

Unique Common Fixed Point Theorem for a Sequence of Mappings

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Abstract

In this paper, a unique common fixed point result for four mappings, two by two occasionally weakly biased of type $(\mathcal{A})$ in a metric space is obtained. Our reason behind this contribution is to investigate the existence and uniqueness of common fixed points in the recent weaker framework of occasionally weakly biased mappings of type $(\mathcal{A})$. Moreover, a suitable example is furnished to highlight our main result. In addition, from our principal theorem, we derived some results, and to generalize our fundamental theorem, a unique common fixed point result of a sequence of mappings is presented.

Keywords: Metric space; normed vector space; occasionally weakly biased mappings of type $(\mathcal{A})$; unique common fixed points.

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1. Introduction and Needed Concepts

Fixed-point theory is considered to be of extreme importance. It is crucial in different fields of sciences, because it models stable and equilibrium situations, guarantees the existence and uniqueness of solutions to complex equations. It also allows concrete modeling in physics, economics, biology, computer science, and so on, because it offers several tools to analyze systems and find numerical solutions. Again, fixed-point theory is vital in different domains especially in medicine because it manages dangerous diseases like cancer, COVID-19 and so on. It offers mathematical frameworks to demonstrate that biological and epidemiological models yield solutions, see for instance [1], [2], [4], [7], [8], [9], [22], [21], and [23].

Recent studies have advanced common fixed-point theory through expansive mappings, occasionally weakly biased maps, and applications in complete b-metric-like spaces, broadening the conditions for establishing the existence and uniqueness of fixed points [10, 11, 12].

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Now, Before arriving and exploring the new spaces, most works sought the existence and often uniqueness of fixed and common fixed points in metric and normed vector spaces. These two ancient spaces are considered as the oldest and best known. In this paper, we will investigate the existence and uniqueness of common fixed points in the new weaker framework of occasionally weakly biased mappings of type $(\mathcal{A})$, in the oldest space; we mean the metric one. Before giving our main results, we need to give the following notions:

Let $\mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of a nonempty set $\mathcal{X}$. If $\mathbb{Y}\rho = \mathbb{Z}\rho$ for some $\rho \in \mathcal{X}$, then ρ is called a coincidence point of $\mathbb{Y}$ and $\mathbb{Z}$. A point $\tau \in \mathcal{X}$ is a common fixed point of $\mathbb{Y}$ and $\mathbb{Z}$ if $\tau = \mathbb{Y}\tau = \mathbb{Z}\tau$.

Now, to highlight the interdependence between the commuting mapping and the fixed point concepts, Jungck [15] initiated the concept of commuting mappings as follows:

Proposition 1.1. *Let $\mathbb{Y}$ be a mapping of a set $\mathcal{X}$ into itself. Then $\mathbb{Y}$ has a fixed point if and only if there is a constant mapping $\mathbb{Z} : \mathcal{X} \rightarrow \mathcal{X}$ which commutes with $\mathbb{Y}$ (i.e., $\mathbb{Z}(\mathbb{Y}(\sigma)) = \mathbb{Y}(\mathbb{Z}(\sigma))$ for all σ in $\mathcal{X}$).*

To Generalize the concept of commuting mappings, Sessa [20] introduced a weaker hypothesis called the concept of weakly commuting mappings.

Definition 1.2. Let $(\mathcal{X}, d)$ be a metric space. Self-mappings $\mathbb{Y}$ and $\mathbb{Z}$ of $\mathcal{X}$ are weakly commuting if

$$d(\mathbb{Y}\mathbb{Z}\sigma, \mathbb{Z}\mathbb{Y}\sigma) \leq d(\mathbb{Y}\sigma, \mathbb{Z}\sigma) \text{ for every } \sigma \in \mathcal{X}.$$

Afterwards, Jungck [13] introduced a weakened commutativity which generalizes the commuting mapping concept.

Definition 1.3. Self-mappings $\mathbb{Y}$ and $\mathbb{Z}$ of a metric space $(\mathcal{X}, d)$ are compatible if and only if

$$\lim_{n \rightarrow \infty} d(\mathbb{Z}\mathbb{Y}(s_n), \mathbb{Y}\mathbb{Z}(s_n)) = 0$$

whenever $\{s_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} \mathbb{Y}(s_n) = \lim_{n \rightarrow \infty} \mathbb{Z}(s_n) = \sigma$ for some $\sigma \in \mathcal{X}$.

According to Pant [19], in a review of [14], Singh produced an example showing that weak commutativity does not imply the existence of a sequence of points satisfying the compatibility condition, for this reason, Pant proposed the concept of R-weak commutativity of mappings which presents a solid generalization of weak commutativity notion.

Definition 1.4. Let $(\mathcal{X}, d)$ be a metric space and let $\mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of $\mathcal{X}$. The mappings $\mathbb{Y}$ and $\mathbb{Z}$ are called R-weakly commuting, provided there exists some positive real number ρ such that

$$d(\mathbb{Y}\mathbb{Z}\sigma, \mathbb{Z}\mathbb{Y}\sigma) \leq \rho d(\mathbb{Y}\sigma, \mathbb{Z}\sigma)$$

for each σ in $\mathcal{X}$. $\mathbb{Y}$ and $\mathbb{Z}$ are called ρ -weakly commuting at a point σ if $d(\mathbb{Y}\mathbb{Z}\sigma, \mathbb{Z}\mathbb{Y}\sigma) \leq \rho d(\mathbb{Y}\sigma, \mathbb{Z}\sigma)$ for some $\rho > 0$.

To produce fixed point theorems for multi-valued noncontinuous functions, Jungck and Rhoades [16] introduced the notion of weakly compatible mappings.

Definition 1.5. Let $(\mathcal{X}, d)$ be a metric space, and let $\mathbb{Y} : \mathcal{X} \rightarrow \mathcal{X}$ and $\mathbb{Z} : \mathcal{X} \rightarrow \mathcal{B}(\mathcal{X})$ (where $\mathcal{B}(\mathcal{X})$ is the set of all nonempty bounded subsets of $\mathcal{X}$). The pair $\{\mathbb{Z}, \mathbb{Y}\}$ is a weakly compatible pair if and only if $\mathbb{Z}\rho = \{\mathbb{Y}\rho\}$ implies that $\mathbb{Y}\mathbb{Z}\rho = \mathbb{Z}\mathbb{Y}\rho$.

In 2008, Al-Thagafi and Shahzad [5] introduced the notion of occasionally weakly compatible mappings as a new class of noncommuting self-mappings which contains the weakly compatible self-mappings as a proper subclass. While the paper [5] was under review, Jungck and Rhoades [17] used the concept of occasionally weakly compatible mappings and proved several results under different contractive conditions [3].

Definition 1.6. Let $\mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of a subset $\mathcal{D}$ of a metric space $\mathcal{X} := (\mathcal{X}, d)$. Then $\mathbb{Y}$ and $\mathbb{Z}$ are called occasionally weakly compatible if $\mathbb{Y}\mathbb{Z}\tau = \mathbb{Z}\mathbb{Y}\tau$ for some $\tau \in \mathcal{C}(\mathbb{Y}, \mathbb{Z})$ (where $\mathcal{C}(\mathbb{Y}, \mathbb{Z})$ is the set of coincidence points of $\mathbb{Y}$ and $\mathbb{Z}$).

Recently in 2022, we introduced the concept of occasionally weakly biased mappings of type $(\mathcal{A})$ [6], and we claimed that weakly compatible and occasionally weakly compatible mappings are occasionally weakly biased of type $(\mathcal{A})$ however the converses are false in general.

Definition 1.7. Let $\mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of a non-empty set $\mathcal{X}$. The pair $(\mathbb{Y}, \mathbb{Z})$ is said to be occasionally weakly $\mathbb{Y}$ -biased of type $(\mathcal{A})$ and occasionally weakly $\mathbb{Z}$ -biased of type $(\mathcal{A})$, respectively, if and only if there exists a point σ in $\mathcal{X}$ such that $\mathbb{Y}\sigma = \mathbb{Z}\sigma$ implies

$$\begin{aligned} d(\mathbb{Y}\mathbb{Y}\sigma, \mathbb{Z}\sigma) &\leq d(\mathbb{Z}\mathbb{Y}\sigma, \mathbb{Y}\sigma), \\ d(\mathbb{Z}\mathbb{Z}\sigma, \mathbb{Y}\sigma) &\leq d(\mathbb{Y}\mathbb{Z}\sigma, \mathbb{Z}\sigma), \end{aligned}$$

respectively.

2. Previous Result

In their paper [18], Pande et al. proved the next theorem:

Theorem 2.1. (Theorem 2.1 of [18]) Let $\mathcal{X}$ be a normed vector space, $\mathbb{S}$, $\mathbb{I}$ and $\mathbb{T}$, $\mathbb{J}$ two pairs of respectively R_1 -weakly and R_2 -weakly commuting mappings on $\mathcal{X}$. Also let C be a closed, convex subset of $\mathcal{X}$ such that

1. $\mathbb{I}(C) \supseteq (1 - k)\mathbb{I}(C) + k\mathbb{S}(C)$,
2. $\mathbb{J}(C) \supseteq (1 - k')\mathbb{J}(C) + k'\mathbb{T}(C)$, where $0 < k, k' \leq 1$ and suppose that
3. for all $x, y \in C$,

$$\begin{aligned} \|\mathbb{S}x - \mathbb{T}y\|^p &\leq \phi(\max\{\|\mathbb{S}x - \mathbb{I}x\|^p + \|\mathbb{I}x - \mathbb{J}y\|^p + \|\mathbb{J}y - \mathbb{T}y\|^p, \\ &\quad \|\mathbb{S}x - \mathbb{I}x\|^p + \|\mathbb{I}x - \mathbb{T}y\|^p, \|\mathbb{S}x - \mathbb{J}y\|^p + \|\mathbb{J}y - \mathbb{T}y\|^p\}), \end{aligned}$$

where $p > 0$ and ϕ is a function which is upper semi-continuous from the right such that $\phi(t) < t$ for each $t > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in $\mathcal{X}$ defined inductively for $n = 1, 2, 3, \dots$, by

4. $\mathbb{I}x_{2n+1} = (1 - a_{2n})\mathbb{I}x_{2n} + a_{2n}Sx_{2n}$,
5. $\mathbb{J}x_{2n+2} = (1 - a_{2n+1})\mathbb{J}x_{2n+1} + a_{2n+1}Tx_{2n+1}$, where $0 < a_n \leq 1$ for all $n \geq 0$ and $\liminf a_n > 0$, converges to a point $z \in C$ and for this $\{x_n\}$, if $\mathbb{I}x_n \rightarrow \mathbb{I}z$ and $\mathbb{J}x_n \rightarrow \mathbb{J}z$ as $n \rightarrow \infty$,

then $S, \mathbb{I}, \mathbb{J}$ and T have the unique common fixed point Tz in C .

Remark 2.2. In the above theorem, the sentence: " $S, \mathbb{I}$ and $T, \mathbb{J}$ two pairs of respectively R_1 -weakly and R_2 -weakly commuting mappings on $\mathcal{X}$ " means S and $\mathbb{I}$ are R -weakly commuting and T and $\mathbb{J}$ are also R -weakly commuting.

In this work, we will represent the above theorem in the recent weaker framework of our recent notion of occasionally weakly biased mappings of type $(\mathcal{A})$, by removing several conditions that we consider superfluous.

3. Main Results

3.1. Unique Common Fixed Point in Metric Space

Let ϕ be a non-decreasing function and let $\delta < 1$ be any real number. Since

$$\begin{aligned} d^p(\mathbb{U}x, \mathbb{V}y) &\leq \phi(\xi, \max\{d^p(\mathbb{U}x, \mathbb{Y}x) + d^p(\mathbb{Y}x, \mathbb{Z}y) + d^p(\mathbb{V}y, \mathbb{Z}y), \\ &\quad d^p(\mathbb{U}x, \mathbb{Y}x) + d^p(\mathbb{Y}x, \mathbb{V}y), d^p(\mathbb{U}x, \mathbb{Z}y) + d^p(\mathbb{V}y, \mathbb{Z}y)\}) \\ &\leq \phi(\max\{d^p(\mathbb{U}x, \mathbb{Y}x) + d^p(\mathbb{Y}x, \mathbb{Z}y) + d^p(\mathbb{V}y, \mathbb{Z}y), \\ &\quad d^p(\mathbb{U}x, \mathbb{Y}x) + d^p(\mathbb{Y}x, \mathbb{V}y), d^p(\mathbb{U}x, \mathbb{Z}y) + d^p(\mathbb{V}y, \mathbb{Z}y)\}). \end{aligned}$$

We give the following result on the recent weaker framework of occasionally weakly biased mappings of type $(\mathcal{A})$ in a metric space.

Theorem 3.1. Let $\mathbb{U}, \mathbb{V}, \mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition:

$$\begin{aligned} d^\sigma(\mathbb{U}x, \mathbb{V}y) &\leq \phi(\xi, \max\{d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y), \\ &\quad d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{V}y), d^\sigma(\mathbb{U}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y)\}) \end{aligned}$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{1+2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{U}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Z}$ are respectively occasionally weakly $\mathbb{Y}$ -biased and $\mathbb{Z}$ -biased of type $(\mathcal{A})$, then, $\mathbb{U}, \mathbb{V}, \mathbb{Y}$ and $\mathbb{Z}$ have a unique common fixed point.

Proof. By hypotheses, since mappings $\mathbb{U}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Z}$ are respectively occasionally weakly $\mathbb{Y}$ -biased and $\mathbb{Z}$ -biased of type $(\mathcal{A})$, then, there are two elements μ and η in $\mathcal{X}$ such that

$$\begin{aligned} \mathbb{U}\mu &= \mathbb{Y}\mu \text{ implies } d(\mathbb{Y}\mathbb{Y}\mu, \mathbb{U}\mu) \leq d(\mathbb{U}\mathbb{Y}\mu, \mathbb{Y}\mu), \\ \mathbb{V}\eta &= \mathbb{Z}\eta \text{ implies } d(\mathbb{Z}\mathbb{Z}\eta, \mathbb{V}\eta) \leq d(\mathbb{V}\mathbb{Z}\eta, \mathbb{Z}\eta). \end{aligned}$$

To prove the existence and uniqueness of the common fixed point, we need four steps.

First step: Assume that $d(U\mu, V\eta)$ is positive, then

$$\begin{aligned} d^\sigma(U\mu, V\eta) &\leq \phi(\xi \max\{d^\sigma(U\mu, Y\mu) + d^\sigma(Y\mu, Z\eta) + d^\sigma(V\eta, Z\eta), \\ &\quad d^\sigma(U\mu, Y\mu) + d^\sigma(Y\mu, V\eta), d^\sigma(U\mu, Z\eta) + d^\sigma(V\eta, Z\eta)\}) \\ &= \phi(\xi d^\sigma(U\mu, V\eta)) \\ &< \xi d^\sigma(U\mu, V\eta) \\ &< d^\sigma(U\mu, V\eta) \end{aligned}$$

which is a contradiction, hence, $U\mu = V\eta$.

Second step: Suppose that $d(UU\mu, U\mu) > 0$, then, we get

$$\begin{aligned} d^\sigma(UU\mu, V\eta) &\leq \phi(\xi \max\{d^\sigma(UU\mu, YU\mu) + d^\sigma(YU\mu, Z\eta) + d^\sigma(V\eta, Z\eta), \\ &\quad d^\sigma(UU\mu, YU\mu) + d^\sigma(YU\mu, V\eta), d^\sigma(UU\mu, Z\eta) + d^\sigma(V\eta, Z\eta)\}) \\ &= \phi(\xi \max\{d^\sigma(UU\mu, YY\mu) + d^\sigma(YY\mu, U\mu), d^\sigma(UU\mu, U\mu)\}) \\ &< \xi \max\{d^\sigma(UU\mu, YY\mu) + d^\sigma(YY\mu, U\mu), d^\sigma(UU\mu, U\mu)\}. \end{aligned}$$

Since U and Y are occasionally weakly Y -biased of type $(\mathcal{A})$, we get

$$\begin{aligned} d^\sigma(YY\mu, U\mu) &\leq d^\sigma(UY\mu, Y\mu) \\ &= d^\sigma(UU\mu, U\mu) \end{aligned}$$

and using the triangle inequality and the relationship between the two mappings, we obtain

$$\begin{aligned} d^\sigma(UU\mu, YY\mu) &\leq (d(UU\mu, U\mu) + d(U\mu, YY\mu))^\sigma \\ &\leq (d(UU\mu, U\mu) + d(Y\mu, UY\mu))^\sigma \\ &= 2^\sigma d^\sigma(UU\mu, U\mu), \end{aligned}$$

hence

$$\begin{aligned} d^\sigma(UU\mu, U\mu) &< \xi \max\{2^\sigma d^\sigma(UU\mu, U\mu) + d^\sigma(UU\mu, U\mu), d^\sigma(UU\mu, U\mu)\} \\ &= \xi \max\{(2^\sigma + 1)d^\sigma(UU\mu, U\mu), d^\sigma(UU\mu, U\mu)\} \\ &= \xi(2^\sigma + 1)d^\sigma(UU\mu, U\mu) \\ &< d^\sigma(UU\mu, U\mu) \end{aligned}$$

a contradiction, thus, $UU\mu = U\mu$, consequently $YU\mu = U\mu$.

Third step: Assume that $d(VV\eta, V\eta)$ is positive, then, we get

$$\begin{aligned} d^\sigma(U\mu, VV\eta) &\leq \phi(\xi \max\{d^\sigma(U\mu, Y\mu) + d^\sigma(Y\mu, ZV\eta) + d^\sigma(VV\eta, ZV\eta), \\ &\quad d^\sigma(U\mu, Y\mu) + d^\sigma(Y\mu, VV\eta), d^\sigma(U\mu, ZV\eta) + d^\sigma(VV\eta, ZV\eta)\}) \\ &= \phi(\xi \max\{d^\sigma(V\eta, ZZ\eta) + d^\sigma(VV\eta, ZZ\eta), d^\sigma(V\eta, VV\eta), \\ &\quad d^\sigma(V\eta, ZZ\eta) + d^\sigma(VV\eta, ZZ\eta)\}) \\ &< \xi \max\{d^\sigma(V\eta, ZZ\eta) + d^\sigma(VV\eta, ZZ\eta), d^\sigma(V\eta, VV\eta)\}. \end{aligned}$$

Using the triangle inequality and the relationship between the mappings, we get

$$\begin{aligned} d^\sigma(\mathbb{V}\mathbb{V}\eta, \mathbb{Z}\mathbb{Z}\eta) &\leq (d(\mathbb{V}\mathbb{V}\eta, \mathbb{V}\eta) + d(\mathbb{V}\eta, \mathbb{Z}\mathbb{Z}\eta))^\sigma \\ &\leq (d(\mathbb{V}\mathbb{V}\eta, \mathbb{V}\eta) + d(\mathbb{Z}\eta, \mathbb{V}\mathbb{Z}\eta))^\sigma \\ &= 2^\sigma d^\sigma(\mathbb{V}\mathbb{V}\eta, \mathbb{V}\eta) \end{aligned}$$

and

$$\begin{aligned} d^\sigma(\mathbb{V}\eta, \mathbb{Z}\mathbb{Z}\eta) &\leq d^\sigma(\mathbb{Z}\eta, \mathbb{V}\mathbb{Z}\eta) \\ &= d^\sigma(\mathbb{V}\mathbb{V}\eta, \mathbb{V}\eta), \end{aligned}$$

consequently,

$$\begin{aligned} d^\sigma(\mathbb{V}\eta, \mathbb{V}\mathbb{V}\eta) &< \xi \max\{d^\sigma(\mathbb{V}\eta, \mathbb{V}\mathbb{V}\eta) + 2^\sigma d^\sigma(\mathbb{V}\eta, \mathbb{V}\mathbb{V}\eta), d^\sigma(\mathbb{V}\eta, \mathbb{V}\mathbb{V}\eta)\} \\ &= \xi(1 + 2^\sigma) d^\sigma(\mathbb{V}\eta, \mathbb{V}\mathbb{V}\eta) \\ &< d^\sigma(\mathbb{V}\eta, \mathbb{V}\mathbb{V}\eta) \end{aligned}$$

a contradiction, hence, $\mathbb{V}\mathbb{V}\eta = \mathbb{V}\eta$, in consequence $\mathbb{Z}\mathbb{V}\eta = \mathbb{V}\eta$. Therefore, $\chi_1 = \mathbb{U}\mu = \mathbb{Y}\mu = \mathbb{V}\eta = \mathbb{Z}\eta$ is a common fixed point of the four mappings.

Fourth step: At the end, assume the existence of another common fixed point (say χ_2). The use of the inequality gives

$$\begin{aligned} d^\sigma(\mathbb{U}\chi_1, \mathbb{V}\chi_2) &\leq \phi(\xi \max\{d^\sigma(\mathbb{U}\chi_1, \mathbb{Y}\chi_1) + d^\sigma(\mathbb{Y}\chi_1, \mathbb{Z}\chi_2) + d^\sigma(\mathbb{V}\chi_2, \mathbb{Z}\chi_2), \\ &\quad d^\sigma(\mathbb{U}\chi_1, \mathbb{Y}\chi_1) + d^\sigma(\mathbb{Y}\chi_1, \mathbb{V}\chi_2), \\ &\quad d^\sigma(\mathbb{U}\chi_1, \mathbb{Z}\chi_2) + d^\sigma(\mathbb{V}\chi_2, \mathbb{Z}\chi_2)\}); \end{aligned}$$

i.e.,

$$\begin{aligned} d^\sigma(\chi_1, \chi_2) &\leq \phi(\xi \max\{d^\sigma(\chi_1, \chi_2), d^\sigma(\chi_1, \chi_2), d^\sigma(\chi_1, \chi_2)\}) \\ &= \phi(\xi d^\sigma(\chi_1, \chi_2)) \\ &< \xi d^\sigma(\chi_1, \chi_2) \\ &< d^\sigma(\chi_1, \chi_2) \end{aligned}$$

a contradiction, thus $\chi_2 = \chi_1$. □

To illustrate our above result, we furnish the following example.

3.2. Illustrative Example

Example 3.2. Equip the set $\mathcal{X} = (0, +\infty)$ with the usual metric $d(x, y) = |x - y|$. Take $\phi(t) = \frac{9}{10}t$, then, $\phi(t) < t$, $\sigma = 1$, $\xi = \frac{1}{4}$, then, $\xi < \frac{1}{1+2^\sigma}$. Now, consider the next four mappings:

$$\begin{aligned} \mathbb{U}x &= \begin{cases} 1 & \text{when } x \in (0, 1], \\ \frac{1}{x} & \text{when } x \in (1, +\infty), \end{cases} & \mathbb{V}x &= \begin{cases} 1 & \text{when } x \in (0, 1], \\ \frac{1}{x} & \text{when } x \in (1, +\infty), \end{cases} \\ \mathbb{Y}x &= \begin{cases} \frac{1}{x} & \text{when } x \in (0, 1], \\ 100 & \text{when } x \in (1, +\infty), \end{cases} & \mathbb{Z}x &= \begin{cases} \frac{1}{x} & \text{when } x \in (0, 1], \\ 500 & \text{when } x \in (1, +\infty). \end{cases} \end{aligned}$$

First of all, it is easy to see that mappings $\mathbb{U}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Z}$ are occasionally weakly $\mathbb{Y}$ -biased (respectively $\mathbb{Z}$ -biased) of type $(\mathcal{A})$.

Now, let the constructed table be as follows:

♠	$0 < x, y \leq 1$	$1 < x, y < +\infty$	$0 < x \leq 1 < y < +\infty$	$0 < y \leq 1 < x < +\infty$
$d(\mathbb{U}x, \mathbb{V}y)$	0	$\left \frac{1}{x} - \frac{1}{y}\right $	$1 - \frac{1}{y}$	$\left \frac{1}{x} - 1\right $
$d(\mathbb{U}x, \mathbb{Y}x)$	$1 - \frac{1}{x}$	$\left \frac{1}{x} - 100\right $	$1 - \frac{1}{x}$	$\left \frac{1}{x} - 100\right $
$d(\mathbb{Y}x, \mathbb{Z}y)$	$\left \frac{1}{x} - \frac{1}{y}\right $	400	$\left \frac{1}{x} - 500\right $	$100 - \frac{1}{y}$
$d(\mathbb{V}y, \mathbb{Z}y)$	$1 - \frac{1}{y}$	$\left \frac{1}{y} - 500\right $	$\frac{1}{y} - 500$	$1 - \frac{1}{y}$
$d(\mathbb{Y}x, \mathbb{V}y)$	$\left \frac{1}{x} - 1\right $	$100 - \frac{1}{y}$	$\left \frac{1}{x} - \frac{1}{y}\right $	99
$d(\mathbb{U}x, \mathbb{Z}y)$	$1 - \frac{1}{y}$	$\left \frac{1}{x} - 500\right $	499	$\left \frac{1}{x} - \frac{1}{y}\right $

Discussion: Looking at the table, we note the following cases:

First case: For $x, y \in (0, 1]$, we have $\mathbb{U}x = 1 = \mathbb{V}y$, $\mathbb{Y}x = \frac{1}{x}$, $\mathbb{Z}y = \frac{1}{y}$ and

$$\begin{aligned}
 d(\mathbb{U}x, \mathbb{V}y) &= 0 \\
 &\leq \frac{9}{10} \left[\frac{1}{4} \max \left\{ \left| 1 - \frac{1}{x} \right| + \left| \frac{1}{x} - \frac{1}{y} \right| + \left| 1 - \frac{1}{y} \right|, 2 \left| 1 - \frac{1}{x} \right|, 2 \left| 1 - \frac{1}{y} \right| \right\} \right] \\
 &= \frac{9}{40} \max \left\{ \left| 1 - \frac{1}{x} \right| + \left| \frac{1}{x} - \frac{1}{y} \right| + \left| 1 - \frac{1}{y} \right|, 2 \left| 1 - \frac{1}{x} \right|, 2 \left| 1 - \frac{1}{y} \right| \right\} \\
 &= \phi(\xi, \max\{d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y), \\
 &\quad d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{V}y), d^\sigma(\mathbb{U}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y)\}).
 \end{aligned}$$

Second case: When $x, y \in (1, +\infty)$, $\mathbb{U}x = \frac{1}{x}$, $\mathbb{V}y = \frac{1}{y}$, $\mathbb{Y}x = 100$, $\mathbb{Z}y = 500$, then,

$$\begin{aligned}
 d(\mathbb{U}x, \mathbb{V}y) &= \left| \frac{1}{x} - \frac{1}{y} \right| \\
 &\leq \frac{9}{10} \left[\frac{1}{4} \max \left\{ \left| \frac{1}{x} - 100 \right| + 400 + \left| \frac{1}{y} - 500 \right|, \right. \right. \\
 &\quad \left. \left| \frac{1}{x} - 100 \right| + \left| 100 - \frac{1}{y} \right|, \left| \frac{1}{x} - 500 \right| + \left| \frac{1}{y} - 500 \right| \right\} \right] \\
 &= \frac{9}{40} \max \left\{ \left| \frac{1}{x} - 100 \right| + 400 + \left| \frac{1}{y} - 500 \right|, \left| \frac{1}{x} - 100 \right| + \left| 100 - \frac{1}{y} \right|, \right. \\
 &\quad \left. \left| \frac{1}{x} - 500 \right| + \left| \frac{1}{y} - 500 \right| \right\} \\
 &= \phi(\xi, \max\{d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y), \\
 &\quad d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{V}y), d^\sigma(\mathbb{U}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y)\}).
 \end{aligned}$$

Third case: Now, for $x \in (0, 1]$ and $y \in (1, +\infty)$, we have $\mathbb{U}x = 1$, $\mathbb{V}y = \frac{1}{y}$, $\mathbb{Y}x = \frac{1}{x}$,

$Zy = 500$ and

$$\begin{aligned}
 d(Ux, Vy) &= \left| 1 - \frac{1}{y} \right| \\
 &\leq \frac{9}{10} \left[\frac{1}{4} \max \left\{ \left| 1 - \frac{1}{x} \right| + \left| \frac{1}{x} - 500 \right| + \left| \frac{1}{y} - 500 \right|, \right. \right. \\
 &\quad \left. \left| 1 - \frac{1}{x} \right| + \left| \frac{1}{x} - \frac{1}{y} \right|, 499 + \left| \frac{1}{y} - 500 \right| \right\} \Big] \\
 &= \frac{9}{40} \max \left\{ \left| 1 - \frac{1}{x} \right| + \left| \frac{1}{x} - 500 \right| + \left| \frac{1}{y} - 500 \right|, \left| 1 - \frac{1}{x} \right| + \left| \frac{1}{x} - \frac{1}{y} \right|, \right. \\
 &\quad \left. 499 + \left| \frac{1}{y} - 500 \right| \right\} \\
 &= \phi(\xi, \max\{d^\sigma(Ux, Yx) + d^\sigma(Yx, Zy) + d^\sigma(Vy, Zy), \\
 &\quad d^\sigma(Ux, Yx) + d^\sigma(Yx, Vy), d^\sigma(Ux, Zy) + d^\sigma(Vy, Zy)\}).
 \end{aligned}$$

Fourth case: Finally, when $x \in (1, +\infty)$ and $y \in (0, 1]$, $Ux = \frac{1}{x}$, $Vy = 1$, $Yx = 100$, $Zy = \frac{1}{y}$ and

$$\begin{aligned}
 d(Ux, Vy) &= \left| \frac{1}{x} - 1 \right| \\
 &\leq \frac{9}{10} \left[\frac{1}{4} \max \left\{ \left| \frac{1}{x} - 100 \right| + \left| 100 - \frac{1}{y} \right| + \left| 1 - \frac{1}{y} \right|, \right. \right. \\
 &\quad \left. \left| \frac{1}{x} - 100 \right| + 99, \left| \frac{1}{x} - \frac{1}{y} \right| + \left| 1 - \frac{1}{y} \right| \right\} \Big] \\
 &= \frac{9}{40} \max \left\{ \left| \frac{1}{x} - 100 \right| + \left| 100 - \frac{1}{y} \right| + \left| 1 - \frac{1}{y} \right|, \left| \frac{1}{x} - 100 \right| + 99, \right. \\
 &\quad \left. \left| \frac{1}{x} - \frac{1}{y} \right| + \left| 1 - \frac{1}{y} \right| \right\} \\
 &= \phi(\xi, \max\{d^\sigma(Ux, Yx) + d^\sigma(Yx, Zy) + d^\sigma(Vy, Zy), \\
 &\quad d^\sigma(Ux, Yx) + d^\sigma(Yx, Vy), d^\sigma(Ux, Zy) + d^\sigma(Vy, Zy)\}).
 \end{aligned}$$

We conclude that the two conditions of our theorem are satisfied and 1 is the unique element which verifies that $U(1) = V(1) = Y(1) = Z(1) = 1$.

Now, we give some results always in the recent weaker framework of occasionally weakly biased mappings of type $(\mathcal{A})$ in metric spaces.

3.3. Some Results

Corollary 3.3. Let U , V , Y and Z be self-mappings of a metric space (X, d) satisfying the following condition:

$$\begin{aligned}
 d^\sigma(Ux, Vy) &\leq \phi(\xi, \max\{d^\sigma(Yx, Zy), d^\sigma(Ux, Yx) + d^\sigma(Vy, Zy), \\
 &\quad \frac{1}{2} (d^\sigma(Ux, Zy) + d^\sigma(Yx, Vy))\})
 \end{aligned}$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{U}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Z}$ are respectively occasionally weakly $\mathbb{Y}$ -biased and $\mathbb{Z}$ -biased of type $(\mathcal{A})$, then, $\mathbb{U}$, $\mathbb{V}$, $\mathbb{Y}$ and $\mathbb{Z}$ have a unique common fixed point.

Our second result is as follows:

Corollary 3.4. Let $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{Y}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition:

$$d^\sigma(\mathbb{U}x, \mathbb{V}y) \leq \phi(\xi(d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{V}y)))$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{1+2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{U}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Y}$ are occasionally weakly $\mathbb{Y}$ -biased of type $(\mathcal{A})$, then, $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{Y}$ have a unique common fixed point.

Now, we give our third result:

Corollary 3.5. Let $\mathbb{V}$, $\mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition:

$$d^\sigma(\mathbb{V}x, \mathbb{V}y) \leq \phi(\xi \max\{d^\sigma(\mathbb{V}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y), \\ d^\sigma(\mathbb{V}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{V}y), d^\sigma(\mathbb{V}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y)\})$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{1+2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{V}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Z}$ are respectively occasionally weakly $\mathbb{Y}$ -biased and $\mathbb{Z}$ -biased of type $(\mathcal{A})$, then, $\mathbb{V}$, $\mathbb{Y}$ and $\mathbb{Z}$ have a unique common fixed point.

In the next lines, we will present our fourth result.

Corollary 3.6. Let $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{Y}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition:

$$d^\sigma(\mathbb{U}x, \mathbb{V}y) \leq \phi(\xi \max\{d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{Y}y) + d^\sigma(\mathbb{V}y, \mathbb{Y}y), \\ d^\sigma(\mathbb{U}x, \mathbb{Y}x) + d^\sigma(\mathbb{Y}x, \mathbb{V}y), d^\sigma(\mathbb{U}x, \mathbb{Y}y) + d^\sigma(\mathbb{V}y, \mathbb{Y}y)\})$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{1+2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{U}$ and $\mathbb{Y}$ as well as $\mathbb{V}$ and $\mathbb{Y}$ are occasionally weakly $\mathbb{Y}$ -biased of type $(\mathcal{A})$, then, $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{Y}$ have a unique common fixed point.

Now, we give our fifth corollary.

Corollary 3.7. Let $\mathbb{V}$ and $\mathbb{Z}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition:

$$d^\sigma(\mathbb{V}x, \mathbb{V}y) \leq \phi(\xi \max\{d^\sigma(\mathbb{V}x, \mathbb{Z}x) + d^\sigma(\mathbb{Z}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y), \\ d^\sigma(\mathbb{V}x, \mathbb{Z}x) + d^\sigma(\mathbb{Z}x, \mathbb{V}y), d^\sigma(\mathbb{V}x, \mathbb{Z}y) + d^\sigma(\mathbb{V}y, \mathbb{Z}y)\})$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{1+2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{V}$ and $\mathbb{Z}$ are occasionally weakly $\mathbb{Z}$ -biased of type $(\mathcal{A})$, then, $\mathbb{V}$ and $\mathbb{Z}$ have a unique common fixed point.

3.4. Unique Common Fixed Point for a Sequence of Mappings in Metric Space

Theorem 3.8. Let $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{V}_n$ for $n = 1, 2, \dots$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following condition:

$$d^\sigma(\mathbb{V}_n x, \mathbb{V}_{n+1} y) \leq \phi(\xi \max\{d^\sigma(\mathbb{V}_n x, \mathbb{Y} x) + d^\sigma(\mathbb{Y} x, \mathbb{Z} y) + d^\sigma(\mathbb{V}_{n+1} y, \mathbb{Z} y), \\ d^\sigma(\mathbb{V}_n x, \mathbb{Y} x) + d^\sigma(\mathbb{Y} x, \mathbb{V}_{n+1} y), \\ d^\sigma(\mathbb{V}_n x, \mathbb{Z} y) + d^\sigma(\mathbb{V}_{n+1} y, \mathbb{Z} y)\})$$

for all $x, y \in \mathcal{X}$, where σ is a positive integer, $\xi < \frac{1}{1+2^\sigma}$, and ϕ is a function such that $\phi(t) < t$ for each $t > 0$. If $\mathbb{V}_n$ and $\mathbb{Y}$ as well as $\mathbb{V}_{n+1}$ and $\mathbb{Z}$ are respectively occasionally weakly $\mathbb{Y}$ -biased and $\mathbb{Z}$ -biased of type $(\mathcal{A})$, then, $\mathbb{V}_n$, $\mathbb{Y}$ and $\mathbb{Z}$ have a unique common fixed point.

4. Conclusion

In this work, we proved the existence and uniqueness of a common fixed point for four occasionally weakly biased mappings of type $(\mathcal{A})$ in a metric space. Also, we furnished a suitable example to illustrate our main result. Moreover, we presented some other results which in my point of view do not need proofs. Again, a unique common fixed point theorem for a sequence of mappings was given. Our results improve some existing results in metric as well as normed vector spaces, for instance, Theorem 2.1 of [18] and all its corollaries. Furthermore, we investigated our results in the recent weaker framework of occasionally weakly biased mappings of type $(\mathcal{A})$. So, some other results in metric and normed vector spaces become as corollaries of our results.

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