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More Properties of Fractional Proportional Differences

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Abstract

The main aim of this paper is to clarify the action of the discrete Laplace transform on the fractional proportional operators. First of all, we recall the nabla fractional sums and differences and the discrete Laplace transform on time scale equivalent to $h\mathbb{Z}$. The discrete h -Laplace transform and its convolution theorem are then used to study the introduced discrete fractional operators.

Keywords: Fractional proportional sum, Caputo fractional proportional difference, Riemann fractional proportional difference, discrete h -Laplace transform.

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1. Introduction

Developing new differential and integral operators that generates the classical operators is an important branch of mathematical analysis [1, 2, 3, 4, 5, 6, 7, 8, 9].

Recently, the discrete fractional operators are investigated thoroughly to develop operators that can better describe some real world problems. In [10, 11], the authors introduced conformable derivatives and integrals which are local-type derivatives and integrals with arbitrary order. The authors in [12, 13] presented a type of proportional derivatives that yields the original function and its derivatives directly when the parameter tends to 0 and 1. In [14, 15, 16, 17, 18, 19], the nonsingular case is studied, authors defined new types of fractional operators with nonsingular exponential and Mittag-Leffler kernels. Fractional h -differences with discrete exponential kernels are discussed along with their monotonicity properties in [20, 21, 22]. The authors in [23], studied nabla fractional

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sums and differences using the discrete Laplace transform on the time scale $h\mathbb{Z}$. They employed a local nabla proportional difference to generate left and right generalized types of fractional differences with memory.

Local type derivatives and integrals are beneficial when generating new types of fractional derivatives and integrals with memory using different types of kernels [24, 25, 26, 27]. In this paper, we reintroduce the discrete Laplace transform on the time scale $h\mathbb{Z}$ and extend the theoretical framework of the nabla discrete version of proportional differences to generate new types of generalized fractional differences and sums presented in [23]. The kernel of the generalized fractional sum and difference operators is then discussed. The convolution theorem for the discrete h-Laplace transform is utilized to introduce the discrete fractional operators and propose a solution for the Cauchy linear fractional difference type problems with step $0 < h \leq 1$.

The paper is organized as follows: In Section 2, we review the nabla fractional sums and differences on the time scale $h\mathbb{Z}$. The generation of the fractional differences and sums with memory is discussed in Section 3. Section 4 dedicated to study Riemann fractional proportional sums and differences using h-discrete Laplace transforms. In Section 5, we present Caputo fractional proportional difference. Section 6 concludes the paper.

2. The nabla fractional sums and differences and Laplace transforms on $h\mathbb{Z}$

This section is devoted to setting some essential definitions and assertions that will be used throughout the remarkaining part of the paper.

Definition 2.1. [5] The following identities are valid.

- (i) Let m be a natural number, then the m rising factorial of t is written as

$$t^{\overline{m}} = \prod_{k=0}^{m-1} (t+k), \quad t^{\overline{0}} = 1. \tag{2.1}$$

- (ii) For any real number, the α rising function becomes

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad \text{such that } t \in \mathbb{C} \setminus \{\dots, -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0 \tag{2.2}$$

In addition, we have

$$\nabla(t^{\overline{\alpha}}) = \alpha t^{\overline{\alpha-1}}. \tag{2.3}$$

Hence $t^{\overline{\alpha}}$ is increasing on \mathbb{N}_0 .

The backward difference operator on $h\mathbb{Z}$ is given by $\nabla_h f(t) = \frac{f(t)-f(t-h)}{h}$ and the forward operator by $\Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$. For $h = 1$, we get the backward and forward difference operators $\nabla f(t) = f(t) - f(t - 1)$ and $\Delta f(t) = f(t + 1) - f(t)$, respectively. The forward jumping operator on the time scale $h\mathbb{Z}$ is $\sigma_h(t) = t + h$ and the backward jumping operator is $\rho_h(t) = t - h$. For $a, b \in \mathbb{R}$ and $h > 0$ we use the notation $\mathbb{N}_{a,h} = \{a, a + h, a + 2h, \dots\}$ and ${}_{b,h}\mathbb{N} = \{b, b - h, b - 2h, \dots\}$.

Definition 2.2. For arbitrary $t, \alpha \in \mathbb{R}$ and $h > 0$, the nabla h -factorial function is defined by

$$t_{\overline{h}}^{\alpha} = h^{\alpha} \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})}.$$

For $h = 1$, we write $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$.

A straightforward verification leads to

$$\nabla_h t_{\overline{h}}^{\alpha} = \alpha t_{\overline{h}}^{\alpha-1}. \tag{2.4}$$

Lemma 2.3. Let $s \in \mathbb{T} = \mathbb{N}_{a,h}$, then for all $t \in \mathbb{T}^k$. Then

$$\nabla_{t,h} \left\{ \frac{(t-s)_{\overline{h}}^{k+1}}{(k+1)!} \right\} = \frac{(t-s)_{\overline{h}}^k}{k!}. \tag{2.5}$$

The proof of the above statement follows by using the definitions and direct calculations. Note that $\nabla_{t,h} f(t,s) = \frac{f(t,s) - f(t-h,s)}{h}$.

Lemma 2.4. For the time scale $\mathbb{T} = \mathbb{N}_{a,h}$, one has the nabla Taylor polynomial

$$\widehat{H}_k(t,s) = \frac{(t-s)_{\overline{h}}^k}{k!}, \quad k \in \mathbb{N}_0 \tag{2.6}$$

The Q -operator $(Qf)(t) = f(a + b - t)$ was used in [7, 8] to connect left and right fractional sums and differences. In our manuscript, we also use the discrete Q -operator to relate left and right h -fractional sums and differences and hence confirm our definitions.

Definition 2.5. (Nabla Discrete Mittag-Leffler) ([7, 8, 9]) For $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $\alpha, \beta, z \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, the nabla discrete Mittag-Leffler functions is

$$E_{\alpha,\beta}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha+\beta-1}}}{\Gamma(\alpha k + \beta)}. \tag{2.7}$$

For $\beta = 1$, we have

$$E_{\overline{\alpha}}(\lambda, z) \triangleq E_{\overline{\alpha},1}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha}}}{\Gamma(\alpha k + 1)}, \quad |\lambda| < 1. \tag{2.8}$$

The following definition generalizes Definition 2.5.

Definition 2.6. (Nabla h -discrete Mittag-Leffler) For $\lambda \in \mathbb{R}$, such that $|\lambda h^{\alpha}| < 1$ and $\alpha, \beta, z \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, the nabla discrete Mittag-Leffler functions is

$${}_h E_{\alpha,\beta}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z_{\overline{h}}^{\overline{k\alpha+\beta-1}}}{\Gamma(\alpha k + \beta)}, \quad |\lambda h^{\alpha}| < 1. \tag{2.9}$$

For $\beta = 1$, we have

$${}_h E_{\overline{\alpha}}(\lambda, z) \triangleq {}_h E_{\overline{\alpha},1}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z_{\overline{h}}^{\overline{k\alpha}}}{\Gamma(\alpha k + 1)}, \quad |\lambda h^{\alpha}| < 1. \tag{2.10}$$

For more details regarding the Mittag–Leffler functions, we refer to [1, 2, 28].

Definition 2.7. (Nabla h –fractional sums) Let $\rho(t) = t - h$, $h > 0$ be backward jump operator. Then, for a function $f : \mathbb{N}_{a,h} = \{a, a + h, a + 2h, \dots\} \rightarrow \mathbb{R}$, the nabla left h –fractional sum of order $\alpha > 0$ is given by

$$\begin{aligned} ({}_a\nabla_h^{-\alpha}f)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t - \rho_h(s))_h^{\overline{\alpha-1}} f(s) \nabla_h \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} (t - \rho_h(kh))_h^{\overline{\alpha-1}} f(kh)h, \quad t \in \mathbb{N}_{a+h,h}. \end{aligned}$$

The nabla right h –fractional sum of order $\alpha > 0$ (ending at b) for $f : {}_{b,h}\mathbb{N} = \{b, b - h, b - 2h, \dots\} \rightarrow \mathbb{R}$ is written as

$$({}_b\nabla_h^{-\alpha}f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s - \rho_h(t))_h^{\overline{\alpha-1}} f(s) \Delta_h s = \frac{1}{\Gamma(\alpha)} \sum_{k=t/h}^{b/h-1} (kh - \rho_h(t))_h^{\overline{\alpha-1}} f(kh)h.$$

Definition 2.8. (Nabla h –RL fractional differences) The nabla left h –fractional difference of order $\alpha > 0$ (starting from a) has the form

$$\begin{aligned} ({}_a\nabla_h^\alpha f)(t) &= (\nabla_h^n {}_a\nabla_h^{-(n-\alpha)} f)(t) \\ &= \frac{\nabla_h^n}{\Gamma(n-\alpha)} \sum_{k=a/h+1}^t (t - \rho_h(kh))_h^{\overline{n-\alpha-1}} f(kh)h, \quad t \in \mathbb{N}_{a+h,h} \end{aligned}$$

and the nabla right h –fractional difference of order $\alpha > 0$ (ending at b) is defined as

$$\begin{aligned} ({}_b\nabla_h^\alpha f)(t) &= ((-1)^n \Delta_h^n {}_b\nabla_h^{-(n-\alpha)} f)(t) \\ &= \frac{(-1)^n \Delta_h^n}{\Gamma(n-\alpha)} \sum_{k=t}^{b/h-1} (kh - \rho_h(t))_h^{\overline{n-\alpha-1}} f(kh)h, \quad t \in {}_{b-h,h}\mathbb{N}. \end{aligned}$$

Definition 2.9. (The h –Caputo fractional differences) Let $\alpha > 0$, $n = [\alpha] + 1$, $h > 0$, $a < b \in \mathbb{R}$, $a_h(\alpha) = a + (n - 1)h$, $b_h(\alpha) = b - (n - 1)h$. Assume f is defined on $\mathbb{N}_{a,h} = \{a, a + h, a + 2h, \dots\}$ and on ${}_{b,h}\mathbb{N} = \{b, b - h, b - 2h, \dots\}$. We usually have $b = a + kh$ for some $k \in \mathbb{N}$. If $0 < \alpha < 1$ then $a_h(\alpha) = a$ and $b_h(\alpha) = b$. Then,

The left h –Caputo fractional difference of order α starting at $a_h(\alpha)$ is defined by

$$({}_a^C\nabla_h^\alpha f)(t) = ({}_{a_h(\alpha)}\nabla_h^{-(n-\alpha)} \nabla_h^n f)(t), \quad t \in \mathbb{N}_{a+n h,h}. \tag{2.11}$$

The right h –Caputo fractional difference of order α ending at $b_h(\alpha)$ is defined by

$$({}_b^C\nabla_h^\alpha f)(t) = (\nabla_h^{-(n-\alpha)} {}_b\nabla_h^n f)(t), \quad t \in {}_{b-n h,h}\mathbb{N}, \tag{2.12}$$

where ${}_\ominus\nabla_h^n = (-1)^n \Delta_h^n$.

For $h = 1$, we obtain the definitions given in [7, 8].

Lemma 2.10. Let $\alpha > 0, \mu > 0, h > 0$. Then,

$${}_a \nabla_h^{-\alpha} (t - a)_{\bar{h}}^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + \alpha)} (t - a)_{\bar{h}}^{\alpha + \mu}, \tag{2.13}$$

$${}_h \nabla_b^{-\alpha} (b - t)_{\bar{h}}^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + \alpha)} (b - t)_{\bar{h}}^{\alpha + \mu}, \tag{2.14}$$

$${}_a \nabla_h^{\alpha} (t - a)_{\bar{h}}^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \alpha)} (t - a)_{\bar{h}}^{\mu - \alpha}, \tag{2.15}$$

and

$${}_h \nabla_b^{\alpha} (b - t)_{\bar{h}}^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \alpha)} (b - t)_{\bar{h}}^{\mu - \alpha}. \tag{2.16}$$

The proof is a modification to the case $h = 1$ in [7, 8] by making use of relation (2.4).

Following the time scale calculus, we have the following definition for the discrete Laplace transform on $\mathbb{N}_{a,h}$.

Definition 2.11. Assume that f is defined on $\mathbb{N}_{a,h}$. Then, the discrete Laplace transform of f is defined by

$$\begin{aligned} \mathcal{N}_{a,h}\{f(t)\}(s) &= \int_a^{\infty} {}_h \widehat{e}_{\ominus s}^{\rho}(t, a) f(t) \nabla_h t \\ &= \int_a^{\infty} \frac{{}_h \widehat{e}_{\ominus s}(t, a)}{1 - hs} f(t) \nabla_h t = \int_a^{\infty} (1 - hs)^{\frac{t-a-h}{h}} f(t) \nabla_h t \end{aligned} \tag{2.17}$$

$$= h \sum_{t=a/h+1}^{\infty} (1 - hs)^{t-a/h-1} f(ht). \tag{2.18}$$

In case $a = 0$ and we write

$$\mathcal{N}_{0,h}\{f(t)\}(s) = \mathcal{N}_h\{f(t)\}(s) = h \sum_{t=1}^{\infty} (1 - hs)^{t-1} f(ht).$$

Definition 2.12. [9] Let $s \in \mathbb{R}, 0 < \alpha < 1$ and $f, g : \mathbb{N}_{a,h} \rightarrow \mathbb{R}$ be a functions. The nabla h -discrete convolution of f with g is defined by

$$(f * g)(t) = \int_a^t g(t - \rho_h(s) + a) f(s) \nabla_h s = h \sum_{k=a/h+1}^{t/h} g(t - \rho_h(kh) + a) f(kh). \tag{2.19}$$

Theorem 2.13. [23] (The h -convolution Theorem) For any $\alpha \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}, s \in \mathbb{R}$ and f, g defined on $\mathbb{N}_{a,h}$, we have

$$\mathcal{N}_{a,h}\{(f * g)(t)\}(s) = \mathcal{N}_{a,h}\{f(t)\}(s) \mathcal{N}_{a,h}\{g(t)\}(s). \tag{2.20}$$

The following Lemma is a generalization of Lemma 2 in [9] to $h\mathbb{Z}$. However, the reader should note that the nabla discrete Laplace which is used here is slightly different.

Lemma 2.14. [23] Let f be defined on $\mathbb{N}_{a,h}$. Then,

$$\mathcal{N}_{a,h}\{\nabla_h f(t)\}(s) = s\mathcal{N}_{a,h}\{f(t)\}(s) - f(a) \tag{2.21}$$

Lemma 2.15. (For $a=0$ see [23]) For any $\alpha \in \mathbb{R} \{ \dots, -2, -1, 0 \}$ and $|1 - hs| < 1$, we have

$$\mathcal{N}_{a,h}\{(t - a)_h^{\overline{\alpha-1}}\}(s) = \frac{\Gamma(\alpha)}{s^\alpha}.$$

Proof. Using Definition 2.11 and identities for hypergeometric functions in [30], we have

$$\begin{aligned} \mathcal{N}_{a,h}\{(t - a)_h^{\overline{\alpha-1}}\}(s) &= h \sum_{k=a/h+1}^{\infty} (1 - hs)^{k-a/h-1} (hk - a)_h^{\overline{\alpha-1}} \\ &= h^\alpha \sum_{k=a/h+1}^{\infty} \frac{(1 - hs)^{k-a/h-1} \Gamma(k - a/h + \alpha - 1)}{\Gamma(k - a/h)} \\ &= h^\alpha \sum_{k=0}^{\infty} \frac{(1 - hs)^k \Gamma(k + \alpha)}{\Gamma(k + 1)} \\ &= h^\alpha \Gamma(\alpha) {}_2F_1(1, \alpha; 1; 1 - hs) \\ &= \frac{h^\alpha}{\Gamma(\alpha - 1)} \int_0^1 \frac{u^{\alpha-1} (1 - u)^{(1-\alpha)-1}}{hs \cdot u + (1 - u)} du \\ &= \frac{h^\alpha}{\Gamma(\alpha - 1)} \frac{\Gamma(\alpha) \Gamma(\alpha - 1)}{(hs)^\alpha} = \frac{\Gamma(\alpha)}{s^\alpha}. \end{aligned}$$

□

Lemma 2.16. [23] For $\alpha, \beta, \lambda \in \mathbb{C}$ ($\text{Re}(\beta) > 0$), and $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, $|\lambda s^{-\alpha}| < 1$, we have

$$\mathcal{N}_h\{ {}_hE_{\alpha,\beta}(\lambda, t) \}(s) = s^{-\beta} [1 - \lambda s^{-\alpha}]^{-1}.$$

The time scale algebraic operations on $h\mathbb{Z}$ [29] can be used to generate some of the present results in this paper. Her we mention some of the useful operations as follows:

$$\begin{aligned} z \oplus w &= z + w - zwh \\ z \ominus w &= z \oplus (\ominus w), \end{aligned}$$

where, $\ominus w = \frac{-w}{1-wh}$. Using the above operations, we can assure that $z \ominus z = 0$ as follows:

$$z \ominus z = z + \ominus z - z(\ominus z)h = z \left[1 - \frac{1}{1-zh} + \frac{zh}{1-zh} \right] = 0. \tag{2.22}$$

Lemma 2.17. Assume u is defined on $\mathbb{N}_{a,h}$ and $\mathcal{N}_{a,h}\{u(t)\}(s) = U_a(s)$. Then,

$$\mathcal{N}_{a,h}\{ {}_h\hat{e}_\lambda(t, a)u(t) \}(s) = \hat{\lambda}_h U_a \left(\frac{\hat{\lambda}_h (hs - 1)}{h} + \frac{1}{h} \right) = \hat{\lambda}_h U_a(s \ominus \lambda), \tag{2.23}$$

$$\mathcal{N}_{\alpha, h}\{ {}_h\widehat{e}_{\ominus\lambda}(t, \alpha)u(t)\}(s) = \widehat{\ominus}\widehat{\lambda}_h U_\alpha(s \oplus \lambda) = (1 - \lambda h)U_\alpha(s \oplus \lambda), \tag{2.24}$$

where $\widehat{\lambda}_h = \frac{1}{1-\lambda h}$, $h > 0$, $\lambda \neq \frac{1}{h}$.

Proof. By Definition 2.11 and using time scale algebraic operations presented in [29], we get

$$\begin{aligned} \mathcal{N}_{\alpha, h}\{ {}_h\widehat{e}_\lambda(t, \alpha)u(t)\}(s) &= h \sum_{t=\alpha/h+1}^{\infty} \left(\frac{1}{1-\lambda h}\right)^{t-\alpha/h} (1-hs)^{t-\alpha/h-1} u(ht) \\ &= h\widehat{\lambda}_h \sum_{t=\alpha/h+1}^{\infty} [\widehat{\lambda}_h(1-hs)]^{t-\alpha/h-1} u(ht) \\ &= h\widehat{\lambda}_h \sum_{t=\alpha/h+1}^{\infty} [1 + \widehat{\lambda}_h - h\widehat{\lambda}_h s - 1]^{t-\alpha/h-1} u(ht) \\ &= h\widehat{\lambda}_h \sum_{t=\alpha/h+1}^{\infty} \left[1 - h\left(\widehat{\lambda}_h s - \frac{\widehat{\lambda}_h}{h} + \frac{1}{h}\right)\right]^{t-\alpha/h-1} u(ht) \\ &= h\widehat{\lambda}_h \sum_{t=\alpha/h+1}^{\infty} \left[1 - h\left(\frac{\widehat{\lambda}_h(hs-1)}{h} + \frac{1}{h}\right)\right]^{t-\alpha/h-1} u(ht) \\ &= \widehat{\lambda}_h U_\alpha\left(\frac{\widehat{\lambda}_h(hs-1)}{h} + \frac{1}{h}\right) = \widehat{\lambda}_h U_\alpha(s \ominus \lambda) \end{aligned}$$

where

$$\begin{aligned} s \ominus \lambda &= s - \frac{\lambda}{1-\lambda h} + \frac{sh\lambda}{1-\lambda h} = s(1 + \lambda h\widehat{\lambda}_h) - \lambda\widehat{\lambda}_h \\ &= \widehat{\lambda}_h s - \frac{\widehat{\lambda}_h - 1}{h} = \frac{\widehat{\lambda}_h(hs-1)}{h} + \frac{1}{h} \end{aligned}$$

verifying (2.24) is straightforward using the same steps when replacing ${}_h\widehat{e}_\lambda(t, \alpha)$ in (2.23) by ${}_h\widehat{e}_{\ominus\lambda}(t, \alpha)$. □

Remark 2.18. In Lemma 2.17 and throughout the article, we assume the domain of the discrete exponential function is the time scale of the form $\mathbb{T} := \{(k + \alpha)h : k = 0, 1, 2, \dots\} \cup \{(k + (n - \alpha))h : k = 0, 1, 2, \dots\}$ which are equivalent to $h\mathbb{Z}$.

3. The proportional differences and sums with memory

In [12], Anderson et al. introduced the modified conformable derivative by

Definition 3.1. (*Modified conformable derivatives*) Let $\rho \in [0, 1]$ and the functions $\kappa_0, \kappa_1 : [0, 1] \times \mathbb{R} \rightarrow [0, \infty)$ be continuous such that for all $t \in \mathbb{R}$, we have

$$\lim_{\rho \rightarrow 0^+} \kappa_1(\rho, t) = 1, \quad \lim_{\rho \rightarrow 0^+} \kappa_0(\rho, t) = 0, \quad \lim_{\rho \rightarrow 1^-} \kappa_1(\rho, t) = 0, \quad \lim_{\rho \rightarrow 1^-} \kappa_0(\rho, t) = 1,$$

where $\kappa_1(\rho, t) \neq 0$, $\rho \in [0, 1)$ and $\kappa_0(\rho, t) \neq 0$, $\rho \in (0, 1]$. Then, the modified conformable differential operator of order ρ is defined by

$$D^\rho f(t) = \kappa_1(\rho, t)f(t) + \kappa_0(\rho, t)f'(t).$$

The derivative given in Definition 3.1 is called a proportional derivative. For more details about the control theory meanings of the proportional derivatives and its component functions κ_0 and κ_1 we refer to [12, 13].

Of special interest, we shall consider in this article the case when $\kappa_1(\rho, t) = 1 - \rho$ and $\kappa_0(\rho, t) = \rho$. That is

$$D^\rho f(t) = (1 - \rho)f(t) + \rho f'(t). \tag{3.1}$$

The reader should note that $\lim_{\rho \rightarrow 0^+} D^\rho f(t) = f(t)$ and $\lim_{\rho \rightarrow 1^-} D^\rho f(t) = f'(t)$ which is an advantage over the conformable derivative since the conformable derivative does not tend to the original function as ρ tends to 0.

In view of (3.1), the h -discrete proportional derivative (proportional difference) of order $0 < \rho \leq 1$ for a function f defined on $\mathbb{N}_{a,h} = \{a, a + h, a + 2h, \dots\}$ is given by [23]

$$(\nabla_h^\rho f)(t) = (1 - \rho)f(t) + \rho(\nabla_h f)(t), \quad t \in \mathbb{N}_{a+h,h}, \quad 1 \geq h > 0, \tag{3.2}$$

where $(\nabla_h f)(t) = f(t) - f(t - h)$ and the regressivity condition insists that $1 - h \frac{\rho-1}{\rho} \neq 0$ or $\rho \neq \frac{h}{1-h}$.

The proportional sum associated to ∇^ρ by

$${}_a \nabla_h^{-1,\rho} f(t) = \frac{1}{\rho} \int_a^t \widehat{e}_p(t-s+h, 0) f(s) \nabla_h s, \quad p = \frac{\rho-1}{\rho},$$

where ${}_a \nabla_h^{0,\rho} f(t) = f(t)$,

Proposition 3.2. [23] For f defined on $\mathbb{N}_{a+h,h}$ and $\rho \in (0, 1]$, $\rho \neq \frac{h}{1-h}$, we have

$${}_a \nabla_h^{-1,\rho} \nabla_h^\rho f(t) = f(t) - \widehat{e}_p(t-a, 0) f(a) = f(t) - \left(\frac{\rho}{\rho - (\rho-1)h} \right)^{\frac{t-a}{h}} f(a).$$

The authors in [23] iterated the local fractional proportional sum to generate the following nonlocal fractional proportional sums and differences.

Definition 3.3. For $\rho > 0$ and $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, we define the left (proportional) fractional sum of f by

$$({}_a \nabla_h^{-\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t \widehat{e}_p(t-\tau+\alpha h, 0) (t-\rho_h(\tau))_h^{\alpha-1} f(\tau) \nabla_h \tau. \tag{3.3}$$

As in the classical fractional calculus, the right fractional sum ending at b can be then defined by

$$({}_h \nabla_b^{-\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_t^b \widehat{e}_p(\tau-t+\alpha h, 0) (\tau-\rho_h(t))_h^{\alpha-1} f(\tau) \Delta_h \tau, \tag{3.4}$$

where $p = \frac{\rho-1}{\rho}$.

Remark 3.4. [23] Using the discrete convolutions, we can express the left proportional fractional sum as

$$({}_a \nabla_h^{-\alpha, \rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \left(\frac{\rho}{\rho - (\rho - 1)h} \right)^{(\alpha-1)} \left(\widehat{e}_\rho(t, a)(t - a)_h^{\overline{\alpha-1}} \right) * f(t). \quad (3.5)$$

For the special case $h = 1$, we have

$$({}_a \nabla^{-\alpha, \rho} f)(t) = \frac{1}{\rho \Gamma(\alpha)} \left(\rho^{t-a} (t - a)^{\overline{\alpha-1}} \right) * f(t). \quad (3.6)$$

Remark 3.5. [23] To deal with the right fractional proportional case we shall use the notation

$$({}_\ominus \Delta^\rho f)(t) = (1 - \rho)f(t) - \rho \Delta_h f(t).$$

We shall also write $(\ominus \Delta_h^{n, \rho} g)(t) = \underbrace{(\ominus \Delta_h^\rho \ominus \Delta_h^\rho \dots \ominus \Delta_h^\rho g)}_{n \text{ times}}(t)$, where $(\Delta_h f)(t) = \frac{f(t+h) - f(t)}{h}$.

Definition 3.6. [23] For $\rho > 0$ and $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, we define the left (proportional) fractional difference of f by

$$\begin{aligned} ({}_a \nabla_h^{\alpha, \rho} f)(t) &= \nabla_h^{n, \rho} {}_a \nabla_h^{-(n-\alpha), \rho} f(t) \\ &= \frac{\nabla_{h,t}^{n, \rho}}{\rho^{n-\alpha} \Gamma(n-\alpha)} \int_a^t \widehat{e}_\rho(t - \tau + h(n-\alpha), 0) \\ &\times (t - \rho_h(\tau))_h^{\overline{n-\alpha-1}} f(\tau) \nabla_h \tau. \end{aligned} \quad (3.7)$$

The right (proportional) fractional difference ending at b is defined by

$$\begin{aligned} ({}_h \nabla_b^{\alpha, \rho} f)(t) &= \ominus \Delta_h^{n, \rho} {}_h \nabla_b^{-(n-\alpha), \rho} f(t) \\ &= \frac{\ominus \Delta_{h,t}^{n, \rho}}{\rho^{n-\alpha} \Gamma(n-\alpha)} \int_t^b \widehat{e}_\rho(\tau - t + h(n-\alpha), 0) \\ &\times (\tau - \rho_h(t))_h^{\overline{n-\alpha-1}} f(\tau) \Delta_h \tau, \end{aligned} \quad (3.8)$$

where $n = [\text{Re}(\alpha)] + 1$.

Theorem 3.7. [23] (The semigroup property for the fractional proportional sums) If $\rho > 0$, $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$. Then, for f defined for $t \in \mathbb{N}_{a,h}$, we have

$${}_a \nabla_h^{-\alpha, \rho} ({}_a \nabla_h^{-\beta, \rho} f)(t) = {}_a \nabla_h^{-\beta, \rho} ({}_a \nabla_h^{-\alpha, \rho} f)(t) = ({}_a \nabla_h^{-(\alpha+\beta), \rho} f)(t). \quad (3.9)$$

Theorem 3.8. [23] Let $0 \leq m < [\text{Re}(\alpha)] + 1$ and f be defined on $\mathbb{N}_{a,h}$. Then

$$\nabla_h^{m, \rho} ({}_a \nabla_h^{-\alpha, \rho} f)(t) = ({}_a \nabla_h^{-(\alpha-m), \rho} f)(t). \quad (3.10)$$

Corollary 3.9. [23] Let $0 < \text{Re}(\beta) < \text{Re}(\alpha)$ and $m - 1 < \text{Re}(\beta) \leq m$. Then,

$${}_a \nabla_h^{\beta, \rho} {}_a \nabla_h^{-\alpha, \rho} f(t) = {}_a \nabla_h^{-(\alpha-\beta), \rho} f(t).$$

Theorem 3.10. [23] For f is defined on $\mathbb{N}_{a,h}$ and $\text{Re}(\alpha) > 0$, $\rho > 0$, $n = [\text{Re}(\alpha)] + 1$, we have ${}_a\nabla_h^{\alpha,\rho} {}_a\nabla_h^{-\alpha,\rho} f(t) = f(t)$.

Proof. From Definition 3.3 and by the help of Theorem 3.7, we have

$${}_a\nabla_h^{\alpha,\rho} {}_a\nabla_h^{-\alpha,\rho} f(t) = \nabla_h^{n,\rho} {}_a\nabla_h^{-(n-\alpha),\rho} {}_a\nabla_h^{-\alpha,\rho} f(t) = \nabla_h^{n,\rho} {}_a\nabla_h^{-n,\rho} f(t) = f(t).$$

□

Lemma 3.11. For $0 < \rho \leq 1$, $\alpha > 0$, and ξ defined on $\mathcal{N}_{a,h}$. Assume ξ is of discrete exponential order $\widehat{e}_c(t,0)$. Then, we have

$$\mathcal{N}_{a,h}\{ {}_a\nabla_h^{-\alpha,\rho} \xi(t) \}(s) = \frac{\widehat{e}_p(\alpha h, 0)}{\rho^\alpha} \frac{\xi_a(s)}{(s \ominus p)^\alpha} = (1 - \rho + \rho s)^{-\alpha} \xi_a(s), \quad s > c. \quad (3.11)$$

where $\mathcal{N}_{a,h}\{\xi(t)\}(s) = \xi_a(s)$ and $p = \frac{\rho-1}{\rho}$.

Proof. First, by Remark 3.4, we observe that

$${}_a\nabla_h^{-\alpha,\rho} \xi(t) = \left(\frac{\rho}{\rho - (\rho - 1)h} \right)^{\alpha-1} \frac{\widehat{e}_p(t, a)(t - a)_{h^{-1}}^{\overline{\alpha-1}} * \xi(t)}{\rho^\alpha \Gamma(\alpha)}. \quad (3.12)$$

Apply the discrete Laplace transform, and make use of Theorem 2.13 and Lemma 3.11, to see that

$$\mathcal{N}_{a,h}\{ {}_a\nabla_h^{-\alpha,\rho} \xi(t) \}(s) = \frac{\widehat{e}_p(\alpha h, 0)}{\rho^\alpha} \frac{\xi_a(s)}{(s \ominus p)^\alpha} = (\rho - (\rho - 1)h)^{-\alpha} \frac{\xi_a(s)}{(s \ominus p)^\alpha}. \quad (3.13)$$

Then, the result follows by using that

$$s \ominus p = \frac{(1 - \rho + \rho s)}{\rho - (\rho - 1)h}, \quad \ominus p = \frac{1 - \rho}{\rho - (\rho - 1)h}, \quad p = \frac{\rho - 1}{\rho}. \quad (3.14)$$

□

If in (3.11), we set $\rho = 1$ then we recover the identity

$$\mathcal{N}_{a,h}\{ {}_a\nabla_h^{-\alpha} \xi(t) \}(s) = s^{-\alpha} \xi_a(s). \quad (3.15)$$

Lemma 3.12. For $\alpha > 0$, $1 \geq \rho > 0$ we have

$$({}_a\nabla_h^{-\alpha,\rho} \nabla_h^\rho \xi)(t) = (\nabla_h^\rho {}_a\nabla_h^{-\alpha,\rho} \xi)(t) - \frac{(\rho - (\rho - 1)h)^{1-\alpha} (t - a)_{h^{-1}}^{\overline{\alpha-1}}}{\Gamma(\alpha)} \widehat{e}_p(t, a) \xi(a) \quad (3.16)$$

Proof. By Lemma 3.11, $\mathcal{N}_{a,h}\{\nabla_h^\rho \xi(t)\}(s) = [(1 - \rho) + \rho s] \xi_a(s) - \rho \xi(a)$ and that $\frac{1}{1 - \rho h} = \frac{\rho}{\rho - (\rho - 1)h}$, we see that

$$\mathcal{N}_{a,h}\{ {}_a\nabla_h^{-\alpha,\rho} \nabla_h^\rho \xi(t) \}(s) = \frac{(\rho - (\rho - 1)h)^{-\alpha}}{(s \ominus p)^\alpha} ([(1 - \rho) + \rho s] \xi_a(s) - \rho \xi(a)), \quad (3.17)$$

and

$$\mathcal{N}_{a,h}\{\nabla_h^\rho \nabla_h^{-\alpha,\rho} \xi(t)\}(s) = \frac{(\rho - (\rho - 1)h)^{-\alpha}}{(s \ominus \rho)^\alpha} [(1 - \rho) + \rho s] \xi_a(s). \tag{3.18}$$

Hence,

$$\mathcal{N}_{a,h}\{\nabla_h^{-\alpha,\rho} \nabla_h^\rho \xi(t)\}(s) = \mathcal{N}_{a,h}\{\nabla_h^\rho \nabla_h^{-\alpha,\rho} \xi(t)\}(s) - \frac{\rho}{\rho - (\rho - 1)h} \frac{(\rho - (\rho - 1)h)^{1-\alpha} \xi(a)}{(s \ominus \rho)^\alpha}. \tag{3.19}$$

Then, applying the inverse discrete Laplace and by making use of (2.23) in Lemma 2.17, we reach (3.16), and hence the proof is completed. \square

Note that Lemma 3.2 in [6] is the ($h = 1, \rho = 1$)–version of (3.16) above.

Lemma 3.13. For $\alpha > 0, 1 \geq \rho > 0$ and m positive integer we have

$$\begin{aligned} & (\nabla_{a+(m-1)h}^{-\alpha,\rho} \nabla_{a+(m-1)h}^{m,\rho} f)(t) = (\nabla_h^{m,\rho} \nabla_{a+(m-1)h}^{-\alpha,\rho} f)(t) - \\ & \sum_{k=0}^{m-1} \frac{(t - (a + (m - 1)h))_h^{\overline{\alpha - m + k}} \eta(h, \rho, \alpha) \widehat{e}_\rho(t - (m - 1)h, a)}{\Gamma(\alpha + k - m + 1) \rho^{\alpha - m + k}} (\nabla_h^{k,\rho} f)(a + (m - 1)h), \end{aligned} \tag{3.20}$$

where $\eta(h, \rho, \alpha) = \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{h(\alpha - 1)}$. In particular, if $m = 1$, then

$$(\nabla_h^{-\alpha,\rho} \nabla_h^\rho f)(t) = (\nabla_h^\rho \nabla_h^{-\alpha,\rho} f)(t) - \frac{(t - a)_h^{\overline{\alpha - 1}} \widehat{e}_\rho(t, a)}{\rho^{\alpha - 1} \Gamma(\alpha)} \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{h(\alpha - 1)} f(a). \tag{3.21}$$

Proof. We only prove (3.21) while (3.20) follows when applying (3.21) inductively through making use of (2.4). The proof of (3.21) is based on the Laplace transform. Indeed, by making use of Theorem 4.1 in the next section and the identity $\mathcal{N}_{a,h} \nabla_h^\rho f(t)(s) = (1 - \rho + \rho s)F_a(s) - \rho f(a)$, where $F_a(s) = \mathcal{N}_{a,h} f(t)(s)$, we have

$$\begin{aligned} \mathcal{N}_{a,h}\{\nabla_h^{-\alpha,\rho} \nabla_h^\rho f(t)\}(s) &= \frac{\mathcal{N}_{a,h}\{\nabla_h^\rho f(t)\}(s)}{(\rho s + 1 - \rho)^\alpha} \\ &= \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{h(\alpha - 1)} \frac{(1 - \rho + \rho s)F_a(s) - \rho f(a)}{(\rho s + 1 - \rho)^\alpha}, \\ \mathcal{N}_{a,h}\{\nabla_h^\rho \nabla_h^{-\alpha,\rho} f(t)\}(s) &= \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{h(\alpha - 1)} (1 - \rho + \rho s) \frac{F_a(s)}{(1 - \rho + \rho s)^\alpha}, \end{aligned}$$

and

$$\mathcal{N}_{a,h}\left\{\frac{(t - a)_h^{\overline{\alpha - 1}} \widehat{e}_\rho(t - a, 0)}{\rho^{\alpha - 1} \Gamma(\alpha)} f(a)\right\}(s) = \frac{f(a)\rho}{(\rho s - \rho + 1)}.$$

\square

Note that Proposition 3.2 is the ($\alpha = 1, \rho = 1$)–version of Lemma 3.13. Also, Lemma 3.2 in [6] is the ($h = 1, \rho = 1$)–version of 3.21 above. More generally, Theorem 3.6 in [6] is the ($\alpha = 1, \rho = 1$)– version of (3.16).

Remark 3.14. We have the following important observation for the discrete proportional operators. For a function f defined on $\mathbb{N}_{a-(n-1)h}$, we have for $\alpha > 0$

$${}_a\nabla_h^{-\alpha,\rho} {}_a\nabla_h^{\alpha,\rho} f(t) = f(t), \text{ if } \alpha \notin \mathbb{N}, \tag{3.22}$$

and if $\alpha = n \in \mathbb{N}$,

$${}_a\nabla_h^{-\alpha,\rho} {}_a\nabla_h^{\alpha,\rho} f(t) = f(t) - \widehat{e}_p(t - (n-1)h, a) \sum_{k=0}^{n-1} \frac{\eta(h, \rho, n)(t-a)_h^{\overline{k}}}{\rho^k} (\nabla_h^k f)(a). \tag{3.23}$$

Alternatively, one can say that when f is only defined on \mathbb{N}_a we start our fractional proportional operators from $a_h(\alpha)$.

4. The h -discrete Laplace transforms and Riemann fractional proportional sums and differences

Theorem 4.1. [23] Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\rho > 0$, $n = [\text{Re}(\alpha)] + 1$. Assume f is of discrete exponential order $\widehat{e}_c(t, 0)$. Then

$$\mathcal{N}_{a,h}\{ {}_a\nabla_h^{-\alpha,\rho} f(t)\}(s) = \left(\frac{\rho}{\rho - (\rho-)h} \right)^{h(\alpha-1)} \frac{\mathcal{N}_{a,h}\{f(t)\}(s)}{(\rho s + 1 - \rho)^\alpha}, s > c \tag{4.1}$$

Proof. From Theorem 2.13, Remark 3.4 and Lemma 2.15 we have

$$\begin{aligned} \mathcal{N}_{a,h}\{ {}_a\nabla_h^{-\alpha,\rho} f(t)\}(s) &= \frac{1}{\rho\Gamma(\alpha)} \left(\frac{\rho}{\rho - (\rho-)h} \right)^{h(\alpha-1)} \mathcal{N}_{a,h}\{\widehat{e}_p(t, 0)t^{\overline{\alpha-1}} * f(t)\}(s) \\ &= \frac{1}{\rho\Gamma(\alpha)} \left(\frac{\rho}{\rho - (\rho-)h} \right)^{h(\alpha-1)} \frac{\Gamma(\alpha)}{(s - \frac{\rho-1}{\rho})^\alpha} \mathcal{N}_{a,h}\{f(t)\}(s) \\ &= \left(\frac{\rho}{\rho - (\rho-)h} \right)^{h(\alpha-1)} \frac{\mathcal{N}_{a,h}\{f(t)\}(s)}{(\rho s + 1 - \rho)^\alpha}. \end{aligned}$$

□

Theorem 4.2. [23] Assume f is defined on $\mathbb{N}_{a,h}$ such that $\nabla^i f(t)$, $i = 1, 2, \dots, n-1$ are of exponential order on each discrete subinterval $\{a, a+h, \dots, a+nh\}$. Then,

$$\mathcal{N}_{a,h}\{\nabla_h^{n,\rho} f(t)\}(s) = (\rho s + 1 - \rho)^n \mathcal{N}_{a,h}\{f(t)\}(s) - \rho \sum_{k=0}^{n-1} (\rho s + 1 - \rho)^{n-1-k} (\nabla_h^{k,\rho} f)(a). \tag{4.2}$$

Proof. By using Lemma 2.14, we have

$$\begin{aligned} \mathcal{N}_{a,h}\{\nabla_h^{n,\rho} f(t)\}(s) &= \mathcal{N}_{a,h}\{(1-\rho)f(t) + \rho\nabla_h f(t)\}(s) \\ &= (\rho s + 1 - \rho)\mathcal{N}_{a,h}\{f(t)\}(s) - \rho f(a). \end{aligned} \tag{4.3}$$

The statement of the theorem follows by applying (4.3) inductively. □

Theorem 4.3. For any $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\rho > 0$, $n = [\text{Re}(\alpha)] + 1$, we have

$$\mathcal{N}_{a,h}\{ {}_a\nabla_h^{\alpha,\rho} f(t) \}(s) = (1 - \rho + \rho s)^\alpha F_a(s), \tag{4.4}$$

where $F_a(s) = \mathcal{N}_{a,h}\{f(t)\}(s)$.

Proof. By applying Theorem 4.2 and Lemma 3.11, we have

$$\begin{aligned} \mathcal{N}_{a,h}\{ {}_a\nabla_h^{\alpha,\rho} f(t) \}(s) &= \mathcal{N}_{a,h}\{ {}_a\nabla_h^{n,\rho} {}_a\nabla_h^{-(n-\alpha),\rho} f(t) \}(s) \\ &= (\rho s + 1 - \rho)^n \mathcal{N}_{a,h}\{ {}_a\nabla_h^{-(n-\alpha),\rho} f(t) \}(s) \\ &\quad - \rho \sum_{k=0}^{n-1} (\rho s + 1 - \rho)^{n-1-k} ({}_a\nabla_h^{-(n-\alpha-k),\rho} f(t))(a) \\ &= (\rho s + 1 - \rho)^n (\rho s + 1 - \rho)^{\alpha-n} F_a(s) \\ &= (1 - \rho + \rho s)^\alpha F_a(s), \end{aligned}$$

where we have used that $({}_a\nabla_h^{-(n-\alpha-k),\rho} f(t))(a) = 0$ for $k = 0, 1, \dots, n - 1$. □

Basing on Remark 3.14, we shall generate the following initial value problem in the sense of Riemann:

$${}_{a-h}\nabla_h^{\alpha,\rho} y(t) = f(t, y(t)) \quad \text{for } t = a + h, a + 2h, \dots, \tag{4.5}$$

$${}_{a-h}\nabla_h^{-(1-\alpha),\rho} y(t)|_{t=a} = \frac{h^{1-\alpha}}{\rho^{1-\alpha}} \left(\frac{\rho}{\rho - (\rho - 1)h} \right)^{1-\alpha} y(a) = c, \tag{4.6}$$

where $0 < \alpha < 1$ and a is any real number. Applying the operator ${}_a\nabla_h^{-\alpha,\rho}$ to each side of the equation (4.5) we obtain

$${}_a\nabla_h^{-\alpha,\rho} {}_{a-h}\nabla_h^{\alpha,\rho} y(t) = {}_a\nabla_h^{-\alpha,\rho} f(t, y(t)).$$

Then using the definition of the fractional difference and sum operators, we obtain

$$\begin{aligned} {}_a\nabla_h^{-\alpha,\rho} \{ \nabla_h^\rho {}_{a-h}\nabla_h^{-(1-\alpha),\rho} y(t) \} &= {}_a\nabla_h^{-\alpha,\rho} f(t, y(t)), \\ {}_a\nabla_h^{-\alpha,\rho} \{ \nabla_h^\rho {}_{a-h}\nabla_h^{-(1-\alpha),\rho} y(t) + \nabla_h^\rho \{ \frac{h(t-a+h)_h^{-\alpha}}{\Gamma(1-\alpha)\rho^{1-\alpha}} \widehat{e}_p(t-a+(1-\alpha)h, 0) y(a) \} \} \\ &= {}_a\nabla_h^{-\alpha,\rho} f(t, y(t)), \end{aligned}$$

or

$$y(t) + {}_a\nabla_h^{-\alpha,\rho} \nabla_h^\rho \{ \frac{h(t-a+h)_h^{-\alpha}}{\Gamma(1-\alpha)\rho^{1-\alpha}} \widehat{e}_p(t-a+(1-\alpha)h, 0) y(a) \} = \nabla_a^{-\alpha,\rho} f(t, y(t)). \tag{4.7}$$

It follows from Lemma 3.13 (3.21) that

$$\begin{aligned} & {}_a\nabla_h^{-\alpha,\rho} \nabla_h^\rho \left(\frac{h(t-a+h)_h^{-\alpha}}{\Gamma(1-\alpha)\rho^{1-\alpha}} \widehat{e}_p(t-a+(1-\alpha)h,0)y(a) \right) = \\ & \nabla_h^\rho {}_a\nabla_h^{-\alpha,\rho} \left(\frac{h(t-a+h)_h^{-\alpha}}{\Gamma(1-\alpha)\rho^{1-\alpha}} \widehat{e}_p(t-a+(1-\alpha)h,0)y(a) \right) - h^{1-\alpha} \frac{(t-a)_h^{\alpha-1}}{\Gamma(\alpha)} \widehat{e}_p(t,a)y(a). \end{aligned}$$

Using that

$${}_a\nabla_h^{-\alpha,\rho} g(t) = {}_{a-h}\nabla_h^{-\alpha,\rho} g(t) - \frac{h\widehat{e}_p(t-a+\alpha h,0)(t-a+h)_h^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)} g(a),$$

we obtain

$$\begin{aligned} & {}_a\nabla_h^{-\alpha,\rho} \left\{ \frac{h(t-a+h)_h^{-\alpha}}{\Gamma(1-\alpha)\rho^{1-\alpha}} \widehat{e}_p(t-a+(1-\alpha)h,0)y(a) \right\} = \\ & {}_{a-h}\nabla_h^{-\alpha,\rho} \left\{ \frac{h(t-a+h)_h^{-\alpha}}{\Gamma(1-\alpha)\rho^{1-\alpha}} \widehat{e}_p(t-a+(1-\alpha)h,0)y(a) \right\} - \frac{h^{2-\alpha}(t-a+h)_h^{\alpha-1} \widehat{e}_p(t+h,a)}{\rho\Gamma(\alpha)} y(a). \end{aligned}$$

Finally, substituting back in (4.7) and making use of the fact that $\nabla_h^\rho c_1 \widehat{e}_p(t, a-h) = 0$, we obtain the solution representation

$$\begin{aligned} y(t) &= \frac{\widehat{e}_p(t,a)y(a)}{\Gamma(\alpha)} \left[h^{2-\alpha}(\alpha-1)(t-a+h)_h^{\alpha-2} + h^{1-\alpha}(t-a)_h^{\alpha-1} \right] + {}_a\nabla_h^{-\alpha,\rho} f(t,y(t)) \\ &= \frac{\widehat{e}_p(t,a)y(a)}{\Gamma(\alpha)} h^{1-\alpha}(t-a+h)_h^{\alpha-1} + {}_a\nabla_h^{-\alpha,\rho} f(t,y(t)) \end{aligned} \tag{4.8}$$

Indeed, we have proved the following theorem.

Theorem 4.4. *y is a solution of the initial value problem, (4.5), (4.6) if and only if it has the integral representation (4.8).*

Remark 4.5. If in (4.8), we let $h = 1$ then we obtain the solution representation

$$\begin{aligned} y(t) &= \frac{\rho^{t-a}y(a)}{\Gamma(\alpha)} \left[(\alpha-1)(t-a+1)^{\alpha-2} + (t-a)^{\alpha-1} \right] + {}_a\nabla^{-\alpha,\rho} f(t,y(t)) \\ &= \frac{\rho^{t-a}y(a)}{\Gamma(\alpha)} (t-a+1)^{\alpha-1} + {}_a\nabla^{-\alpha,\rho} f(t,y(t)). \end{aligned} \tag{4.9}$$

If we let $\rho = 1$ in (4.9), then we recover the solution representation (5.9) in [6].

5. The Caputo fractional proportional difference

Definition 5.1. [23] For $\rho \in (0, 1]$ and $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$. Assume f is defined on $\mathbb{N}_{a,h} = \{a, a+h, a+2h, \dots\}$ and on ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \dots\}$. We usually have $b = a + kh$ for some $k \in \mathbb{N}$. Then, we define the left Caputo fractional proportional difference starting at a by

$$\begin{aligned} ({}^C_a\nabla_h^{\alpha,\rho} f)(t) &= {}_{a_h(\alpha)}\nabla_h^{-(n-\alpha),\rho}(\nabla_h^{n,\rho} f)(t) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_{a_h(\alpha)}^t \widehat{e}_\rho(t-s+h(n-\alpha))(t-\rho_h(s))_h^{\overline{n-\alpha-1}} \\ &\times (\nabla_h^{n,\rho} f)(s)\nabla_h s. \end{aligned} \tag{5.1}$$

The right Caputo fractional proportional difference ending at b

$$\begin{aligned} ({}^C_h\nabla_b^{\alpha,\rho} f)(t) &= {}_h\nabla_{b_h(\alpha)}^{-(n-\alpha),\rho}({}_\ominus\Delta_h^{n,\rho} f)(t) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_t^{b_h(\alpha)} \widehat{e}_\rho(s-t+h(n-\alpha))(s-\rho_h(t))_h^{\overline{n-\alpha-1}} \\ &\times ({}_\ominus\Delta_h^{n,\rho} f)(s)\Delta_h s. \end{aligned} \tag{5.2}$$

We always consider $n = [\text{Re}(\alpha)] + 1$, $a_h(\alpha) = a + (n-1)h$ and $b_h(\alpha) = b - (n-1)h$.

Remark 5.2. [23] Alternatively, if in Definition 5.1 we assume that f is defined on $\mathbb{N}_{a-(n-1)h,h}$ for the left case and on ${}_{b+(n-1)h,h}\mathbb{N}$ for the right case. Then, we define

$$({}^C_a\nabla_h^{\alpha,\rho} f)(t) = {}_a\nabla_h^{-(n-\alpha),\rho}(\nabla_h^{n,\rho} f)(t),$$

and

$$({}^C_h\nabla_b^{\alpha,\rho} f)(t) = {}_h\nabla_b^{-(n-\alpha),\rho}({}_\ominus\Delta_h^{n,\rho} f)(t).$$

Example 5.3. [23] Let $\alpha, \beta \in \mathbb{C}$ be such that $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$. Then, for any $\rho > 0$ and $n = [\text{Re}(\alpha)] + 1$ we have

1. $({}^C_a\nabla_h^{\alpha,\rho} \widehat{e}_\rho(t,0)(t-a_h(\alpha))_h^{\overline{\beta-1}})(x) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} \widehat{e}_\rho(x-\alpha h,0)(x-a_h(\alpha))_h^{\overline{\beta-\alpha-1}}$, $\text{Re}(\beta) > n$.
2. $({}^C_h\nabla_b^{\alpha,\rho} \widehat{e}_\rho(b_h(\alpha)-t,0)(b_h(\alpha)-t)_h^{\overline{\beta-1}})(x) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} \widehat{e}_\rho(b_h(\alpha)-x-\alpha h,0)(b_h(\alpha)-x)_h^{\overline{\beta-\alpha-1}}$, $\text{Re}(\beta) > n$.

For $k = 0, 1, \dots, n-1$, we have $({}^C_a\nabla_h^{\alpha,\rho} \widehat{e}_\rho(t,0)(t-a_h(\alpha))_h^{\overline{k}})(x) = 0$ and $({}^C_h\nabla_b^{\alpha,\rho} \widehat{e}_\rho(b_h(\alpha)-t,0)(b_h(\alpha)-t)_h^{\overline{k}})(x) = 0$. In particular, $({}^C_a\nabla_h^{\alpha,\rho} \widehat{e}_\rho(t,0))(x) = 0$ and $({}^C_h\nabla_b^{\alpha,\rho} \widehat{e}_\rho(b_h(\alpha)-t,0))(x) = 0$.

Theorem 5.4. [23] Assume that f is defined on $\mathbb{N}_{a,h}$ for the left case and is defined on ${}_{b,h}\mathbb{N}$ for the right case. Then, for $\rho > 0$ and $n = [\text{Re}(\alpha)] + 1$, we have

$${}_{a_h(\alpha)}\nabla_h^{-\alpha,\rho}({}^C_a\nabla_h^{\alpha,\rho} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(\nabla_h^{k,\rho} f)(a_h(\alpha))_h^{\overline{k}}}{\rho^k k!} (t-a_h(\alpha))_h^{\overline{k}} \widehat{e}_\rho(t, a_h(\alpha)). \tag{5.3}$$

In the right case, we get

$${}_h\nabla_{b_h(\alpha)}^{-\alpha,\rho}({}_h^C\nabla_b^{\alpha,\rho}f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{({}_\ominus\Delta_h^{k,\rho}f)(b_h(\alpha))}{\rho^k k!} (b_h(\alpha) - t) {}_h^{\bar{k}}\hat{e}_p(b_h(\alpha), t). \quad (5.4)$$

Proof. By the help of (3.23) in Remark 3.14, we have

$$\begin{aligned} {}_{a_h(\alpha)}\nabla_h^{-\alpha,\rho}({}_a^C D^{\alpha,\rho}f)(t) &= {}_{a_h(\alpha)}\nabla_h^{-\alpha,\rho}({}_{a_h(\alpha)}\nabla_h^{-(n-\alpha),\rho}\nabla_h^{n,\rho}f)(t) \\ &= ({}_{a_h(\alpha)}\nabla_h^{-n,\rho}\nabla_h^{n,\rho}f)(t) \\ &= f(t) - \sum_{k=0}^{n-1} \frac{(\nabla_h^{k,\rho}f)(a_h(\alpha))}{\rho^k k!} (t - a_h(\alpha)) {}_h^{\bar{k}}\hat{e}_p(t, a_h(\alpha)). \end{aligned}$$

In the proof, we have used that $\eta(h, \rho, n) = \hat{e}_p(h(n-1), 0)$. For the proof of (5.4), one may either follow similar arguments or use the action of the Q-operator. \square

Theorem 5.5. Assume that f is defined on $\mathbb{N}_{a-(n-1)h,h}$. Let $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\rho > 0$, $n = [\text{Re}(\alpha)] + 1$. If $F_\alpha(s) = \mathcal{N}_{a,h}\{f(t)\}(s)$, then

$$\begin{aligned} \mathcal{N}_{a,h}\{({}_a^C\nabla_h^{\alpha,\rho}f)(t)\}(s) &= (\rho s + 1 - \rho)^\alpha F_\alpha(s) \\ &\quad - \rho \sum_{k=0}^{n-1} (\rho s + 1 - \rho)^{\alpha-1-k} (\nabla_h^{k,\rho}f)(a). \end{aligned} \quad (5.5)$$

Proof. By using Lemma 3.11 and Theorem 4.2, we obtain

$$\begin{aligned} \mathcal{N}_{a,h}\{({}_a^C\nabla_h^{\alpha,\rho}f)(t)\}(s) &= \mathcal{N}_{a,h}\{({}_a\nabla_h^{-(n-\alpha),\rho}\nabla_h^{n,\rho}f)(t)\}(s) \\ &= (\rho s - (\rho - 1))^{\alpha-n} \mathcal{N}_{a,h}\{(\nabla_h^{n,\rho}f)(t)\}(s) \\ &= (\rho s - (\rho - 1))^{\alpha-n} \\ &\quad \times \left[(\rho s + 1 - \rho)^n F_\alpha(s) - \rho \sum_{k=0}^{n-1} (\rho s + 1 - \rho)^{n-1-k} (\nabla_h^{k,\rho}f)(a) \right]. \end{aligned}$$

From which (5.5) follows and the proof is completed. \square

By using Theorem 5.5, Theorem 4.3, and Lemma 3.11, we can state the following relation between Caputo and Riemann fractional proportional derivatives.

Proposition 5.6. For any $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\rho > 0$, $n = [\text{Re}(\alpha)] + 1$ we have

$$\begin{aligned} ({}_a^C\nabla_h^{\alpha,\rho}f)(t) &= ({}_a\nabla_h^{\alpha,\rho}f)(t) \\ &\quad - \sum_{k=0}^{n-1} \frac{(\rho - (\rho - 1)h)^{\alpha-k}}{\Gamma(k+1-\alpha)} (t-a) {}_h^{\bar{k}-\alpha}\hat{e}_p(t, a) (\nabla_h^{k,\rho}f)(a). \end{aligned} \quad (5.6)$$

Similarly, by applying the Q-operator, we have

$$\begin{aligned} ({}_h^C\nabla_b^{\alpha,\rho}f)(t) &= ({}_h\nabla_b^{\alpha,\rho}f)(t) \\ &\quad - \sum_{k=0}^{n-1} \frac{(\rho - (\rho - 1)h)^{\alpha-k}}{\Gamma(k+1-\alpha)} (b-t) {}_h^{\bar{k}-\alpha}\hat{e}_p(b, t) ({}_\ominus\Delta_h^{k,\rho}f)(b). \end{aligned} \quad (5.7)$$

Example 5.7. Consider the linear Caputo fractional proportional initial value problem

$$({}^C\nabla_h^{\alpha,\rho}y(t) - \rho^\alpha \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{-\alpha h} \lambda y(t) = f(t), y(a) = y_a, 0 < \alpha \leq 1. \tag{5.8}$$

Then, $y(t)$ is a solution of (5.8) if and only if it satisfies the integral equation

$$y(t) = y_a \widehat{e}_p(t, a) {}_hE_{\alpha, \alpha}(\lambda, t - a) + \rho^{-\alpha} \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{\alpha h} \times \int_a^t {}_hE_{\alpha, \alpha}(\lambda, t - s + h) \widehat{e}_p(t - s + h, 0) f(s) \nabla_h s. \tag{5.9}$$

In fact, if we apply $\mathcal{N}_{a,h}$ to (5.8) and make use of Theorem 5.5 with $n = 1$, then we have

$$\left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{-\alpha h} ((\rho s + 1 - \rho)^\alpha - \lambda \rho^\alpha) Y_a(s) = \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{-\alpha h} \times \rho y_a (\rho s + 1 - \rho)^{\alpha - 1} + F_a(s).$$

Hence,

$$Y_a(s) = \frac{(s - \frac{\rho - 1}{\rho})^{\alpha - 1} y_a}{(s - \frac{\rho - 1}{\rho})^\alpha - \lambda} + \left(\frac{\rho}{\rho - (\rho - 1)h}\right)^{\alpha h} \frac{\rho^{-\alpha} F_a(s)}{(\frac{\rho - 1}{\rho} - s)^\alpha - \lambda}.$$

Applying the inverse of $\mathcal{N}_{a,h}$ and using Theorem 2.13 and Lemma 2.16, we reach the representation (5.9). Conversely, if $y(t)$ has the representation (5.9), then by the help of Example 5.3 it satisfies (5.8).

Remark 5.8. If in Example 5.7 we let $\rho = h = 1$, then the solution representation (116) in [7] is recovered.

6. Conclusions

Proposing new fractional operators has become one of the most important tackles in the field of mathematical analysis. This is due to the need of different types of fractional operators that serve to study some real world phenomena. The proportional fractional operators, both in continuous and discrete versions were recently proposed. In this work, we presented the discrete fractional operators defined on $h\mathbb{Z}$. These operators covers some known discrete fractional operators such as the Riemann-Liouville fractional sum, Riemann-Liouville fractional difference and the Caputo fractional difference as ρ tends to 1. In addition, we found the discrete h -Laplace transforms of the fractional operators discussed and used them to solve some linear type problems. On the top of this, we corrected a mistake done previously in the literature concerning the h -Laplace transforms of these operators.

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