6.34010 e −07
0.7	1.80625 e −03	2.34570 e −06
0.8	3.66560 e −03	7.36485 e −06
0.9	6.88273 e −03	2.04115 e −05
1.0	1.21580 e −02	5.12793 e −05

Example 4.4. Consider the following nonlinear integro—differential equation

$$v^{(4)}(x) = e^{x} - \frac{1}{2}e^{-2x} + \frac{1}{2} + \int_0^x v(t)v''(t)\,dt, \;\; 0 \leqslant t, x \leqslant 1, \tag{4.18}$$

with initial conditions

$$v(0) = v'(0) = v''(0) = v^{(3)}(0) = 1.$$

By the same manipulation as in Example 4.1 and Example 4.2, we have

$$v_0 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^7}{1680} - \frac{x^8}{5760} - \frac{x^9}{24192} - \frac{31x^{10}}{3628800} - \frac{x^{11}}{633600} - \frac{127x^{12}}{479001600} - \frac{17x^{13}}{415134720}$$

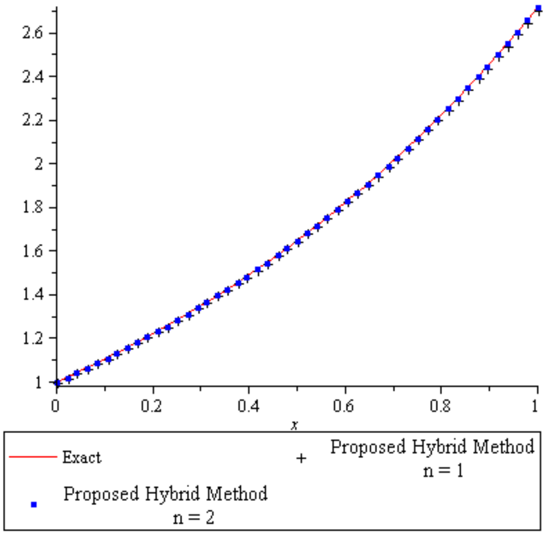
$$\begin{aligned} v_1 = {} & \frac{x^5}{120} + \frac{x^6}{360} + \frac{x^7}{1260} + \frac{x^8}{5760} + \frac{x^9}{36288} + \frac{x^{10}}{518400} \\ & - \frac{13x^{11}}{19958400} - \frac{43x^{12}}{119750400} - \frac{23x^{13}}{222393600} - \frac{113x^{14}}{4843238400} \\ & - \frac{1427x^{15}}{326918592000} - \frac{401x^{16}}{597793996800} - \frac{5303x^{17}}{71137485619200} \\ & - \frac{8549x^{18}}{6402373705728000} + \frac{280727x^{19}}{121645100408832000} + \frac{350899x^{20}}{405483668029440000} + \ldots \end{aligned}$$

The absolute errors at different values of x for one term $v_0$and two terms $v_0 + v_1$of the proposed hybrid method are reported in Table 3, where the exact solution of (4.18) is $v(x) = e^{x}$.

Table 3: The comparison of absolute errors for Example 4

The Figure 3 demonstrates the convergence to the exact solution.

Figure 3: Comparison of exact and approximate solutions for Example (4)



5. Conclusion

In this work, the Kharrat—Toma transform method and the homotopy perturbation method are successfully combined to form a new semi—analytical method for solving linear and nonlinear integro—differential equations. The proposed technique leads to an analytical solution expressed as a series that converges rapidly to an exact solution. The semi—analytical approach is successfully applied and the approximate or exact solution of many integro—differential equations arising in applied sciences is obtained. Thus, the proposed hybrid method is powerful and efficient to solve the linear and nonlinear integro—differential equations.

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