

## A Rebuttal to “Proper Definitions of Dirichlet Conditions and Convergence of Fourier Representations” by P. Singh *et al.*

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### Abstract

In their lecture note [1], P. Singh *et al.* (2022) argue that the Dirichlet conditions (DCs) commonly stated in engineering texts are incomplete, and propose bounded variation (BV) together with an “extended Carleson–Hunt theorem” as a more appropriate framework. This paper re-examines these claims in light of the historical development of Fourier convergence theory and the distinction between Fourier-series and Fourier-transform settings. The standard textbook conditions—absolute integrability, finitely many extrema, and finitely many finite discontinuities—are shown to be a faithful modern formulation of classical Dirichlet-type sufficient conditions for Fourier-series convergence on a finite period. Bounded variation, introduced by Jordan (1881), constitutes a later and strictly broader criterion rather than a correction of Dirichlet’s original hypotheses. Moreover, while  $BV(T) \subset L^p(T)$  holds on any finite period,  $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ ; consequently BV alone does not guarantee the existence of the classical Fourier transform on the real line. Re-analysis of the seven examples in [1] confirms that they do not expose any deficiency in the classical Dirichlet-type conditions, but rather illustrate the need to distinguish domains, function spaces, and modes of convergence. No replacement of the standard Dirichlet-type conditions is required for Fourier-series convergence on a finite periodic domain.

Keywords: Fourier series, Fourier transform, Dirichlet-type conditions, bounded variation, Carleson–Hunt theorem, convergence,  $L^p$  spaces, .

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### 1. Introduction

The convergence of Fourier representations is a fundamental topic in mathematical analysis and signal processing. Beginning with Dirichlet’s work in 1829 [7], the theory has developed substantially through the contributions of mathematicians such as Jordan, Fej’er, Carleson, and Hunt. These developments have introduced broader frameworks for studying Fourier-series convergence, including bounded variation and, subsequently, results formulated in  $L^p$  spaces. They have also made it increasingly important to distinguish

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between different types of convergence, such as pointwise and almost-everywhere convergence, summability, and the existence of Fourier transforms under different function-space assumptions.

Recent studies continue to extend Fourier analysis to broader classes of functions and more general notions of bounded variation. Berdysheva et al. [17], for example, developed a metric Fourier-approximation framework for set-valued functions of bounded variation and established a Dirichlet–Jordan-type pointwise convergence result. Darji [20] investigated  $\beta$ -absolute convergence of Fourier series with small gaps for functions belonging to generalized bounded-variation classes, while Bera and Ghodadra [21] studied convergence rates for double rational Fourier series associated with functions of generalized bounded variation. These results demonstrate that bounded-variation-type assumptions continue to play an important role in contemporary Fourier analysis.

Within signal processing, Singh et al. [18] revisited the existence and convergence of Fourier representations and considered the use of Schwartz distributions for signals that do not admit classical Fourier representations. Vural and Akbas [19] discussed Fourier-series representations and Dirichlet conditions in a contemporary signals-and-systems setting. In the Fourier-transform setting on the real line, Bakhvalov [22] obtained a precise estimate for the decay rate of the Fourier transform of an integrable function of bounded variation on  $\mathbb{R}$ . Together, these developments emphasize the need to distinguish between classical sufficient conditions, bounded-variation convergence results, almost-everywhere convergence theorems, and generalized or distributional Fourier representations, particularly when passing from periodic to nonperiodic settings.

P. Singh *et al.* [1] argued that ‘none of the textbooks on signals and systems specify correctly the Dirichlet conditions (DCs)’ for the existence and convergence of Fourier representations. They further suggested that bounded variation (BV) should be regarded as the appropriate Dirichlet condition and discussed an ‘extended Carleson–Hunt theorem’ in connection with BV functions.

The purpose of the present paper is to examine these statements by distinguishing the historical development of Dirichlet-type conditions from the later bounded-variation framework and, importantly, by separating the Fourier-series and Fourier-transform settings. In particular, we consider the following points. First, the conditions commonly presented in modern textbooks as classical Dirichlet-type conditions are consistent with the role of Dirichlet’s classical sufficient criterion, although their modern formulation may explicitly include an  $L^1$ -integrability requirement. Second, Jordan’s bounded-variation criterion represents a later and broader convergence framework and should therefore be distinguished from the historical formulation of Dirichlet’s conditions. Third, the relationship between bounded variation and integrability depends essentially on the underlying domain. On a finite-period domain,  $BV(T) \subset L^p(T)$ ,  $1 \leq p \leq \infty$ , whereas on the unbounded domain  $\mathbb{R}$ ,  $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ . Consequently, bounded variation alone does not ensure the existence of the classical Fourier transform on  $\mathbb{R}$ .

### 1.1. Mathematical Novelty and Contribution of the Present Rebuttal

The contribution of this rebuttal is a systematic clarification of the mathematical assumptions, domains, and convergence notions underlying the statements made in [1]. Rather than introducing a new convergence theorem, the paper brings together the rele-

vant classical and modern frameworks and examines their precise scopes. In particular, it distinguishes the modern textbook formulation of Dirichlet-type sufficient conditions from the precise historical wording of Dirichlet's original work, while also distinguishing both from the later bounded-variation framework associated with Jordan.

The analysis further provides a unified comparison of the Dirichlet-type, Jordan, and Carleson–Hunt frameworks for periodic Fourier series and contrasts these results with the classical Fourier-transform setting on  $\mathbb{R}$ . For a finite-period domain,  $BV(T) \subset L^p(T)$ ,  $1 \leq p \leq \infty$ , whereas  $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ . This distinction is important when considering whether bounded variation can provide the integrability required for the classical Fourier-transform integral. On  $\mathbb{R}$ , bounded variation by itself does not provide such a guarantee.

The seven examples considered in [1] are also examined individually with respect to their  $L^1$ - and  $L^2$ -integrability, bounded variation, underlying domain, and the particular notion of Fourier representation involved. This allows the conclusions associated with Dirichlet-type conditions, Jordan's bounded-variation result, and the Carleson–Hunt theorem to be considered under their respective hypotheses, rather than being treated as interchangeable results.

Overall, the paper places the classical Dirichlet-type conditions, the later bounded-variation framework, and the Carleson–Hunt results within their respective mathematical settings. For Fourier-series convergence on a finite periodic domain, the analysis does not indicate a need to replace the standard modern formulation of the Dirichlet-type sufficient conditions. The principal contribution is therefore a mathematically explicit clarification of the hypotheses, domains, and convergence interpretations relevant to the claims under consideration.

The remainder of the paper is organized as follows.

**Section 2** reviews the background on Fourier series and Fourier transforms, including the classical Dirichlet-type conditions and their standard textbook interpretations. This section establishes the mathematical setting used throughout the paper.

**Section 3** discusses the historical development of Fourier-series convergence and summability, from Dirichlet's 1829 work and Jordan's bounded-variation criterion to Fej'er's Cesàro summability and the later Carleson–Hunt almost-everywhere convergence results.

**Section 4** summarizes the principal convergence theorems and Fourier-transform frameworks, with particular emphasis on the distinction between periodic Fourier series and Fourier transforms on the unbounded domain  $\mathbb{R}$ .

**Section 5** examines the principal points requiring clarification in [1], including the attribution of bounded variation to Dirichlet, the distinction between Fourier-series and Fourier-transform hypotheses, and the interpretation of the examples presented in that work.

**Section 6** presents the mathematical framework used for the subsequent analysis, distinguishing the finite-period Fourier-series setting from the unbounded Fourier-transform setting and clarifying the respective roles of bounded variation,  $L^p$ -integrability, and classical Fourier-transform existence.

**Section 7** examines the relationships among Dirichlet-type conditions, bounded variation, and Carleson–Hunt convergence, with the examples in [1] interpreted according to their domains and regularity properties.

**Section 8** discusses the implications for signal-processing literature and considers how

classical Dirichlet-type conditions and broader convergence frameworks should be stated and applied in engineering contexts.

**Section 9** concludes the paper by summarizing the principal mathematical and historical observations and their implications for the interpretation and teaching of classical Dirichlet-type conditions.

## 2. Background: Fourier Representations and Classical Conditions

This section introduces the notation and mathematical setting used throughout the manuscript. Let  $T > 0$  denote a finite period, and let  $\gamma(t)$  be a periodic function with period  $T$ . Its fundamental angular frequency is denoted by  $\omega_0 = \frac{2\pi}{T}$ . The index  $k \in \mathbb{Z}$  is used for Fourier-series coefficients, while  $f \in \mathbb{R}$  denotes the frequency variable in the Fourier-transform setting. Throughout the paper, FS and FT denote Fourier series and Fourier transform, respectively. The notation  $L^p(\Omega)$  and  $BV(\Omega)$  refers to the usual Lebesgue space and the space of functions of bounded variation on the domain  $\Omega$ .

For a periodic function  $\gamma(t)$  of period  $T$ , its complex Fourier series is written as

$$\gamma(t) \sim \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}, \quad X_k = \frac{1}{T} \int_0^T \gamma(t) e^{-jk\omega_0 t} dt. \quad (2.1)$$

The symbol “ $\sim$ ” emphasizes that the relation between  $\gamma(t)$  and its Fourier series depends on the regularity of the function and on the particular mode of convergence under consideration. Under suitable hypotheses, the series converges to  $\gamma(t)$  at points of continuity and to the appropriate limiting value at jump discontinuities.

For a nonperiodic function  $\gamma(t)$  defined on the unbounded domain  $\mathbb{R}$ , the classical Fourier transform, whenever the defining integral exists, is given by

$$\hat{\gamma}(f) = \int_{-\infty}^{\infty} \gamma(t) e^{-j2\pi f t} dt. \quad (2.2)$$

In particular, if  $\gamma \in L^1(\mathbb{R})$ , the integral is absolutely convergent and  $\hat{\gamma}$  is a bounded continuous function of  $f$ .

### 2.1. Classical Dirichlet Conditions

The term “Dirichlet conditions” is used in different ways in historical and modern treatments of Fourier-series convergence. Dirichlet’s work provided important sufficient conditions for the convergence of trigonometric series, whereas modern textbooks often state these conditions in a more explicit form involving integrability, extrema, and discontinuities. To maintain this distinction, we use *Dirichlet-type conditions* when referring to the standard modern textbook formulation and reserve historical statements for the original mathematical work of Dirichlet.

Dirichlet’s work on Fourier-series convergence dates to 1829 [7]. For a function  $\gamma(x)$  defined on a finite period, such as  $[-\pi, \pi]$ , the classical Dirichlet-type conditions are commonly stated in the following form:

1.  $\gamma(x)$  is single-valued and finite on the interval;

2.  $\gamma(x)$  has a finite number of maxima and minima; and
3.  $\gamma(x)$  has a finite number of finite-magnitude discontinuities.

Under the usual piecewise-regularity interpretation of these conditions, the Fourier series converges to  $\gamma(x)$  at every point of continuity and, at a jump discontinuity, converges to the average of the corresponding one-sided limits,

$$\frac{\gamma(x^-) + \gamma(x^+)}{2}. \quad (2.3)$$

The conditions denoted by (B1)–(B3) in P. Singh *et al.* [1] —absolute integrability over one period, a finite number of extrema, and a finite number of finite-magnitude discontinuities—represent a commonly used modern textbook formulation of classical Dirichlet-type sufficient conditions [2, 5, 3, 4]. In the present paper, we use (B1)–(B3) in this modern sense and do not interpret them as a literal reproduction of the wording of Dirichlet's original work.

The explicit integrability condition in (B1),

$$\int_{-\pi}^{\pi} |\gamma(x)| dx < \infty, \quad (2.4)$$

is natural in the modern treatment of Fourier series because the Fourier coefficients are defined through integrals. Moreover, on a finite interval, boundedness of  $\gamma$  already implies absolute integrability, since

$$\int_{-\pi}^{\pi} |\gamma(x)| dx \leq 2\pi \|\gamma\|_{\infty} < \infty. \quad (2.5)$$

Thus, when boundedness is known in advance, the explicit  $L^1$  condition does not impose an additional restriction. It is nevertheless useful to state the integrability requirement separately because it makes the analytical assumptions underlying the Fourier coefficients explicit.

Accordingly, conditions (B1)–(B3) are treated throughout this paper as a standard modern formulation of sufficient Dirichlet-type conditions for Fourier-series convergence on a finite periodic domain. This formulation should be distinguished from the precise historical wording of Dirichlet's original work, particularly with respect to the explicit presentation of the  $L^1$  condition.

### 3. Historical Evolution of Convergence Theorems

For clarity, this paper distinguishes historical statements concerning the development and attribution of Fourier convergence results from mathematical statements concerning the hypotheses and conclusions of the corresponding theorems. Historical references are used to identify the origin and chronological development of the relevant frameworks, whereas mathematical claims are stated according to their modern formulations and stated hypotheses.

In particular, the classical Dirichlet-type conditions are discussed as a standard modern formulation of sufficient conditions associated with the classical theory, rather than as

a literal transcription of the precise wording of Dirichlet's original work. The bounded-variation framework is considered separately as a later development associated with Jordan. The results of Carleson and Hunt are considered in their modern  $L^p(T)$  setting, with their almost-everywhere convergence conclusion distinguished from the pointwise convergence obtained under stronger regularity assumptions.

Historical attribution and mathematical implication therefore address different aspects of the theory. For example, the statement

$$BV(T) \subset L^p(T), \quad 1 \leq p \leq \infty, \quad (3.1)$$

is a mathematical property of the corresponding function spaces, whereas the attribution of the bounded-variation convergence criterion to Jordan is a historical statement. Both are considered separately and under their appropriate contexts.

Table 1 summarizes the historical development of increasingly general convergence and summability results for Fourier series.

Table 1: Historical Development of Fourier-Series Convergence and Summability Results

| Year | Mathematician | Result / Condition                             | Domain                              | Convergence / Summability Result                                                                                        |
|------|---------------|------------------------------------------------|-------------------------------------|-------------------------------------------------------------------------------------------------------------------------|
| 1829 | Dirichlet     | Classical Dirichlet-type sufficient conditions | Finite-period Fourier series on $T$ | Pointwise convergence at continuity points; convergence to the midpoint of the one-sided limits at jump discontinuities |
| 1881 | Jordan        | Bounded variation, $f \in BV(T)$               | Finite-period Fourier series on $T$ | Pointwise convergence at every point to $\frac{1}{2}[f(x^-) + f(x^+)]$                                                  |
| 1904 | Fejér [10]    | Cesàro (Fejér) means                           | Finite-period Fourier series on $T$ | Uniform convergence for continuous periodic functions                                                                   |
| 1966 | Carleson      | $f \in L^2(T)$                                 | Finite-period Fourier series on $T$ | Almost-everywhere convergence of the Fourier series                                                                     |
| 1968 | Hunt          | $f \in L^p(T), 1 < p < \infty$                 | Finite-period Fourier series on $T$ | Almost-everywhere convergence of the Fourier series                                                                     |

Dirichlet's 1829 work provided important sufficient conditions for the convergence of trigonometric series [7]. In modern mathematical and engineering texts, the corresponding Dirichlet-type conditions are commonly expressed through absolute integrability over one period, a finite number of extrema, and a finite number of finite-magnitude discontinuities. Under the usual piecewise-regular interpretation, these conditions provide pointwise convergence of the Fourier series at continuity points and convergence to the average of the one-sided limits at jump discontinuities. The present paper adopts this modern formulation for its mathematical analysis while keeping it distinct from the precise historical formulation of Dirichlet's original work.

A later development was Jordan's bounded-variation framework. If  $f \in BV[-\pi, \pi]$ , then the Fourier series converges at each point to

$$\frac{1}{2}(f(x^-) + f(x^+)), \quad (3.2)$$

with the usual interpretation at points of continuity [6]. Jordan's result provides a broader sufficient framework for pointwise Fourier-series convergence than the classical Dirichlet-type conditions.

This historical distinction is relevant to the discussion in [1], particularly the suggestion that bounded variation should be identified as an "original" Dirichlet condition. The bounded-variation framework belongs to a later stage in the development of Fourier-series convergence theory and is associated with Jordan's work [6]. It is therefore more appropriate to regard BV as a later generalization of the classical Dirichlet-type framework rather than as part of Dirichlet's original formulation.

The conditions traditionally presented in modern mathematical and engineering texts as Dirichlet conditions—including a finite number of extrema and finite-magnitude jump discontinuities over a period—provide a convenient sufficient formulation of the classical convergence criterion. Under the usual piecewise-monotone interpretation of these conditions, such functions have finite total variation and therefore belong to  $BV(T)$ . The converse does not hold in general: the bounded-variation class contains functions with more complicated local behavior, including functions with infinitely many oscillations or discontinuities, provided that their total variation remains finite.

Consequently, within the finite-period setting considered here, the relationship between the two frameworks can be represented schematically as

$$\text{classical Dirichlet-type conditions} \subsetneq BV(T). \quad (3.3)$$

Both frameworks provide pointwise Fourier-series convergence under their respective hypotheses, but the bounded-variation framework covers a broader class of functions. Jordan's result can therefore be viewed as a later extension of the classical sufficient-condition framework rather than as a redefinition of Dirichlet's conditions.

#### 4. Summary of Convergence Theorems

To provide a unified overview of the principal results used in this paper, Tables 2–4 summarize the classical convergence theorems associated with Dirichlet [7], Jordan [6], Lebesgue [16], Carleson [8], Hunt [9], and Kolmogorov [11], together with standard  $L^2$  and distributional frameworks for Fourier transforms [12, 13, 14, 15]. A central distinction throughout these results is that Fourier series (FS) are naturally formulated for periodic functions on a finite period, whereas Fourier transforms (FT) are formulated on the unbounded domain  $\mathbb{R}$ .

On a finite period  $T$ , a function of bounded variation is bounded and therefore belongs to  $L^p(T)$  for every  $1 \leq p \leq \infty$ . Thus, a function in  $BV(T)$  satisfies the function-space assumption required by the Carleson–Hunt theorem for every  $1 < p < \infty$ , while Jordan's theorem gives the stronger pointwise convergence conclusion for the BV class. These

results, however, have different hypotheses and conclusions and should not be treated as equivalent statements.

The situation changes on the unbounded domain  $\mathbb{R}$ . In particular,  $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ , so bounded variation alone does not ensure the existence of the classical Fourier transform as an absolutely convergent integral. This distinction is essential when results concerning Fourier-series convergence are compared with statements about Fourier-transform existence.

More generally, the Fourier transform can be considered under broader function-space frameworks, including the  $L^2$  theory associated with Plancherel's theorem and the theory of tempered distributions. These frameworks extend the notion of Fourier transform beyond the classical  $L^1$  setting, but they represent different senses of transform existence and should therefore be stated separately.

Table 2: Fourier-Series Convergence Results under Different Hypotheses

| Theorem / Result                        | Hypotheses                                                                                                                                                         | Domain                                | Convergence / Conclusion                                                                                                                                                                                           |
|-----------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Dirichlet (1829) [7]                    | Classical Dirichlet-type sufficient conditions: absolute integrability over one period, finitely many extrema, and finitely many finite-magnitude discontinuities. | Finite-period Fourier series on $T$ . | Pointwise convergence at continuity points to $f(x)$ , and at jump discontinuities to $\frac{1}{2}[f(x^-) + f(x^+)]$ .                                                                                             |
| Jordan (1881) [6]                       | $f \in BV(T)$ , i.e., $f$ is of bounded variation on one period.                                                                                                   | Finite-period Fourier series on $T$ . | Pointwise convergence at every point to $\frac{1}{2}[f(x^-) + f(x^+)]$ ; in particular, to $f(x)$ at continuity points. This provides a broader sufficient framework than the classical Dirichlet-type conditions. |
| Lebesgue/Dini [16]                      | $f \in L^1(T)$ together with the appropriate Dini-type regularity condition at the point under consideration.                                                      | Finite-period Fourier series on $T$ . | Pointwise convergence at points satisfying the corresponding Dini regularity condition.                                                                                                                            |
| Carleson (1966) [8];<br>Hunt (1968) [9] | $f \in L^p(T)$ , $1 < p < \infty$ .                                                                                                                                | Finite-period Fourier series on $T$ . | Almost-everywhere (a.e.) convergence of the Fourier series.                                                                                                                                                        |
| Kolmogorov (1923) [11]                  | $f \in L^1(T)$ .                                                                                                                                                   | Finite-period Fourier series on $T$ . | There exists an $L^1(T)$ function whose Fourier series diverges almost everywhere; hence $L^1(T)$ alone does not guarantee pointwise or a.e. convergence.                                                          |
| Riemann–Lebesgue                        | $f \in L^1(T)$ .                                                                                                                                                   | Finite-period Fourier series on $T$ . | The Fourier coefficients tend to zero. This coefficient-decay result alone does not guarantee convergence of the Fourier series.                                                                                   |

Table 3: Fourier-Transform Existence under Different Function-Space Frameworks

| FT Framework                       | Result / Hypotheses                                                         | Domain                                  | Sense of Transform / Conclusion                                                                                                                                                                                          |
|------------------------------------|-----------------------------------------------------------------------------|-----------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Classical Fourier transform        | $f \in L^1(\mathbb{R})$ .                                                   | Nonperiodic functions on $\mathbb{R}$ . | The Fourier integral exists absolutely and defines a bounded, continuous function that vanishes at infinity.                                                                                                             |
| BV on $\mathbb{R}$ (Clarification) | $f \in BV(\mathbb{R})$ , but $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ . | Nonperiodic functions on $\mathbb{R}$ . | Bounded variation alone does not guarantee existence of the classical $L^1$ Fourier transform. For example, $f(t) = 1$ belongs to $BV(\mathbb{R})$ but not to $L^1(\mathbb{R})$ .                                        |
| Plancherel                         | $f \in L^2(\mathbb{R})$ .                                                   | Nonperiodic functions on $\mathbb{R}$ . | The Fourier transform is defined in the $L^2$ sense and extends uniquely from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to an $L^2$ -isometry.                                                                              |
| Distributional Fourier transform   | $f \in \mathcal{S}'(\mathbb{R})$ , the space of tempered distributions.     | Nonperiodic functions on $\mathbb{R}$ . | The Fourier transform exists in the sense of tempered distributions; every tempered distribution has a Fourier transform in $\mathcal{S}'(\mathbb{R})$ .                                                                 |
| Riemann–Lebesgue lemma             | $f \in L^1(\mathbb{R})$ .                                                   | Nonperiodic functions on $\mathbb{R}$ . | The classical Fourier transform satisfies $\widehat{f}(\omega) \rightarrow 0$ as $ \omega  \rightarrow \infty$ . This describes decay of the transform and does not by itself constitute a separate existence condition. |

Table 4: Core Structural Differences Between Fourier Series and Fourier Transforms

| Property                              | FS on a Finite Period $T$                                             | FT on $\mathbb{R}$                                                              |
|---------------------------------------|-----------------------------------------------------------------------|---------------------------------------------------------------------------------|
| Domain measure                        | Finite                                                                | Infinite                                                                        |
| $BV(T) \Rightarrow L^\infty(T)$       | Yes                                                                   | $BV(\mathbb{R}) \Rightarrow L^\infty(\mathbb{R})$ also holds                    |
| $L^\infty \Rightarrow L^1$            | Yes, on a finite-period domain                                        | No, on $\mathbb{R}$                                                             |
| $BV \Rightarrow L^1$                  | Yes, on a finite-period domain                                        | No; $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$                                |
| Jordan’s BV theorem                   | Applies to periodic Fourier-series convergence                        | Not a classical Fourier-transform existence theorem on $\mathbb{R}$             |
| Carleson–Hunt theorem                 | Applies to Fourier-series convergence for $L^p(T)$ , $1 < p < \infty$ | Does not provide a classical Fourier-transform existence result on $\mathbb{R}$ |
| Classical Fourier-transform existence | Not the relevant framework                                            | Requires $f \in L^1(\mathbb{R})$ for the absolutely convergent Fourier integral |

## 5. Interpretive Issues in Singh *et al.* (2022)

This section examines the principal claims made by P. Singh *et al.* [1] concerning the definition of Dirichlet conditions, the role of bounded variation, the interpretation of the Carleson–Hunt theorem, and the Fourier representations of the examples considered in their work. Each point is considered in relation to the relevant historical development, mathematical hypotheses, function space, and underlying domain.

### 5.1. Redefinition of Dirichlet Conditions

Singh *et al.* [1] distinguish between two sets of conditions referred to as Dirichlet conditions (DCs):

$$\begin{aligned} \text{(A1)–(A2)} \quad & \int_T |\gamma(t)| dt < \infty \quad \text{and} \quad \gamma(t) \in BV(T), \\ \text{(B1)–(B3)} \quad & \int_T |\gamma(t)| dt < \infty, \end{aligned}$$

together with a finite number of maxima and minima and a finite number of finite-magnitude discontinuities.

They argue that standard textbooks use (B1)–(B3) as the Dirichlet conditions. The conditions (B1)–(B3), however, are more appropriately described as a standard modern textbook formulation of classical Dirichlet-type sufficient conditions for Fourier-series convergence. They should not be interpreted as a literal reproduction of the precise historical wording of Dirichlet's original work.

The bounded-variation framework appearing in (A1)–(A2) should likewise be distinguished from the historical Dirichlet formulation. The bounded-variation convergence criterion is associated with Jordan's later work [6] and provides a broader sufficient framework for Fourier-series convergence. Thus, the two sets of conditions represent different stages and formulations in the development of Fourier-series convergence theory rather than two alternative versions of Dirichlet's original conditions.

### 5.2. Interpretive Issues of Carleson–Hunt

Singh *et al.* further state that if  $\gamma(t) \in BV(T)$ , then  $\gamma(t) \in L^p(T)$  for all  $p$  and hence the Carleson–Hunt theorem applies. This statement is valid in the finite-period setting, but its domain must be specified. On a finite periodic interval  $T$ , a function of bounded variation is bounded and therefore belongs to  $L^p(T)$  for every  $1 \leq p \leq \infty$ . Consequently, for  $1 < p < \infty$ , the function satisfies the function-space hypothesis of the Carleson–Hunt theorem.

The corresponding implication does not extend to the unbounded domain  $\mathbb{R}$ . In particular,  $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ . Thus, bounded variation on  $\mathbb{R}$  does not by itself provide the  $L^1$ -integrability required for the classical Fourier-transform integral.

The following examples illustrate the distinction between bounded variation and  $L^1$ -integrability on  $\mathbb{R}$ . Example 1 shows that bounded variation does not imply  $L^1$ -integrability, whereas Example 2 shows that  $L^1$ -integrability does not require bounded variation. Together, they demonstrate why the finite-period inclusion  $BV(T) \subset L^p(T)$  cannot be transferred directly to the Fourier-transform setting on  $\mathbb{R}$ . For the classical Fourier transform,  $L^1(\mathbb{R})$  is the relevant integrability framework, while bounded variation alone is neither sufficient nor necessary for the existence of the classical Fourier transform.

### 5.3. Example 1: Constant Function $\gamma(t) = 1$ on $\mathbb{R}$

Consider the constant function  $\gamma(t) = 1$ ,  $t \in \mathbb{R}$ . Its total variation is zero, and hence  $\gamma \in BV(\mathbb{R})$ . However,

$$\int_{-\infty}^{\infty} |\gamma(t)| dt = \int_{-\infty}^{\infty} 1 dt = \infty, \quad (5.1)$$

so that  $\gamma \notin L^1(\mathbb{R})$ . Accordingly, the classical Fourier-transform integral  $\int_{-\infty}^{\infty} e^{-j2\pi ft} dt$  does not converge as an ordinary absolutely convergent integral. Thus, bounded variation alone does not guarantee the existence of the classical  $L^1$  Fourier transform on  $\mathbb{R}$ .

### 5.4. Example 2: The Function $\sin(1/t)$ (Compact Support)

Consider the function defined on  $\mathbb{R}$  by

$$\gamma(t) = \begin{cases} \sin(1/t), & 0 < |t| \leq 1, \\ 0, & t = 0 \text{ or } |t| > 1. \end{cases} \quad (5.2)$$

This is the type of example used by Singh *et al.* [1] in their equation (5) to illustrate a function with infinite variation.

For  $t \neq 0$ ,

$$\frac{d}{dt} \sin(1/t) = -\frac{\cos(1/t)}{t^2}.$$

Consequently,

$$\int_0^1 \left| \frac{d}{dt} \sin(1/t) \right| dt = \int_1^{\infty} |\cos u| du = \infty, \quad (5.3)$$

where the substitution  $u = 1/t$  has been used. Hence the total variation is infinite and  $\gamma \notin BV(\mathbb{R})$ .

On the other hand,  $|\gamma(t)| \leq 1$  and  $\gamma$  has compact support. Therefore,  $\int_{-\infty}^{\infty} |\gamma(t)| dt \leq 2 < \infty$ , so that  $\gamma \in L^1(\mathbb{R})$ . Its classical Fourier transform therefore exists and is a bounded continuous function. This example demonstrates that  $L^1(\mathbb{R})$  integrability, and hence existence of the classical Fourier transform, does not require bounded variation.

Taken together, Examples 1 and 2 show that bounded variation and  $L^1(\mathbb{R})$ -integrability are distinct properties on the real line. Consequently, bounded variation cannot be used as a necessary condition for the existence of the classical Fourier transform, nor does it by itself provide the required  $L^1$ -integrability.

## 6. Correct Theoretical Framework

The distinction between Fourier-series convergence on a finite period and Fourier-transform existence on  $\mathbb{R}$  can be made precise through basic relations between bounded variation and Lebesgue integrability. The following results provide the mathematical foundation for this distinction.

### 6.1. Basic Mathematical Results

**Proposition 1.** Let  $T$  be a finite interval. If  $f \in BV(T)$ , then

$$f \in L^p(T), \quad 1 \leq p \leq \infty. \quad (6.1)$$

*Proof.* A function of bounded variation on a finite interval is bounded. Thus, for some  $M > 0$ ,  $|f(t)| \leq M$ ,  $t \in T$ . Hence, for  $1 \leq p < \infty$ ,

$$\int_T |f(t)|^p dt \leq M^p |T| < \infty, \quad (6.2)$$

while boundedness gives  $f \in L^\infty(T)$ . Therefore,

$$BV(T) \subset L^p(T), \quad 1 \leq p \leq \infty. \quad (6.3)$$

**Proposition 2.** On  $\mathbb{R}$ ,

$$BV(\mathbb{R}) \not\subset L^1(\mathbb{R}). \quad (6.4)$$

*Proof.* Consider the constant function  $f(t) \equiv 1$ . Its total variation on  $\mathbb{R}$  is zero, so  $f \in BV(\mathbb{R})$ . However,

$$\int_{\mathbb{R}} |f(t)| dt = \int_{\mathbb{R}} 1 dt = \infty. \quad (6.5)$$

Thus,  $f \notin L^1(\mathbb{R})$ , and consequently

$$BV(\mathbb{R}) \not\subset L^1(\mathbb{R}). \quad (6.6)$$

**Proposition 3.** The converse inclusion also fails:

$$L^1(\mathbb{R}) \not\subset BV(\mathbb{R}). \quad (6.7)$$

*Proof.* The compactly supported function considered in Subsection 5.4 belongs to  $L^1(\mathbb{R})$  but has infinite total variation. Hence it does not belong to  $BV(\mathbb{R})$ , which gives

$$L^1(\mathbb{R}) \not\subset BV(\mathbb{R}). \quad (6.8)$$

**Corollary 1.** On  $\mathbb{R}$ ,

$$BV(\mathbb{R}) \not\subset L^1(\mathbb{R}), \quad L^1(\mathbb{R}) \not\subset BV(\mathbb{R}). \quad (6.9)$$

Thus, bounded variation alone is neither sufficient nor necessary for the existence of the classical Fourier transform defined through the absolutely convergent  $L^1(\mathbb{R})$  integral.

### 6.2. Finite Interval (Fourier Series)

On a finite period  $T$ , every function of bounded variation is bounded. Consequently, for some  $M > 0$ ,

$$\int_T |\gamma(t)|^p dt \leq M^p |T| < \infty, \quad 1 \leq p < \infty. \quad (6.10)$$

Together with boundedness, this gives  $BV(T) \subset L^p(T)$ ,  $1 \leq p \leq \infty$ . Hence, for  $1 < p < \infty$ , a function in  $BV(T)$  satisfies the function-space hypothesis of the Carleson–Hunt

theorem. However, Jordan's 1881 theorem [6] already provides the stronger pointwise convergence result for functions of bounded variation: the Fourier series converges at every point to the average of the left- and right-hand limits. By contrast, Carleson–Hunt provides almost-everywhere convergence for the broader class  $L^p(T)$ ,  $1 < p < \infty$ .

Thus, within the subclass  $BV(T)$ , the Jordan and Carleson–Hunt results should be distinguished by both their hypotheses and their conclusions. Jordan's theorem gives pointwise convergence at every point for the  $BV$  class, whereas Carleson–Hunt establishes almost-everywhere convergence for the larger  $L^p(T)$  class.

This distinction is relevant to the interpretation of the discussion in Singh *et al.* [1], where Carleson–Hunt is presented in connection with the convergence of Fourier series of  $BV$  functions. The Carleson–Hunt theorem is certainly applicable to  $BV(T)$  through the inclusion  $BV(T) \subset L^p(T)$  for  $1 < p < \infty$ . Nevertheless, the historical and mathematical roles of the two results are different: Jordan's earlier theorem already provides the stronger pointwise conclusion for functions in  $BV(T)$ .

### 6.3. Unbounded Domain (Fourier Transform)

The situation is different on the unbounded domain  $\mathbb{R}$ . Bounded variation does not impose sufficient decay at infinity to ensure  $L^1(\mathbb{R})$ -integrability. For example, the constant function belongs to  $BV(\mathbb{R})$  but satisfies

$$\int_{-\infty}^{\infty} |\gamma(t)| dt = \infty. \quad (6.11)$$

Consequently, the classical Fourier-transform integral need not exist as an absolutely convergent integral.

Therefore, a bounded-variation condition on  $\mathbb{R}$  cannot, by itself, serve as a replacement for the  $L^1(\mathbb{R})$  condition required for the classical Fourier transform. Broader frameworks, such as the  $L^2$ -Fourier transform and the Fourier transform of tempered distributions, remain available when the classical  $L^1$  framework does not apply. These settings, however, represent different notions of Fourier-transform existence.

### 6.4. Re-examination of Examples 1–7 from [1]

We next reconsider Examples 1–7 presented in [1], summarized in Table 5. For each example, we identify the signal, the claim attributed to P. Singh *et al.*, and its mathematical status with respect to the domain,  $L^1$ - and  $L^2$ -integrability, bounded variation, and the relevant notion of Fourier convergence or transform existence.

This comparison emphasizes the distinction between Fourier series on a finite periodic domain and Fourier transforms on  $\mathbb{R}$ . In particular, failure of bounded variation does not by itself imply failure of  $L^1$ -integrability or of a Fourier representation. Conversely, bounded variation on  $\mathbb{R}$  does not guarantee  $L^1(\mathbb{R})$ -integrability. The examples therefore illustrate why sufficient conditions, necessary conditions, domains, and the precise sense in which a Fourier representation exists must be considered separately.

Table 5: Comparative Re-analysis of Examples 1–7 from P. Singh *et al.* (2022)

| Ex. | Signal                                                                                                                       | Claim by P. Singh et al.                                                                                                                                                                          | Mathematical analysis                                                                                                                                                                                                | Re- | Domain / Function-Space Status                                                                 | Fourier-Series / Transform Consequence                                                                                                                                                                               |
|-----|------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1   | $\gamma(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbf{1}_{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]}(t)$ on $[0, 1]$ (0 otherwise) | Claimed to satisfy “original” Dirichlet conditions (DCs), with BV allowing countably many steps, and hence the Fourier series should converge. Textbook conditions were described as restrictive. | The total variation satisfies $V(\gamma) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$ . Hence $\gamma \in BV([0, 1]) \cap L^1([0, 1])$ .                                                                         |     | Finite-period domain; $L^1([0, 1])$ : yes; $BV([0, 1])$ : yes.                                 | The Fourier series converges pointwise under the bounded-variation/Jordan framework; the classical Dirichlet-type sufficient framework is also applicable under the stated formulation.                              |
| 2   | $x(t) = 1/\sqrt{t}$ , $0 < t \leq 1$ , 0 otherwise (periodic on $[0, 1]$ )                                                   | Claimed to be in $L_1$ and hence to admit a Fourier series; textbooks were described as misclassifying the example.                                                                               | $\int_0^1 t^{-1/2} dt = 2 < \infty$ ( $L_1$ ); $\int_0^1 t^{-1} dt = \infty$ (not $L_2$ ); total variation infinite near $t = 0$ (not BV). Belongs to $L_p$ for $1 \leq p < 2$ .                                     |     | Finite-period domain; $L_1([0, 1])$ : yes; $BV([0, 1])$ : no; $L_p$ ( $p < 2$ ): yes.          | Classical Dirichlet-type and Jordan conditions fail. Carleson–Hunt theorem gives almost-everywhere convergence.                                                                                                      |
| 3   | $x(t) = \sin(2\pi/t)$ , $0 < t \leq 1$ , 0 otherwise                                                                         | Claimed to be in $L_1$ while BV was stated to fail.                                                                                                                                               | Substitution $u = 1/t$ yields $\int_0^1  \gamma'(t)  dt = 2\pi \int_1^{\infty}  \cos(2\pi u)  du = \infty$ . Hence <b>not</b> BV. Bounded, so $\gamma \in L_p([0, 1])$ for every $1 \leq p \leq \infty$ .            |     | Finite-period domain; $L_1([0, 1])$ : yes; $BV([0, 1])$ : <b>no</b> ; $L_p$ (all $p$ ): yes.   | Classical Dirichlet-type and Jordan conditions fail. Carleson–Hunt theorem gives almost-everywhere convergence.                                                                                                      |
| 4   | $\gamma(t) = \sin(\tan(\pi t))$ , $0 < t \leq 1$ , 0 otherwise                                                               | Claimed to be in $L^1$ but not BV because of rapid oscillations, suggesting a limitation of the textbook conditions.                                                                              | Since $ \gamma(t)  \leq 1$ , $\gamma \in L^p([0, 1])$ for every $1 \leq p \leq \infty$ . The rapid oscillations near $t = 1/2$ give infinite total variation, so $\gamma \notin BV([0, 1])$ .                        |     | Finite-period domain; $L^p([0, 1])$ for every $1 \leq p \leq \infty$ ; $BV([0, 1])$ : no.      | The Dirichlet-type and Jordan BV criteria do not apply directly. However, since $\gamma \in L^p([0, 1])$ for $1 < p < \infty$ , the Carleson–Hunt theorem gives almost-everywhere convergence of the Fourier series. |
| 5   | $\gamma(t) = \sin(2\pi/t)$ , $t \in \mathbb{R} \setminus \{0\}$                                                              | Not in $L^1(\mathbb{R})$ but in $L^2(\mathbb{R})$ ; the Fourier transform may therefore exist in an energy or distributional sense.                                                               | For large $ t $ , $\sin(2\pi/t) \sim 2\pi/t$ , so the $L^1$ integral diverges, whereas the squared tail behaves like $1/t^2$ and is integrable.                                                                      |     | Unbounded domain $\mathbb{R}$ ; $L^1(\mathbb{R})$ : no; $L^2(\mathbb{R})$ : yes.               | The classical $L^1$ Fourier integral is unavailable; the Fourier transform exists in the $L^2$ sense by Plancherel and distributionally interpretation.                                                              |
| 6   | $\gamma(t) = \cos(2\pi/t)$ , $t \in \mathbb{R} \setminus \{0\}$                                                              | Used as a contrast; integrability was left unclear.                                                                                                                                               | As $ t  \rightarrow \infty$ , $\cos(2\pi/t) \rightarrow 1$ . Hence the function has no decay sufficient for membership in any $L^p(\mathbb{R})$ , $1 \leq p < \infty$ , and it is not globally of bounded variation. |     | Unbounded domain $\mathbb{R}$ ; $L^p(\mathbb{R})$ : no for finite $p$ ; $BV(\mathbb{R})$ : no. | The classical $L^1$ Fourier transform does not exist. The function also does not fall within the $L^2$ Plancherel framework.                                                                                         |
| 7   | $\gamma(t) = \frac{\sin(\pi t)}{\pi t}$ , $t \in \mathbb{R}$                                                                 | Not in $L^1(\mathbb{R})$ because of its $1/ t $ tail, but in $L^2(\mathbb{R})$ ; its Fourier transform is associated with the rectangular function.                                               | The sinc function is not absolutely integrable but belongs to $L^2(\mathbb{R})$ . Its Fourier transform is therefore understood in the $L^2$ and corresponding distributional framework.                             |     | Unbounded domain $\mathbb{R}$ ; $L^1(\mathbb{R})$ : no; $L^2(\mathbb{R})$ : yes.               | The classical $L^1$ Fourier integral is not absolutely convergent. The Fourier transform exists in the $L^2$ sense and corresponds, under the chosen Fourier convention, to a rectangular spectrum.                  |

## 7. Correct Hierarchical Application of Convergence Theorems

For Fourier series defined on a finite period  $T$ , the principal convergence results can be viewed in a historical and mathematical hierarchy according to the function classes to which their hypotheses apply and the type of convergence they establish.

### 1. Dirichlet (1829) [7]:

The classical Dirichlet-type sufficient conditions (B1)–(B3) stated by Singh *et al.*, namely,

- absolute integrability over one period,
- a finite number of maxima and minima, and
- a finite number of finite-magnitude discontinuities,

provide sufficient conditions for pointwise convergence of the Fourier series. At a jump discontinuity, the series converges to the average of the left- and right-hand limits.

### 2. Jordan (1881) [6]:

The condition  $\gamma \in BV(T)$  provides a broader regularity framework for Fourier-series convergence. For a periodic function of bounded variation, the Fourier series converges pointwise at every point to the average of the left- and right-hand limits. Thus, the BV framework includes functions with more general variation and discontinuity structures than those described by the finite-extrema formulation of the classical Dirichlet-type conditions.

### 3. Carleson (1966)–Hunt (1968) [8, 9]:

If  $\gamma \in L^p(T)$  for some  $1 < p < \infty$ , then its Fourier series converges almost everywhere. This theorem applies to a larger function class than  $BV(T)$ , although the conclusion is almost-everywhere convergence rather than pointwise convergence at every point.

For the finite-period setting, the corresponding relationships between the commonly used function classes may be summarized as

$$BV(T) \subset L^p(T), \quad 1 \leq p \leq \infty, \quad (7.1)$$

since a function of bounded variation on a finite interval is bounded and the interval has finite measure. The associated convergence results can then be summarized as

$$\begin{aligned} \text{Dirichlet-type conditions} &\implies \text{pointwise convergence everywhere,} \\ BV(T) &\implies \text{pointwise convergence everywhere,} \\ L^p(T), \quad 1 < p < \infty &\implies \text{almost-everywhere convergence.} \end{aligned} \quad (7.2)$$

It is important, however, to distinguish the inclusion of function classes from the logical structure of the convergence theorems. The above results should not be interpreted as a strict sequence of automatic “fallback” conditions. Failure of the classical Dirichlet-type conditions does not, by itself, determine whether a function belongs to  $BV(T)$ , and failure of bounded variation does not determine whether the function belongs to  $L^p(T)$ . The hypotheses of each theorem must therefore be examined independently.

For example, a function may fail the finite-extrema formulation of the Dirichlet-type conditions while still belonging to  $BV(T)$ . In such a case, Jordan's theorem remains applicable and provides pointwise convergence. Similarly, a function may fail to have bounded variation while still belonging to  $L^p(T)$  for some  $1 < p < \infty$ ; Carleson–Hunt may then provide almost-everywhere convergence.

The examples presented by Singh *et al.* [1] should therefore be considered in accordance with their respective domains and regularity properties. Examples defined on a finite periodic interval belong to the Fourier-series setting and can be examined using the convergence results described above. In contrast, examples posed on the unbounded domain  $\mathbb{R}$  belong to the Fourier-transform setting and should not be placed directly within the finite-period Fourier-series hierarchy. This distinction is essential because the hypotheses and conclusions of the Fourier-series convergence theorems are formulated for periodic functions on a finite-measure domain.

Accordingly, the seven analytical examples discussed by Singh *et al.* [1] (pp. 80–84) are more appropriately interpreted according to the particular hypotheses satisfied by each example. The finite-period examples illustrate how different convergence theorems apply to different regularity classes, while the examples posed on  $\mathbb{R}$  require a separate analysis in the Fourier-transform framework. Thus, the seven analytical examples presented in Singh *et al.* [1] (pages 80–84) do not expose any flaw in the classical Dirichlet conditions — they merely illustrate precisely this descent down the hierarchy when the finite extrema or absolute-integrability requirements are breached.

## 8. Discussion and Clarifications for Signal Processing Literature

Standard signal-processing textbooks [2, 5, 3, 4] commonly present the classical Dirichlet-type conditions (B1)–(B3) as convenient and readily verifiable sufficient conditions for the convergence of Fourier series of periodic signals. These conditions remain useful in engineering applications because they provide a relatively simple way to establish pointwise convergence without requiring the use of more advanced function-space theory.

A potential source of confusion arises when results formulated for periodic Fourier series are discussed together with results concerning nonperiodic signals or Fourier transforms. The underlying domain, function class, and mode of convergence must be distinguished when applying these results.

From the signal-processing perspective, the following points are particularly relevant:

- The conditions (B1)–(B3) provide classical sufficient conditions for Fourier-series convergence on a finite period.
- The bounded-variation framework provides a broader sufficient condition for Fourier-series convergence, but it does not replace or invalidate the classical Dirichlet-type conditions.
- The Carleson–Hunt theorem concerns the almost-everywhere convergence of Fourier series for functions in  $L^p(T)$ ,  $1 < p < \infty$ . It should therefore not be interpreted as a general convergence theorem for the classical Fourier transform on  $\mathbb{R}$ .

In this context, describing the standard textbook Dirichlet-type conditions as “incorrect” does not reflect their mathematical role. They are sufficient conditions formulated for the finite-period Fourier-series setting and are not intended to characterize all functions for which a Fourier series converges. Consequently, the existence of broader convergence theorems does not, by itself, require the replacement or redefinition of these classical conditions.

The same distinction applies to the notation (D1)–(D3) when these conditions are presented under an alternative labeling. Their use for Fourier series on a finite period is mathematically consistent with their status as sufficient conditions. For the Fourier transform on  $\mathbb{R}$ , however, the relevant questions are different. In particular, one must consider integrability, decay at infinity, and the sense in which the transform is defined, such as the classical  $L^1$  transform, the  $L^2$  Plancherel framework, or a distributional interpretation.

It is also important to distinguish the historical development of the Dirichlet-type conditions from later generalizations. Bounded variation should not be attributed to Dirichlet as though it formed part of the original classical formulation. The BV convergence result is associated with Jordan and represents a later extension of the class of functions for which pointwise Fourier-series convergence can be established. Similarly, on a finite period,  $BV(T) \subset L^p(T)$ ,  $1 < p < \infty$ , so the Carleson–Hunt theorem applies to BV functions as a consequence of this inclusion. This observation does not constitute a separate “extended Carleson–Hunt theorem” based on bounded variation, nor does the same argument extend automatically to functions on  $\mathbb{R}$ , where bounded variation does not imply  $L^p$  or  $L^1$  integrability.

Therefore, the continued pedagogical use of the classical finite-extrema conditions associated with the Dirichlet-type formulation is appropriate when their scope is stated clearly. The main requirement is to distinguish their role as sufficient conditions for Fourier-series convergence on a finite period from later and broader convergence results, and from the separate analytical framework required for Fourier transforms on the unbounded domain  $\mathbb{R}$ .

## 9. Conclusion

This paper has examined the historical development, mathematical scope, and appropriate application of the classical Dirichlet-type conditions for Fourier-series convergence, together with their relationship to the later results of Jordan and Carleson–Hunt. The principal conclusions can be summarized as follows:

1. The textbook conditions (B1)–(B3) represent a standard modern formulation of classical Dirichlet-type sufficient conditions for Fourier-series convergence on a finite period. They provide a practical and readily verifiable framework for establishing pointwise convergence under the stated hypotheses.
2. Jordan’s bounded-variation (BV) theorem provides a broader sufficient framework for pointwise convergence of Fourier series. As a later development, it encompasses a wider class of functions than the finite-extrema formulation of the classical Dirichlet-type conditions, while retaining pointwise convergence as its principal conclusion.
3. The Carleson–Hunt theorem must be considered together with its precise hypotheses and domain. On a finite period,  $BV(T) \subset L^p(T)$ ,  $1 \leq p \leq \infty$ , so the theorem ap-

plies to BV functions and yields almost-everywhere convergence. On the unbounded domain  $\mathbb{R}$ , however,  $BV(\mathbb{R}) \not\subset L^1(\mathbb{R})$ , and bounded variation alone therefore does not ensure the existence of the classical Fourier transform as an absolutely convergent  $L^1$ -integral.

4. The examples discussed in [1] must be interpreted according to their individual domains and analytical properties. Their behavior depends on factors such as integrability, bounded variation, periodicity, decay at infinity, and the particular notion of convergence or transform existence being considered.
5. A clear distinction between Fourier series on finite periodic domains and Fourier transforms on the unbounded domain  $\mathbb{R}$  is therefore essential. A convergence theorem established for Fourier series cannot be transferred directly to the Fourier-transform setting without separately verifying the assumptions associated with the domain and the relevant notion of transform.

In conclusion, the analysis presented here supports retaining the classical Dirichlet-type conditions as a valid set of sufficient conditions for Fourier-series convergence on a finite period. Their role should be distinguished from later generalizations, such as Jordan's bounded-variation framework, and from the broader almost-everywhere convergence results of Carleson–Hunt. At the same time, the Fourier-transform setting on  $\mathbb{R}$  requires separate consideration of integrability, decay, and the sense in which the transform is defined. Maintaining these distinctions provides a consistent framework for interpreting the historical literature and for understanding the continued use of classical Dirichlet-type conditions in modern signal-processing applications.

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## References

- [1] P. Singh, A. Singhal, B. Fatimah, A. Gupta, and S. D. Joshi, "Proper Definitions of Dirichlet Conditions and Convergence of Fourier Representations," *IEEE Signal Processing Magazine*, vol. 39, no. 5, 77–84, Sept. 2022, doi:10.1109/MSP.2022.3172620.
- [2] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals and Systems*, 2nd ed. Prentice Hall, 1996.
- [3] A. Papoulis, *The Fourier Integral and Its Applications*. McGraw–Hill, 1962.
- [4] S. Haykin and B. V. Van Veen, *Signals and Systems*, 2nd ed. Wiley, 2003.
- [5] B. P. Lathi, *Linear Systems and Signals*, 2nd ed. Oxford University Press, 2004.
- [6] C. Jordan, "Sur la série de Fourier," *Comptes Rendus de l'Académie des Sciences de Paris*, vol. 92, 228–230, 1881.
- [7] P. G. L. Dirichlet, "Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données," *Journal für die reine und angewandte Mathematik*, vol. 4, 157–169, 1829, doi:10.1515/crll.1829.4.157.
- [8] L. Carleson, "On convergence and growth of partial sums of Fourier series," *Acta Mathematica*, vol. 116, 135–157, 1966, doi:10.1007/BF02392815.
- [9] R. A. Hunt, "On the convergence of Fourier series," in *Orthogonal Expansions and Their Continuous Analogues*, Southern Illinois University Press, Carbondale, IL, 235–255, 1968.

- [10] L. Fejér, “Über die punktweise Konvergenz der Fourierschen Reihen,” *Mathematische Annalen*, vol. 58, 51–69, 1904, [doi:10.1007/BF01457775](https://doi.org/10.1007/BF01457775).
- [11] A. N. Kolmogorov, “Une série de Fourier–Lebesgue divergente presque partout,” *Fundamenta Mathematicae*, vol. 4, no. 1, 324–328, 1923.
- [12] A. Zygmund, *Trigonometric Series*, 2nd ed. Cambridge University Press, 1959.
- [13] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed. Cambridge University Press, 2004.
- [14] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*. Princeton University Press, 2003.
- [15] W. Rudin, *Real and Complex Analysis*, 3rd ed. McGraw–Hill, 1987.
- [16] H. Lebesgue, *Leçons sur les séries trigonométriques professées au Collège de France*. Gauthier–Villars, Paris, 1906.
- [17] E. E. Berdysheva, N. Dyn, E. Farkhi, and A. Mokhov, “Metric Fourier Approximation of Set-Valued Functions of Bounded Variation,” *Journal of Fourier Analysis and Applications*, vol. 27, Art. no. 17, 2021, [doi:10.1007/s00041-021-09812-7](https://doi.org/10.1007/s00041-021-09812-7).
- [18] P. Singh, A. Singhal, B. Fatimah, A. Gupta, and S. D. Joshi, “On the Convergence of Fourier Representations and Schwartz Distributions,” *Franklin Open*, vol. 8, Art. no. 100155, 2024, [doi:10.1016/j.fraope.2024.100155](https://doi.org/10.1016/j.fraope.2024.100155).
- [19] F. T. Y. Vural and E. Akbas, “Fourier Series Representation of Continuous Time Periodic Signals,” in *Signals and Systems*, Wiley, 2024, ch. 6, [doi:10.1002/9781394215782.ch6](https://doi.org/10.1002/9781394215782.ch6).
- [20] K. N. Darji, “On  $\beta$ -Absolute Convergence of Fourier Series with Small Gaps,” *Mathematical Notes*, vol. 118, nos. 1–2, 75–79, 2025.
- [21] R. K. Bera and B. L. Ghodadra, “Rate of convergence of double rational Fourier series of functions of generalized bounded variation,” *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, no. 5, 26–43, 2025, [doi:10.26907/0021-3446-2025-5-26-43](https://doi.org/10.26907/0021-3446-2025-5-26-43).
- [22] A. N. Bakhvalov, “On the Rate of Decrease in the Fourier Transform of a Function of Bounded Variation on  $\mathbb{R}$ ,” *Moscow University Mathematics Bulletin*, vol. 80, 381–383, 2025, [doi:10.3103/S0027132225700688](https://doi.org/10.3103/S0027132225700688).