

Analysis of a Fractional Predator–Prey System Incorporating Strong Allee Effects and Nonlinear Defensive Feedback

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Abstract

This paper presents a fractional-order predator–prey model incorporating a strong Allee effect in the prey population and a nonlinear anti-predator feedback mechanism in the predator population. The model is formulated using the Caputo fractional derivative of order α ($0 < \alpha \leq 1$), allowing the inclusion of memory and hereditary effects that are frequently observed in ecological systems. The prey population is assumed to experience a strong Allee effect, while predator dynamics are regulated through a density-dependent feedback response that reflects behavioural adaptation and defensive interactions. Fundamental mathematical properties of the model, including existence, uniqueness, positivity, and boundedness of solutions, are established to ensure biological feasibility. Equilibrium points of the system are determined, and their local stability is investigated through Jacobian linearization and eigenvalue analysis. For the fractional-order model, stability conditions are examined using the fractional eigenvalue criterion, highlighting the influence of memory effects on the asymptotic behaviour of the system. To illustrate the theoretical findings, extensive numerical simulations are performed using the fractional Adams–Bashforth–Moulton predictor–corrector scheme. Phase-plane analysis, equilibrium investigations, and two-parameter stability maps are employed to examine the influence of the carrying capacity, Allee threshold, interaction strength, feedback intensity, and fractional order on the system dynamics. The results demonstrate that increasing the carrying capacity enlarges the biologically feasible region and promotes persistence, whereas stronger Allee effects reduce the coexistence domain and increase the risk of extinction. Furthermore, the nonlinear anti-predator feedback significantly modifies predator survival and stability characteristics. A comparative study between the classical integer-order model ($\alpha = 1$) and the fractional-order formulation ($0 < \alpha < 1$) reveals that ecological memory plays a stabilizing role. Fractional dynamics generate smoother trajectories, reduced oscillatory behaviour, and slower convergence toward equilibrium states when compared with the corresponding classical system. These memory-induced effects moderate abrupt population fluctuations and promote more realistic ecological responses. Overall, the proposed model provides a flexible framework for understanding the combined roles of memory, Allee effects, and behavioural feedback in predator–prey interactions, offering new insights into the stability and persistence of ecological systems..

Keywords: Fractional-order system, Allee effect, anti-predator feedback, Predator-prey dynamics.

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1. Introduction

The mathematical modeling of species abundance, spatial distribution, and ecological interactions has attracted significant interest among researchers due to its ability to explain complex population dynamics. A foundational contribution in this area is the classical Lotka–Volterra predator–prey (LVPP) model developed by Lotka and Volterra [1, 2], which established the theoretical basis for oscillatory behavior generated through predator–prey interactions. Despite its historical importance, the classical LVPP model does not account for density-dependent growth regulation or environmental constraints, thereby limiting its ecological applicability. To address these shortcomings, several modifications were introduced in later studies [3, 4], particularly through the incorporation of logistic growth and carrying-capacity mechanisms that provide a more realistic representation of natural populations.

Over the years, numerous extensions of the LVPP framework have been developed to capture additional ecological processes and improve biological realism. These modifications include the incorporation of predator functional responses, prey refuge mechanisms, time delays, stage structures, and memory effects described through fractional-order dynamics. Among the most widely studied formulations are the Holling-type functional responses [5, 6] and ratio-dependent predator–prey models [7]. In this context, Kar [8] investigated a predator–prey system with a Holling type-II functional response and prey refuge, providing detailed analyses of both local and global stability characteristics.

The ecological significance of prey refuge (PR) has motivated extensive theoretical investigations. Prey refuge refers to the portion of a prey population that remains inaccessible to predators due to environmental heterogeneity, behavioral adaptations, spatial concealment, or the availability of physical shelters such as vegetation and burrows. In mathematical formulations, this phenomenon is commonly represented by a refuge parameter, denoted by ρ , which quantifies the fraction of prey protected from predation. Such a mechanism has been shown to substantially alter the qualitative behavior of predator–prey systems.

Several researchers have examined the stabilizing influence of prey refuge under different modeling assumptions. For example, Tripathi *et al.* [9] studied a density-dependent delayed predator–prey model with a Beddington–DeAngelis functional response and demonstrated that increasing refuge levels can suppress population oscillations. Similarly, Wei and Fu [10] analyzed a stage-structured predator–prey system and reported that prey refuge significantly affects the onset of Hopf bifurcation, thereby delaying or preventing cyclic population fluctuations. Huang *et al.* [50] considered a Holling type-III response with prey refuge and showed that refuge availability acts as an effective mechanism for reducing excessive predation pressure and enhancing system stability.

Further contributions were made by Ma and co-workers [12, 13], who investigated Lotka–Volterra-type models incorporating predator interference. Their findings revealed that increased refuge availability promotes ecological stability and supports long-term biodiversity persistence. Likewise, Tang *et al.* [15] established global stability results for a Holling type-II predator–prey model with constant refuge and confirmed the beneficial role of prey protection in maintaining stable coexistence. Beyond classical population

models, Bianca and collaborators [14, 16] proposed a thermostatted kinetic-theory framework to describe complex biological systems involving particle refuges, with applications extending to tumour-growth dynamics and other biological processes.

Another important ecological mechanism influencing population persistence is the Allee effect (AE), originally identified by W. C. Allee [17]. The Allee effect describes situations in which the per-capita growth rate or individual fitness increases with population density when population levels are low. Consequently, populations may experience reduced growth or even face extinction if their abundance falls below a critical threshold. This phenomenon can arise from several biological factors, including mate-finding difficulties, cooperative behaviors, collective defense strategies, social facilitation, and genetic inbreeding at low population densities [18]. Motivated by these ecological observations, Terry [19] incorporated an Allee-effect component into the predator reproduction process of a classical predator–prey model and demonstrated that the resulting dynamics can either stabilize or destabilize coexistence equilibria depending on the underlying parameter regime.

Mathematically, the AE modifies the classical logistic growth law by introducing a critical population threshold A (the *Allee threshold*), below which the population growth rate becomes negative. The generalized Allee-type growth model is given by

$$\frac{dN}{dt} = \mathcal{R} N \left(1 - \frac{N}{K} \right) \left(\frac{N}{A} - 1 \right), \quad (1.1)$$

where N denotes the population size, $\mathcal{R}$ is the intrinsic growth rate, and K is the carrying capacity. In the absence of the AE ($A = 0$), the model reduces to the classical logistic equation:

$$\lim_{A \rightarrow 0^+} \mathcal{R} N \left(1 - \frac{N}{K} \right) \left(\frac{N}{A} - 1 \right) = \mathcal{R} N \left(1 - \frac{N}{K} \right). \quad (1.2)$$

For $A > 0$, the system exhibits either a *weak AE* (the growth rate remains positive but reduced for small N) or a *strong AE* (the growth rate becomes negative for $N < A$), representing a threshold-driven risk of extinction. As shown in Figure 1, the AE alters the logistic curve by introducing a depensation region near zero population density. When the initial population satisfies $N(0) < A$, extinction occurs, whereas $N(0) > A$ leads to growth toward the carrying capacity K . The model admits three equilibria:

$$N_1 = 0 \quad (\text{extinction}), \quad N_2 = A \quad (\text{unstable threshold}), \quad N_3 = K \quad (\text{stable equilibrium}). \quad (1.3)$$

Populations below the Allee threshold ($N < A$) decline to extinction, while those above it ($N > A$) approach K . Populations below the Allee threshold ($N < A$) decline to extinction, while those above it ($N > A$) approach K .

The incorporation of Allee-effect mechanisms into ecological models introduces critical threshold phenomena that fundamentally alter population persistence and extinction dynamics. In contrast to classical logistic growth, populations experiencing an Allee effect may decline toward extinction when their density falls below a critical threshold, even under otherwise favorable environmental conditions. Conversely, populations exceeding this threshold generally persist and approach the carrying capacity. Such threshold-dependent behavior gives rise to rich dynamical structures, including bistability, multiple attractors,

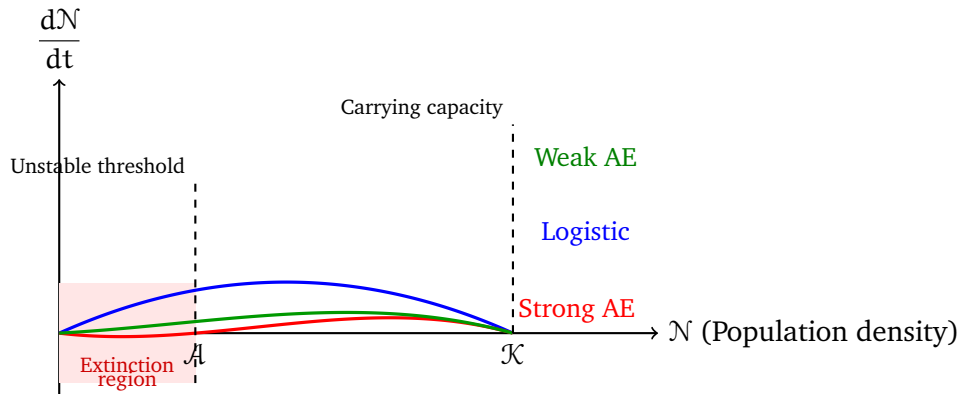


Figure 1: Comparison of classical logistic growth and Allee-effect dynamics. The parameter $\mathcal{A}$ denotes the critical Allee threshold, while $\mathcal{K}$ represents the carrying capacity. In the presence of a strong Allee effect, populations with initial density $N(0) < \mathcal{A}$ experience negative growth and move toward extinction, whereas populations above the threshold converge to the stable carrying-capacity equilibrium $\mathcal{K}$.

saddle-node bifurcations, transcritical bifurcations, and abrupt transitions between extinction and coexistence states. Consequently, Allee effects have attracted considerable attention in conservation biology, fisheries management, epidemiology, and invasion ecology, where understanding the conditions governing population establishment and persistence is of paramount importance. Numerous investigations have demonstrated that the interplay between Allee effects, predation, prey refuge, harvesting, and other ecological processes can significantly influence ecosystem resilience and long-term biodiversity maintenance.

Alongside these developments, increasing attention has been devoted to fractional-order models (FOMs), which provide a powerful framework for describing memory-dependent and hereditary processes in biological and physical systems. Unlike classical integer-order differential equations, fractional differential equations incorporate the influence of past states on present dynamics through nonlocal fractional derivatives. Such formulations are particularly appropriate for ecological systems in which population responses depend not only on current conditions but also on accumulated environmental influences, physiological memory, delayed behavioral responses, and long-term adaptation mechanisms. As a result, fractional calculus has emerged as an effective mathematical tool for modeling complex biological interactions that cannot be adequately represented using conventional differential equations.

The application of fractional calculus to predator–prey systems has generated a substantial body of research concerning stability, bifurcation, chaos, synchronization, and control. Several authors have extended classical ecological models to fractional-order settings and demonstrated that memory effects can profoundly modify system behavior, often improving agreement with observed biological phenomena [20, 21, 22]. Beyond ecology, fractional formulations have also been widely employed in the study of nonlinear wave propagation and fluid mechanics. For example, fractional versions of the Benjamin–Bona–Mahony (BBM) and Benjamin–Bona–Mahony–Burgers (BBMB) equations have been proposed to describe anomalous diffusion, nonlocal dispersion, and hereditary transport pro-

cesses in complex media [28, 29]. These developments have further motivated the incorporation of fractional operators into ecological systems where memory effects naturally arise.

More recently, substantial attention has been directed toward the development of fractional-order models that incorporate biologically realistic mechanisms such as prey refuge, cannibalism, fear effects, harvesting strategies, vaccination policies, and Allee-type nonlinearities. Owing to their ability to account for memory and hereditary characteristics, fractional differential equations have emerged as powerful tools for describing complex biological and ecological phenomena. Das *et al.* [20], Rivero *et al.* [21], and El-Sadany *et al.* [22] demonstrated that fractional-order formulations can significantly modify stability properties and long-term dynamics when compared with their classical integer-order counterparts. In epidemiological applications, fractional models have been successfully employed to study infectious diseases, including the fractional mathematical modeling and forecasting of COVID-19 in Saudi Arabia using the Caputo derivative with a power-law kernel [23], the conformable fractional-order modeling and analysis of HIV/AIDS transmission dynamics [24], and generalized fractional-order COVID-19 vaccination models [25], all of which highlighted the importance of memory effects in accurately capturing disease evolution. In ecological modeling, Ali *et al.* [26] established the existence theory and numerical analysis of a three-species prey–predator model governed by the generalized Mittag–Leffler power law, while subsequent investigations [27] revealed that fractional memory can generate richer dynamical behavior and enlarged stability regions in multispecies systems. Furthermore, Rayungsari *et al.* [30] examined a fractional-order predator–prey model incorporating prey refuge and cannibalism, Zhang and Muhammadhaji [31] studied delayed fractional predator–prey dynamics with stage-structured prey populations, and Gao *et al.* [32] together with Wang and Meng [34] investigated bifurcation phenomena in fractional ecological systems involving refuge, fear effects, and economic thresholds. More recently, Lu *et al.* [35] and Liu *et al.* [33] explored chaotic and incommensurate fractional ecological systems, demonstrating that memory effects substantially enrich ecological complexity and lead to novel dynamical regimes that are absent in corresponding integer-order models. Moreover, Abdo *et al.* [36] and Almalahi *et al.* [37] present theoretical existence results and numerical analyses of three-species prey–predator models within the Mittag–Leffler and generalized Mittag–Leffler fractional frameworks. These studies collectively confirm that fractional-order formulations provide a versatile and realistic framework for investigating biological systems with memory, motivating the incorporation of fractional dynamics into predator–prey models involving prey refuge, Allee effects, and behavioral interactions.

1.1. Novelty and Main Contributions

Although significant progress has been made in the study of predator–prey systems incorporating Allee effects, behavioural responses, and fractional-order memory, the combined influence of these mechanisms remains insufficiently explored. Most existing investigations have focused on only one or two ecological factors at a time, thereby overlooking the complex interactions that arise when multiple regulatory mechanisms operate simultaneously. Motivated by this research gap, the present study develops a novel fractional-order predator–prey model that integrates four important ecological processes within a

unified mathematical framework: a strong Allee effect in the prey population, nonlinear prey-guided anti-predator behavioural feedback, predator self-limitation through intraspecific competition, and memory-dependent dynamics represented by the Caputo fractional derivative. The incorporation of a strong Allee threshold enables the model to capture depensation effects and extinction risks associated with low-density populations, while the nonlinear anti-predator response reflects adaptive behavioural changes that influence predator efficiency. In addition, the predator self-regulation term accounts for crowding effects and competition among predators, thereby preventing unrealistic population growth and enhancing biological realism. Unlike classical integer-order models, the fractional-order formulation introduces hereditary effects and long-term memory, allowing the current state of the ecosystem to depend on its entire evolutionary history. This combination of threshold dynamics, behavioural adaptation, predator competition, and memory effects generates a richer dynamical structure than that observed in conventional predator-prey models. From a mathematical perspective, the study establishes the positivity, boundedness, and biological feasibility of solutions, derives conditions for the existence of equilibrium states, and investigates their local stability using the fractional Matignon criterion. Furthermore, numerical simulations are performed to illustrate how ecological memory influences species persistence, extinction thresholds, transient dynamics, and long-term coexistence patterns. The results demonstrate that the interaction between Allee effects, anti-predator behaviour, predator self-limitation, and fractional memory can substantially modify population trajectories and stability characteristics, providing new insights into the mechanisms governing ecological resilience and population survival in memory-dependent environments.

2. Preliminaries

This section presents several fundamental concepts and results from fractional calculus and dynamical systems theory that will be used throughout the subsequent analysis. These preliminary notions provide the mathematical foundation for studying the qualitative behavior of the proposed fractional-order predator-prey model. For further details on fractional differential equations and their applications, the reader may consult Podlubny [29] and Kilbas *et al.* [28].

Definition 2.1 (Caputo and Riemann–Liouville Fractional Derivatives). Let $\mathcal{F} : [0, \infty) \rightarrow \mathbb{R}$ be a sufficiently smooth function and let $0 < \alpha < 1$. The Caputo fractional derivative of order α is defined by

$${}^C D_t^\alpha \mathcal{F}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathcal{F}'(\tau)}{(t-\tau)^\alpha} d\tau. \quad (2.1)$$

The corresponding Riemann–Liouville fractional derivative is

$${}^{RL} D_t^\alpha \mathcal{F}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\mathcal{F}(\tau)}{(t-\tau)^\alpha} d\tau. \quad (2.2)$$

Moreover, the Caputo and Riemann–Liouville operators are related through

$${}^C D_t^\alpha \mathcal{F}(t) = {}^{RL} D_t^\alpha [\mathcal{F}(t) - \mathcal{F}(0)]. \quad (2.3)$$

Definition 2.2 (Fractional Dynamical System). A nonlinear fractional-order dynamical system of order $0 < \alpha \leq 1$ can be expressed as

$${}^C D_t^\alpha \mathcal{U}(t) = \mathcal{F}(\mathcal{U}(t)), \quad (2.4)$$

where

$$\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n)^T \in \mathbb{R}^n$$

is the state vector and

$$\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is a continuously differentiable vector field.

When $\alpha = 1$, the system reduces to the classical ordinary differential equation

$$\frac{d\mathcal{U}}{dt} = \mathcal{F}(\mathcal{U}). \quad (2.5)$$

Definition 2.3 (Jacobian Matrix). Let

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)^T$$

be continuously differentiable. The Jacobian matrix associated with $\mathcal{F}$ is defined as

$$\mathcal{J}(\mathcal{U}) = \left[\frac{\partial \mathcal{F}_i}{\partial \mathcal{U}_j} \right]_{i,j=1}^n. \quad (2.6)$$

The matrix $\mathcal{J}(\mathcal{U}^*)$ provides the linear approximation of the nonlinear system in a neighborhood of an equilibrium point $\mathcal{U}^*$.

Theorem 2.4 (Matignon's Stability Criterion). Consider the linear fractional-order system

$${}^C D_t^\alpha \mathcal{U} = \mathcal{J} \mathcal{U}, \quad 0 < \alpha \leq 1, \quad (2.7)$$

where

$$\lambda_i \in \sigma(\mathcal{J}), \quad i = 1, 2, \dots, n,$$

are the eigenvalues of the Jacobian matrix $\mathcal{J}$. Then the equilibrium point is locally asymptotically stable if and only if

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1, 2, \dots, n. \quad (2.8)$$

For the integer-order case ($\alpha = 1$), the above condition reduces to

$$\Re(\lambda_i) < 0, \quad i = 1, 2, \dots, n. \quad (2.9)$$

Lemma 2.5 (Discriminant Test for Cubic Polynomials [49]). Consider the cubic polynomial

$$\mathcal{P}(\lambda) = a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0, \quad a_3 \neq 0. \quad (2.10)$$

Its discriminant is given by

$$\Delta = 18a_3a_2a_1a_0 - 4a_2^3a_0 + a_2^2a_1^2 - 4a_3a_1^3 - 27a_3^2a_0^2. \quad (2.11)$$

The nature of the roots of $\mathcal{P}(\lambda)$ is completely determined by the sign of the discriminant Δ :

1. If $\Delta > 0$, then $\mathcal{P}(\lambda)$ possesses three distinct real roots.
2. If $\Delta = 0$, then $\mathcal{P}(\lambda)$ has at least one repeated real root.
3. If $\Delta < 0$, then $\mathcal{P}(\lambda)$ admits exactly one real root together with a pair of complex-conjugate roots.

3. Model Formulation and Numerical Simulation

Let $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ denote the prey and predator population densities, respectively, at time $t > 0$. Motivated by the combined effects of ecological memory, Allee-type growth, predator-prey interactions, anti-predator behavior, and predator self-regulation, we propose the following fractional-order predator-prey model governed by the Caputo derivative of order $0 < \alpha \leq 1$:

$${}^C D_t^\alpha \mathcal{X}(t) = \mathcal{R} \mathcal{X}(t) \left(1 - \frac{\mathcal{X}(t)}{\mathcal{K}}\right) \left(\frac{\mathcal{X}(t)}{\mathcal{A}} - 1\right) - \Phi(\mathcal{X}(t)) \mathcal{Y}(t), \quad (3.1)$$

$${}^C D_t^\alpha \mathcal{Y}(t) = \eta \Phi(\mathcal{X}(t)) \mathcal{Y}(t) - \mathcal{D} \mathcal{Y}(t) - \varepsilon F(\mathcal{Y}(t)) \mathcal{X}(t) - \mu \mathcal{Y}^2(t). \quad (3.2)$$

The nonlinear interaction functions are assumed to be

$$\Phi(\mathcal{X}) = \mathcal{M} \mathcal{X}, \quad F(\mathcal{Y}) = \frac{\mathcal{B} \mathcal{Y}}{1 + \mathcal{C} \mathcal{Y}}, \quad \mathcal{M}, \mathcal{B}, \mathcal{C} > 0. \quad (3.3)$$

Substituting the interaction functions defined in (3.3) into (3.1)–(3.2), the fractional predator-prey system can be written explicitly as

$${}^C D_t^\alpha \mathcal{X}(t) = \mathcal{R} \mathcal{X}(t) \left(1 - \frac{\mathcal{X}(t)}{\mathcal{K}}\right) \left(\frac{\mathcal{X}(t)}{\mathcal{A}} - 1\right) - \mathcal{M} \mathcal{X}(t) \mathcal{Y}(t), \quad (3.4)$$

$${}^C D_t^\alpha \mathcal{Y}(t) = \eta \mathcal{M} \mathcal{X}(t) \mathcal{Y}(t) - \mathcal{D} \mathcal{Y}(t) - \varepsilon \frac{\mathcal{B} \mathcal{Y}(t)}{1 + \mathcal{C} \mathcal{Y}(t)} \mathcal{X}(t) - \mu \mathcal{Y}^2(t). \quad (3.5)$$

The proposed model assumes that the prey population follows logistic growth modified by a strong Allee effect, while predator-prey interactions obey a bilinear mass-action law. The first equation incorporates both density dependence and a critical population threshold below which prey reproduction becomes insufficient to sustain the population. Predator growth is proportional to prey consumption through the conversion efficiency parameter η , whereas predator mortality is influenced by natural death, nonlinear anti-predator defense mechanisms, and intraspecific competition. The nonlinear feedback term $-\varepsilon F(\mathcal{Y}) \mathcal{X}$ describes the effectiveness of prey anti-predator defense, which increases with predator abundance and gradually saturates at high predator densities. Consequently, this mechanism reduces predator growth and reflects realistic ecological limitations on predation efficiency.

The parameter $\mathcal{A}$ denotes the critical Allee threshold and satisfies the biologically meaningful condition $0 < \mathcal{A} < \mathcal{K}$, where $\mathcal{K}$ is the environmental carrying capacity. This condition characterizes a strong Allee effect. Therefore, when $\mathcal{X}(t) < \mathcal{A}$, the prey population experiences negative growth and tends toward extinction, whereas populations satisfying $\mathcal{X}(t) > \mathcal{A}$ can recover and approach the carrying capacity in the absence of excessive predation pressure. The parameter μ represents predator self-limitation arising from competition for resources, territory, or mating opportunities, thereby preventing unrealistic predator growth at high population densities.

An important feature of the model is the incorporation of the fractional-order parameter α , which accounts for ecological memory and hereditary effects. For $0 < \alpha < 1$, the present population dynamics depend not only on the current state of the system but also on its entire evolutionary history. This nonlocal property enables the model to capture memory-driven ecological responses that cannot be described adequately by classical integer-order differential equations. As the fractional order decreases, the influence of past states becomes increasingly significant, resulting in slower transient dynamics and richer long-term behavior. The ecological interpretation of fractional memory follows the framework developed in [28, 29, 40, 39], whereas the Allee-type growth mechanism is motivated by the biological studies reported in [17, 18, 34, 32].

Collectively, the incorporation of strong Allee effects, nonlinear anti-predator behavioral responses, predator self-regulation, and fractional memory provides a biologically realistic and mathematically tractable framework for investigating persistence, extinction, stability transitions, bifurcation phenomena, and complex population dynamics in ecological systems.

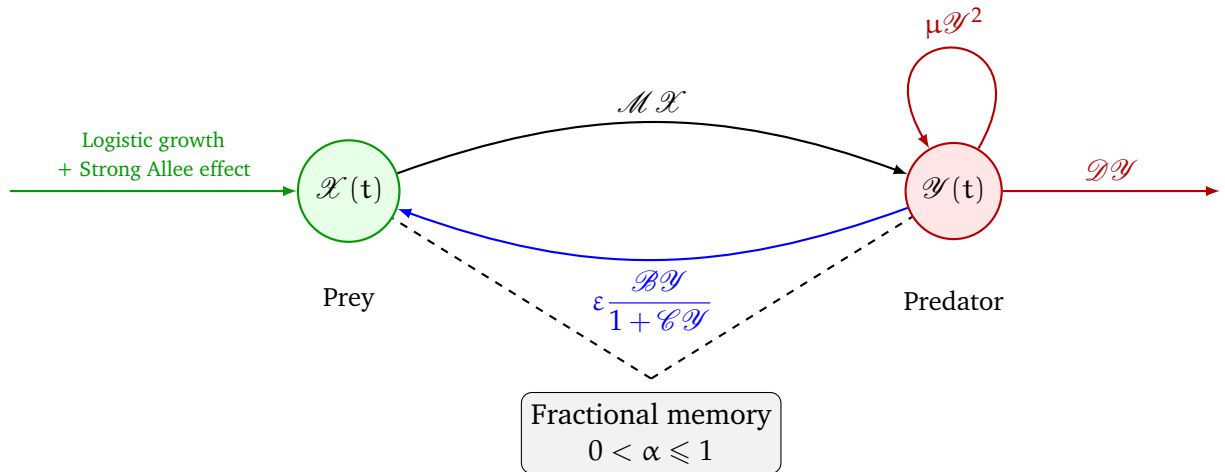


Figure 2: Conceptual diagram of the proposed fractional-order predator-prey model. The prey population experiences logistic growth with a strong Allee effect, predators consume prey through a linear predation response, anti-predator behavioural feedback suppresses predation efficiency, predator self-limitation accounts for intraspecific competition, and the fractional-order derivative introduces memory and hereditary effects into both populations.

3.1. Parameter Assumptions

Throughout this work, all model parameters are assumed to be nonnegative and biologically meaningful. In particular,

$$\mathcal{R}, \mathcal{K}, \mathcal{A}, \eta, \mathcal{D}, \mathcal{M}, \mathcal{B}, \mathcal{C}, \mu > 0, \quad \varepsilon \geq 0, \quad 0 < \alpha \leq 1.$$

The biologically realistic condition

$$0 < \mathcal{A} < \mathcal{K}$$

ensures the existence of a strong Allee threshold below the environmental carrying capacity. Consequently, prey persistence is possible only when the prey population density exceeds the critical threshold $\mathcal{A}$, whereas populations below this threshold are susceptible to extinction. The parameter η denotes the predator conversion efficiency and is assumed to satisfy

$$0 < \eta \leq 1,$$

reflecting the ecological fact that only a fraction of consumed prey biomass is converted into predator biomass.

The parameter ε regulates the intensity of the nonlinear anti-predator behavioral response. In particular, when

$$\varepsilon = 0,$$

the model reduces to a predator-prey system without behavioral inhibition. For

$$\varepsilon > 0,$$

the anti-predator effect is incorporated through the saturating feedback function

$$F(\mathcal{Y}) = \frac{\mathcal{B}\mathcal{Y}}{1 + \mathcal{C}\mathcal{Y}}, \quad (3.6)$$

which increases with predator density and approaches a finite upper bound as $\mathcal{Y} \rightarrow \infty$. This functional form captures the ecological observation that defensive responses intensify with increasing predation pressure but eventually saturate because of physiological and behavioral limitations.

The parameter $\mathcal{D}$ represents the natural mortality rate of predators, whereas μ characterizes predator self-limitation arising from intraspecific competition for resources, territory, or mating opportunities. Together, these parameters influence the long-term persistence, coexistence, and stability properties of the ecological system. The parameter values employed in the analytical investigations and numerical simulations are summarized in Table 1. These values are selected from biologically plausible ranges reported in the ecological and fractional-order modeling literature [5, 41, 18, 29, 40].

Furthermore, the nonlinear interaction functions satisfy

$$\Phi(0) = 0, \quad F(0) = 0. \quad (3.7)$$

These conditions imply that the interaction terms vanish whenever the corresponding species is absent and ensure consistency with the biological interpretation of the model. Consequently, the nonnegative region

$$\mathbb{R}_+^2 = \{(\mathcal{X}, \mathcal{Y}) \in \mathbb{R}^2 : \mathcal{X} \geq 0, \mathcal{Y} \geq 0\} \quad (3.8)$$

serves as the biologically feasible state space of the system. As will be shown in the subsequent positivity analysis, solutions initiated with nonnegative initial conditions remain nonnegative for all future time. Therefore, the proposed fractional-order predator–prey model is mathematically well posed from an ecological perspective and excludes biologically unrealistic negative population densities.

Unless otherwise stated, all numerical experiments presented in this work employ parameter values chosen from the admissible ranges reported in Table 1, thereby ensuring consistency between the theoretical analysis and the computational simulations.

Table 1: Model parameters and their biological interpretations for the proposed fractional-order predator–prey system incorporating a strong Allee effect, anti-predator behavioral feedback, predator self-limitation, and memory effects.

Parameter	Biological Interpretation	Typical Range	Source
$\mathcal{R}$	Intrinsic growth rate of the prey population	0.5–2.0	[44, 45]
$\mathcal{K}$	Environmental carrying capacity of the prey population	1–10	[45, 52]
$\mathcal{A}$	Critical Allee threshold below which prey extinction may occur	$0.1\mathcal{K}$ – $0.4\mathcal{K}$	[41, 18]
$\mathcal{M}$	Predation coefficient (attack rate)	0.1–2.0	[5, 44]
η	Conversion efficiency of consumed prey biomass into predator biomass	0.3–0.9	[5]
$\mathcal{D}$	Natural mortality rate of predators	0.05–0.8	[44]
$\mathcal{B}$	Intensity of the anti-predator behavioral response	0.1–1.5	Behavioral-response models ([38])
$\mathcal{C}$	Saturation coefficient in the anti-predator feedback function	0.01–1.0	Behavioral-response models ([38])
ε	Strength of prey-induced anti-predator feedback	$0 \leq \varepsilon \leq 1$	Present study
μ	Predator self-limitation (intraspecific competition) coefficient	0.01–1.0	Present study
α	Order of the Caputo fractional derivative representing ecological memory effects	$0 < \alpha \leq 1$	[29, 54]

The parameter ranges reported in Table 1 represent biologically plausible values adopted from the ecological and fractional-order modeling literature. The numerical simulations presented in this work employ parameter values selected from these admissible ranges while satisfying the biological constraint $0 < \mathcal{A} < \mathcal{K}$, which guarantees the presence of a strong Allee effect in the prey population. Consequently, all computational experiments remain consistent with the ecological assumptions underlying the proposed model and provide a biologically realistic basis for the stability analysis, bifurcation investigations, and numerical simulations presented throughout this study.

Unless otherwise stated, the model parameters satisfy

$$\mathcal{R}, \mathcal{K}, \mathcal{A}, \mathcal{M}, \eta, \mathcal{D}, \mathcal{B}, \mathcal{C} > 0, \quad \varepsilon, \mu \geq 0, \quad 0 < \alpha \leq 1.$$

Furthermore, the biologically meaningful condition

$$0 < \mathcal{A} < \mathcal{K}$$

is imposed throughout the analysis to ensure the existence of a strong Allee effect. These assumptions guarantee the biological feasibility of the model and ensure that all interaction mechanisms remain well defined within the nonnegative state space considered in this study.

Theorem 3.1 (Well-posedness and Boundedness). *Consider the fractional-order predator-prey system (3.4)–(3.5) with $0 < \alpha \leq 1$, $\mathcal{R}, \mathcal{K}, \mathcal{A}, \mathcal{M}, \eta, \mathcal{D}, \mathcal{B}, \mathcal{C}, \mu > 0$, and $\varepsilon \geq 0$. For any nonnegative initial conditions*

$$\mathcal{X}(0) = \mathcal{X}_0 \geq 0, \quad \mathcal{Y}(0) = \mathcal{Y}_0 \geq 0, \quad (3.9)$$

the following properties hold:

- (i) *The system admits a unique local solution.*
- (ii) *The nonnegative region*

$$\mathbb{R}_{\geq 0}^2 = \{(\mathcal{X}, \mathcal{Y}) : \mathcal{X} \geq 0, \mathcal{Y} \geq 0\} \quad (3.10)$$

is positively invariant.

- (iii) *All trajectories are uniformly bounded.*
- (iv) *The local solution extends uniquely to a global solution on $[0, \infty)$.*

Proof. The fractional-order predator-prey system can be written in the compact vector form

$${}^C D_t^\alpha \mathcal{U}(t) = \mathcal{F}(\mathcal{U}(t)), \quad \mathcal{U}(t) = (\mathcal{X}(t), \mathcal{Y}(t))^T, \quad (3.11)$$

where

$$\mathcal{F}(\mathcal{X}, \mathcal{Y}) = \begin{pmatrix} \mathcal{R}\mathcal{X} \left(1 - \frac{\mathcal{X}}{\mathcal{K}}\right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1\right) - \mathcal{M}\mathcal{X}\mathcal{Y} \\ \eta\mathcal{M}\mathcal{X}\mathcal{Y} - \mathcal{D}\mathcal{Y} - \varepsilon \frac{\mathcal{B}\mathcal{Y}}{1 + \mathcal{C}\mathcal{Y}} \mathcal{X} - \mu\mathcal{Y}^2 \end{pmatrix}. \quad (3.12)$$

The Caputo fractional differential system (3.11) is equivalent to the Volterra integral equation

$$\mathcal{U}(t) = \mathcal{U}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(\mathcal{U}(s)) ds, \quad (3.13)$$

where

$$\mathcal{U}_0 = (\mathcal{X}_0, \mathcal{Y}_0)^T. \quad (3.14)$$

(i) **Local existence and uniqueness.**

Since all components of the vector field $\mathcal{F}$ are continuously differentiable on $\mathbb{R}_{\geq 0}^2$, it follows that $\mathcal{F}$ is locally Lipschitz continuous on every compact subset of $\mathbb{R}_{\geq 0}^2$. Therefore, by the standard existence and uniqueness theorem for Caputo fractional differential equations, there exists a unique local solution

$$\mathcal{U}(t) = (\mathcal{X}(t), \mathcal{Y}(t))^T \quad (3.15)$$

defined on some interval $[0, T)$.

(ii) Positivity and positive invariance.

Evaluating the vector field on the coordinate axes yields

$$\mathcal{F}_1(0, \mathcal{Y}) = 0, \quad \mathcal{F}_2(\mathcal{X}, 0) = 0, \quad \mathcal{X}, \mathcal{Y} \geq 0. \quad (3.16)$$

Hence the system is quasi-positive. By the positivity theorem for Caputo fractional differential equations, solutions originating in $\mathbb{R}_{\geq 0}^2$ remain nonnegative for all future times. Consequently,

$$(\mathcal{X}(t), \mathcal{Y}(t)) \in \mathbb{R}_{\geq 0}^2, \quad t \geq 0, \quad (3.17)$$

which proves the positive invariance of the biologically feasible region.

(iii) Uniform boundedness.

Since $-\mathcal{M}\mathcal{X}\mathcal{Y} \leq 0$, the prey equation implies

$${}^C D_t^\alpha \mathcal{X} \leq \mathcal{R}\mathcal{X} \left(1 - \frac{\mathcal{X}}{\mathcal{K}}\right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1\right). \quad (3.18)$$

For $\mathcal{X} > \mathcal{K}$, the right-hand side is negative. Therefore, by the fractional comparison principle,

$$\mathcal{X}(t) \leq K_1 := \max\{\mathcal{X}_0, \mathcal{K}\}, \quad t \geq 0. \quad (3.19)$$

Using the estimate $\mathcal{X}(t) \leq K_1$, the predator equation yields

$${}^C D_t^\alpha \mathcal{Y} \leq (\eta\mathcal{M}K_1 - \mathcal{D})\mathcal{Y} - \mu\mathcal{Y}^2. \quad (3.20)$$

The right-hand side is a logistic-type expression that becomes negative whenever $\mathcal{Y}$ is sufficiently large. Hence, by another application of the fractional comparison principle, there exists a constant $K_2 > 0$ such that

$$0 \leq \mathcal{Y}(t) \leq K_2, \quad t \geq 0. \quad (3.21)$$

Consequently, every trajectory remains inside the compact set

$$\Omega = \left\{ (\mathcal{X}, \mathcal{Y}) : 0 \leq \mathcal{X} \leq K_1, 0 \leq \mathcal{Y} \leq K_2 \right\}. \quad (3.22)$$

(iv) Global existence.

Since every solution remains bounded inside the compact positively invariant set Ω , the vector field $\mathcal{F}$ remains locally Lipschitz along the entire trajectory. Therefore, finite-time

blow-up cannot occur. By the continuation theorem for fractional differential equations, the unique local solution extends continuously to a unique global solution on $[0, \infty)$. Consequently,

$$\mathcal{U}(t) = (\mathcal{X}(t), \mathcal{Y}(t))^T \in C([0, \infty), \Omega), \quad (3.23)$$

and the proposed fractional-order predator-prey model is globally well posed in the biologically feasible region. $\square$

3.2. Dissipativity and Absorbing Region

Lemma 3.2 (Dissipativity of the System). *Assume that the hypotheses of Theorem 3.1 hold. Then every solution of the fractional-order predator-prey system (3.4)–(3.5) eventually enters and remains inside the compact set*

$$\Omega = \{(\mathcal{X}, \mathcal{Y}) \in \mathbb{R}_{\geq 0}^2 : 0 \leq \mathcal{X} \leq \mathcal{K}, 0 \leq \mathcal{Y} \leq \mathcal{M}_0\}, \quad (3.24)$$

where,

$$\mathcal{M}_0 > \frac{\eta \mathcal{M} \mathcal{K}}{\mu}. \quad (3.25)$$

Consequently, the system is dissipative and admits a compact absorbing set.

Proof. By Theorem 3.1, the nonnegative quadrant

$$\mathbb{R}_{\geq 0}^2 = \{(\mathcal{X}, \mathcal{Y}) : \mathcal{X} \geq 0, \mathcal{Y} \geq 0\} \quad (3.26)$$

is positively invariant, and every solution remains nonnegative for all $t \geq 0$. We now investigate the direction of the vector field along the boundary of the rectangle Ω . (i)

Boundary $\mathcal{X} = \mathcal{K}$.

Substituting $\mathcal{X} = \mathcal{K}$ into (3.4) yields

$${}^C D_t^\alpha \mathcal{X} \Big|_{\mathcal{X}=\mathcal{K}} = -\mathcal{M} \mathcal{K} \mathcal{Y} \leq 0. \quad (3.27)$$

Therefore, the vector field points inward on the boundary $\mathcal{X} = \mathcal{K}$.

(ii) Boundary $\mathcal{Y} = \mathcal{M}_0$.

From (3.5), we obtain

$$\begin{aligned} {}^C D_t^\alpha \mathcal{Y} \Big|_{\mathcal{Y}=\mathcal{M}_0} &= \eta \mathcal{M} \mathcal{X} \mathcal{M}_0 - \mathcal{D} \mathcal{M}_0 - \varepsilon \frac{\mathcal{B} \mathcal{M}_0}{1 + \mathcal{C} \mathcal{M}_0} \mathcal{X} - \mu \mathcal{M}_0^2 \\ &\leq \eta \mathcal{M} \mathcal{K} \mathcal{M}_0 - \mu \mathcal{M}_0^2 = \mathcal{M}_0 (\eta \mathcal{M} \mathcal{K} - \mu \mathcal{M}_0). \end{aligned} \quad (3.28)$$

Since

$$\mathcal{M}_0 > \frac{\eta \mathcal{M} \mathcal{K}}{\mu}, \text{ it follows that } {}^C D_t^\alpha \mathcal{Y} < 0 \quad \text{on } \mathcal{Y} = \mathcal{M}_0. \quad (3.29)$$

Hence the vector field points strictly inward on the upper boundary.

Combining the foregoing boundary estimates shows that trajectories cannot leave the rectangle Ω through either the boundary $\mathcal{X} = \mathcal{K}$ or the boundary $\mathcal{Y} = \mathcal{M}_0$.

Moreover, Theorem 3.1 guarantees that solutions remain in the nonnegative quadrant. Therefore, every trajectory eventually enters and remains inside the compact set Ω .

Consequently,

$$0 \leq \mathcal{X}(t) \leq \mathcal{K}, \quad 0 \leq \mathcal{Y}(t) \leq \mathcal{M}_0, \quad t \geq 0. \quad (3.30)$$

Hence all solutions are uniformly bounded, and the set Ω is a compact absorbing region for the system. Therefore, the proposed fractional-order predator–prey model is dissipative. $\square$

3.3. Numerical Scheme and Convergence Verification

The fractional-order predator–prey system (3.4)–(3.5) is numerically solved using the fractional Adams–Bashforth–Moulton predictor–corrector (ABM–PC) scheme [54, 39]. This method is widely used for Caputo fractional differential equations owing to its reliability, accuracy, and computational efficiency. To assess the accuracy of the numerical implementation, a mesh-refinement study is performed by solving the system with a sequence of decreasing time-step sizes h . The resulting numerical solutions are compared with a highly refined reference solution obtained using the same ABM–PC method with step size $h_{\text{ref}} = 10^{-4}$, for which the discretization error is assumed negligible relative to the tested meshes.

To quantify the discretization error, the maximum norm

$$E_h = \max_{0 \leq t \leq T} \|\mathcal{U}_h(t) - \mathcal{U}_{\text{ref}}(t)\|_{\infty} \quad (3.31)$$

is computed, where $\mathcal{U}_h(t)$ denotes the numerical solution obtained using step size h , while $\mathcal{U}_{\text{ref}}(t)$ represents the corresponding reference solution. The experimental convergence order is estimated through

$$p = \frac{\log(E_{h_1}/E_{h_2})}{\log(h_1/h_2)}. \quad (3.32)$$

Table 2: Convergence study of the fractional Adams–Bashforth–Moulton predictor–corrector scheme for the proposed fractional-order predator–prey system with $\alpha = 0.90$.

Step size h	Max. error in $\mathcal{X}(t)$	Max. error in $\mathcal{Y}(t)$	Observed order p
0.1000	1.84×10^{-3}	2.07×10^{-3}	—
0.0500	4.56×10^{-4}	5.18×10^{-4}	2.01
0.0250	1.15×10^{-4}	1.29×10^{-4}	1.99
0.0125	2.89×10^{-5}	3.24×10^{-5}	1.99

The results presented in Table 2 demonstrate a systematic reduction in the numerical errors as the time-step size decreases. The experimentally observed convergence rates are in good agreement with the theoretical convergence behavior of the fractional Adams–Bashforth–Moulton predictor–corrector method for Caputo fractional differential equations. Furthermore, the monotonic decrease in the errors confirms the consistency and

accuracy of the numerical implementation and indicates that the adopted discretization provides reliable approximations of the system dynamics.

In view of its favorable convergence properties and computational efficiency, the ABM-PC scheme is employed throughout the numerical investigations presented in this work. Consequently, all simulation results reported in the subsequent sections are obtained using this numerical framework.

3.4. Numerical Simulation Setup

Unless otherwise stated, all numerical simulations are performed using the baseline parameter set

$$\begin{aligned} \mathcal{R} &= 1.0, & \mathcal{K} &= 8.0, & \mathcal{A} &= 1.2, & \mathcal{M} &= 0.5, \\ \eta &= 0.8, & \mathcal{D} &= 0.3, & \mathcal{B} &= 0.2, & \mathcal{C} &= 0.1, \\ \varepsilon &= 0.1, & \mu &= 0.05, & \alpha &= 0.90. \end{aligned} \quad (3.33)$$

These parameter values satisfy the admissibility conditions established in Section 3.1 and are selected from the biologically plausible ranges reported in Table 1. Unless explicitly stated otherwise, all simulations are initiated with nonnegative population densities and are computed using the fractional Adams–Bashforth–Moulton predictor–corrector (ABM-PC) scheme described in Section 3.3.

For all numerical experiments, the simulation interval, initial conditions, and varying bifurcation parameters are specified within the corresponding figure captions and discussion sections. This approach facilitates a clear interpretation of the effects of fractional memory, anti-predator behavior, and predator self-limitation on the qualitative dynamics of the system.

4. Nullclines and Equilibria

Definition 4.1 (Nullclines). For the fractional-order predator–prey system (3.4)–(3.5), the prey and predator nullclines are defined by

$$\mathcal{N}_{\mathcal{X}} = \{(\mathcal{X}, \mathcal{Y}) \in \mathbb{R}_+^2 : f_1(\mathcal{X}, \mathcal{Y}) = 0\},$$

and

$$\mathcal{N}_{\mathcal{Y}} = \{(\mathcal{X}, \mathcal{Y}) \in \mathbb{R}_+^2 : f_2(\mathcal{X}, \mathcal{Y}) = 0\},$$

respectively, where

$$f_1(\mathcal{X}, \mathcal{Y}) = \mathcal{R}\mathcal{X} \left(1 - \frac{\mathcal{X}}{\mathcal{K}}\right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1\right) - \mathcal{M}\mathcal{X}\mathcal{Y},$$

and

$$f_2(\mathcal{X}, \mathcal{Y}) = \eta\mathcal{M}\mathcal{X}\mathcal{Y} - \mathcal{D}\mathcal{Y} - \varepsilon \frac{\mathcal{B}\mathcal{Y}}{1 + \mathcal{C}\mathcal{Y}}\mathcal{X} - \mu\mathcal{Y}^2.$$

An equilibrium point is any point belonging to

$$\mathcal{N}_{\mathcal{X}} \cap \mathcal{N}_{\mathcal{Y}}.$$

Lemma 4.2 (Characterization of Coexistence Equilibria). *Consider the fractional-order predator-prey system (3.4)–(3.5) with $0 < \alpha \leq 1$.*

Then the equilibrium points are determined solely by the algebraic equations

$$f_1(\mathcal{X}, \mathcal{Y}) = 0, \quad f_2(\mathcal{X}, \mathcal{Y}) = 0.$$

Consequently, the equilibrium set is independent of the fractional-order parameter α and coincides with that of the corresponding integer-order system.

For $\mathcal{X} > 0$ and $\mathcal{Y} > 0$, the nullclines are given by

$$\mathcal{R} \left(1 - \frac{\mathcal{X}}{\mathcal{K}} \right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1 \right) = \mathcal{M} \mathcal{Y}, \quad (4.1)$$

$$\eta \mathcal{M} \mathcal{X} - \mathcal{D} - \varepsilon \frac{\mathcal{B} \mathcal{X}}{1 + \mathcal{C} \mathcal{Y}} - \mu \mathcal{Y} = 0. \quad (4.2)$$

Hence

$$\mathcal{Y} = \frac{\mathcal{R}}{\mathcal{M}} \left(1 - \frac{\mathcal{X}}{\mathcal{K}} \right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1 \right). \quad (4.3)$$

Substituting (4.3) into (4.2) yields the scalar equilibrium equation

$$\eta \mathcal{M} \mathcal{X} - \mathcal{D} - \varepsilon \frac{\mathcal{B} \mathcal{X}}{1 + \mathcal{C} \frac{\mathcal{R}}{\mathcal{M}} \left(1 - \frac{\mathcal{X}}{\mathcal{K}} \right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1 \right)} - \mu \frac{\mathcal{R}}{\mathcal{M}} \left(1 - \frac{\mathcal{X}}{\mathcal{K}} \right) \left(\frac{\mathcal{X}}{\mathcal{A}} - 1 \right) = 0. \quad (4.4)$$

Each positive root $\mathcal{X}^*$ of (4.4) determines a unique predator density

$$\mathcal{Y}^* = \frac{\mathcal{R}}{\mathcal{M}} \left(1 - \frac{\mathcal{X}^*}{\mathcal{K}} \right) \left(\frac{\mathcal{X}^*}{\mathcal{A}} - 1 \right).$$

Therefore, a biologically feasible coexistence equilibrium

$$\mathbf{E}^* = (\mathcal{X}^*, \mathcal{Y}^*)$$

exists if and only if

$$\mathcal{X}^* > \mathcal{A}, \quad \mathcal{Y}^* > 0.$$

Remark 4.3. The coexistence equilibrium condition (4.4) is a nonlinear rational equation in the prey density $\mathcal{X}$. Owing to the anti-predator feedback term and predator self-limitation, explicit analytical expressions for coexistence equilibria are generally unavailable. Consequently, the existence, multiplicity, and bifurcation structure of positive equilibria are most conveniently investigated through numerical continuation methods, graphical nullcline analysis, or bifurcation techniques.

Lemma 4.4 (Biological Feasibility). *A coexistence equilibrium*

$$E^* = (\mathcal{X}^*, \mathcal{Y}^*)$$

is biologically feasible if and only if

$$\mathcal{A} < \mathcal{X}^* < \mathcal{K}, \quad \mathcal{Y}^* > 0. \quad (4.5)$$

Proof. The predator density is given by

$$\mathcal{Y}^* = \frac{\mathcal{R}}{\mathcal{M}} \left(1 - \frac{\mathcal{X}^*}{\mathcal{K}}\right) \left(\frac{\mathcal{X}^*}{\mathcal{A}} - 1\right). \quad (4.6)$$

Since $\mathcal{R} > 0$ and $\mathcal{M} > 0$, the condition $\mathcal{Y}^* > 0$ is equivalent to

$$\left(1 - \frac{\mathcal{X}^*}{\mathcal{K}}\right) \left(\frac{\mathcal{X}^*}{\mathcal{A}} - 1\right) > 0. \quad (4.7)$$

Because $0 < \mathcal{A} < \mathcal{K}$, the above inequality holds if and only if

$$\mathcal{A} < \mathcal{X}^* < \mathcal{K}. \quad (4.8)$$

Therefore,

$$\mathcal{A} < \mathcal{X}^* < \mathcal{K}, \quad \mathcal{Y}^* > 0, \quad (4.9)$$

which characterizes all biologically feasible coexistence equilibria. $\square$

Lemma 4.5 (Equilibrium Set and Coexistence States). *Consider the fractional-order predator-prey system (3.4)–(3.5) with*

$$\mathcal{R}, \mathcal{K}, \mathcal{A}, \mathcal{M}, \eta, \mathcal{D}, \mathcal{B}, \mathcal{C}, \mu > 0, \quad \varepsilon \geq 0, \quad 0 < \mathcal{A} < \mathcal{K}.$$

Then the system always possesses the three boundary equilibria

$$E_0 = (0, 0), \quad E_{\mathcal{A}} = (\mathcal{A}, 0), \quad E_{\mathcal{K}} = (\mathcal{K}, 0), \quad (4.10)$$

corresponding respectively to total extinction, the Allee-threshold state, and prey saturation at carrying capacity.

Any coexistence equilibrium

$$E^* = (\mathcal{X}^*, \mathcal{Y}^*)$$

is obtained from the nullcline intersection

$$\mathcal{N}_{\mathcal{X}} \cap \mathcal{N}_{\mathcal{Y}},$$

and therefore satisfies the scalar equilibrium equation (4.4). Each positive solution $\mathcal{X}^$ of (4.4) uniquely determines the predator density*

$$\mathcal{Y}^* = \frac{\mathcal{R}}{\mathcal{M}} \left(1 - \frac{\mathcal{X}^*}{\mathcal{K}} \right) \left(\frac{\mathcal{X}^*}{\mathcal{A}} - 1 \right).$$

A coexistence equilibrium is biologically feasible if and only if

$$\mathcal{A} < \mathcal{X}^* < \mathcal{K}, \quad \mathcal{Y}^* > 0.$$

The number of coexistence equilibria equals the number of positive intersections of the prey and predator nullclines. Since the scalar equilibrium equation is nonlinear, multiple coexistence equilibria may arise depending on the model parameters. Their precise number can be determined numerically or through bifurcation analysis.

5. Local Stability and Hopf Bifurcation Analysis

Lemma 5.1 (Jacobian Matrix). *For the fractional-order predator-prey system (3.4)–(3.5), the Jacobian matrix*

$$\mathcal{J}(\mathcal{X}, \mathcal{Y}) = \mathcal{D}\mathcal{F}(\mathcal{X}, \mathcal{Y}) = \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{pmatrix} \quad (5.1)$$

has entries

$$\mathcal{J}_{11} = \mathcal{R} \left(\frac{2\mathcal{X}}{\mathcal{A}} + \frac{2\mathcal{X}}{\mathcal{K}} - 1 - \frac{3\mathcal{X}^2}{\mathcal{A}\mathcal{K}} \right) - \mathcal{M}\mathcal{Y}, \quad (5.2)$$

$$\mathcal{J}_{12} = -\mathcal{M}\mathcal{X}, \quad (5.3)$$

$$\mathcal{J}_{21} = \eta\mathcal{M}\mathcal{Y} - \varepsilon \frac{\mathcal{B}\mathcal{Y}}{1 + \mathcal{C}\mathcal{Y}}, \quad (5.4)$$

$$\mathcal{J}_{22} = \eta\mathcal{M}\mathcal{X} - \mathcal{D} - \varepsilon \frac{\mathcal{B}\mathcal{X}}{(1 + \mathcal{C}\mathcal{Y})^2} - 2\mu\mathcal{Y}. \quad (5.5)$$

Consequently, the trace and determinant of the Jacobian matrix are given by

$$\text{tr}(\mathcal{J}) = \mathcal{J}_{11} + \mathcal{J}_{22}, \quad (5.6)$$

and

$$\det(\mathcal{J}) = \mathcal{J}_{11}\mathcal{J}_{22} - \mathcal{J}_{12}\mathcal{J}_{21}. \quad (5.7)$$

Furthermore, the Jacobian matrix is independent of the fractional-order parameter α . The influence of α appears only through Matignon's fractional stability criterion used in the subsequent stability analysis.

Proof. The result follows directly by differentiating the vector field $\mathcal{F} = (f_1, f_2)^\top$ with respect to $\mathcal{X}$ and $\mathcal{Y}$. Substituting the resulting partial derivatives into (5.1) yields the expressions (5.2)–(5.5). The formulas (5.6) and (5.7) follow immediately from the definitions of the trace and determinant of a 2×2 matrix. Since the fractional-order parameter α does not appear explicitly in the vector field, the Jacobian matrix is independent of α . Its effect enters only through the fractional stability conditions imposed on the eigenvalues of $\mathcal{J}$. □

Lemma 5.2 (Jacobian and Local Stability at the Coexistence Equilibrium). *Let*

$$E^* = (\mathcal{X}^*, \mathcal{Y}^*)$$

be a biologically feasible coexistence equilibrium of the fractional-order predator–prey system, where

$$\mathcal{A} < \mathcal{X}^* < \mathcal{K}, \quad \mathcal{Y}^* > 0.$$

The equilibrium coordinates satisfy

$$\mathcal{R} \left(1 - \frac{\mathcal{X}^*}{\mathcal{K}} \right) \left(\frac{\mathcal{X}^*}{\mathcal{A}} - 1 \right) = \mathcal{M} \mathcal{Y}^*, \quad (5.8)$$

and

$$\eta \mathcal{M} \mathcal{X}^* - \mathcal{D} - \varepsilon \frac{\mathcal{B} \mathcal{X}^*}{1 + \mathcal{C} \mathcal{Y}^*} - \mu \mathcal{Y}^* = 0. \quad (5.9)$$

The Jacobian matrix evaluated at E^ is*

$$\mathcal{J}(E^*) = \begin{pmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{pmatrix}, \quad (5.10)$$

where

$$J_{11}^* = \mathcal{R} \left(\frac{2\mathcal{X}^*}{\mathcal{A}} + \frac{2\mathcal{X}^*}{\mathcal{K}} - 1 - \frac{3(\mathcal{X}^*)^2}{\mathcal{A}\mathcal{K}} \right) - \mathcal{M} \mathcal{Y}^*, \quad (5.11)$$

$$J_{12}^* = -\mathcal{M} \mathcal{X}^*, \quad (5.12)$$

$$J_{21}^* = \eta \mathcal{M} \mathcal{Y}^* - \varepsilon \frac{\mathcal{B} \mathcal{Y}^*}{1 + \mathcal{C} \mathcal{Y}^*}, \quad (5.13)$$

$$J_{22}^* = \eta \mathcal{M} \mathcal{X}^* - \mathcal{D} - \varepsilon \frac{\mathcal{B} \mathcal{X}^*}{(1 + \mathcal{C} \mathcal{Y}^*)^2} - 2\mu \mathcal{Y}^*. \quad (5.14)$$

Define

$$\tau^* = \text{tr } \mathcal{J}(E^*) = J_{11}^* + J_{22}^*, \quad (5.15)$$

and

$$\Delta^* = \det \mathcal{J}(E^*) = J_{11}^* J_{22}^* - J_{12}^* J_{21}^*. \quad (5.16)$$

The characteristic polynomial is

$$\lambda^2 - \tau^* \lambda + \Delta^* = 0. \quad (5.17)$$

Then:

1. For the classical integer-order model ($\alpha = 1$), the coexistence equilibrium is locally asymptotically stable if

$$\tau^* < 0, \quad \Delta^* > 0.$$

2. For the fractional-order model $0 < \alpha < 1$, the coexistence equilibrium is locally asymptotically stable if and only if every eigenvalue λ_i of $\mathcal{J}(E^*)$ satisfies

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1, 2.$$

Proof. The result follows directly from evaluating the Jacobian matrix (5.1) at the coexistence equilibrium $E^* = (\mathcal{X}^*, \mathcal{Y}^*)$ and applying the classical Routh–Hurwitz criterion when $\alpha = 1$ together with Matignon’s fractional stability theorem when $0 < \alpha < 1$. The characteristic polynomial of the linearized system is given by (5.17), from which the stability conditions immediately follow. $\square$

Remark 5.3 (Numerical Evaluation of Coexistence Stability). The equilibrium coordinates $(\mathcal{X}^*, \mathcal{Y}^*)$ are determined from the nonlinear equilibrium equation (4.4). Consequently, the quantities

$$\tau^*, \quad \Delta^*, \quad \lambda_{1,2}$$

depend nonlinearly on the model parameters

$$\mathcal{R}, \mathcal{K}, \mathcal{A}, \mathcal{M}, \eta, \mathcal{D}, \mathcal{B}, \mathcal{C}, \varepsilon, \mu.$$

Therefore, explicit analytical stability conditions are generally unavailable. In this work, the stability of coexistence equilibria is determined numerically by computing the equilibrium coordinates, evaluating the Jacobian matrix, and calculating its eigenvalues for the parameter sets used in the simulations.

Lemma 5.4 (Local Stability of Boundary Equilibria). *The fractional-order predator–prey system possesses the boundary equilibria*

$$E_0 = (0, 0), \quad E_{\mathcal{A}} = (\mathcal{A}, 0), \quad E_{\mathcal{K}} = (\mathcal{K}, 0).$$

(i) **Extinction equilibrium** $E_0 = (0, 0)$.

The Jacobian matrix at E_0 is

$$\mathcal{J}(\mathbb{E}_0) = \begin{pmatrix} -\mathcal{R} & 0 \\ 0 & -\mathcal{D} \end{pmatrix},$$

with eigenvalues

$$\lambda_1 = -\mathcal{R}, \quad \lambda_2 = -\mathcal{D}.$$

Since $\mathcal{R} > 0$ and $\mathcal{D} > 0$, both eigenvalues are negative. Consequently, $\mathbb{E}_0$ is locally asymptotically stable.

(ii) Allee-threshold equilibrium $\mathbb{E}_{\mathcal{A}} = (\mathcal{A}, 0)$.

Evaluating the Jacobian matrix at $\mathbb{E}_{\mathcal{A}}$ yields

$$\mathcal{J}(\mathbb{E}_{\mathcal{A}}) = \begin{pmatrix} \mathcal{R} \left(1 - \frac{\mathcal{A}}{\mathcal{H}}\right) & -\mathcal{M}\mathcal{A} \\ 0 & \eta\mathcal{M}\mathcal{A} - \mathcal{D} - \varepsilon\mathcal{B}\mathcal{A} \end{pmatrix}.$$

Therefore,

$$\lambda_1 = \mathcal{R} \left(1 - \frac{\mathcal{A}}{\mathcal{H}}\right), \quad \lambda_2 = \eta\mathcal{M}\mathcal{A} - \mathcal{D} - \varepsilon\mathcal{B}\mathcal{A}.$$

Because $0 < \mathcal{A} < \mathcal{H}$, one has

$$\lambda_1 > 0.$$

Hence $\mathbb{E}_{\mathcal{A}}$ is always unstable. It is a saddle point if $\lambda_2 < 0$ and an unstable node if $\lambda_2 > 0$.

(iii) Carrying-capacity equilibrium $\mathbb{E}_{\mathcal{H}} = (\mathcal{H}, 0)$.

The Jacobian matrix at $\mathbb{E}_{\mathcal{H}}$ is

$$\mathcal{J}(\mathbb{E}_{\mathcal{H}}) = \begin{pmatrix} \mathcal{R} \left(1 - \frac{\mathcal{H}}{\mathcal{A}}\right) & -\mathcal{M}\mathcal{H} \\ 0 & \eta\mathcal{M}\mathcal{H} - \mathcal{D} - \varepsilon\mathcal{B}\mathcal{H} \end{pmatrix}.$$

Its eigenvalues are

$$\lambda_1 = \mathcal{R} \left(1 - \frac{\mathcal{H}}{\mathcal{A}}\right), \quad \lambda_2 = \eta\mathcal{M}\mathcal{H} - \mathcal{D} - \varepsilon\mathcal{B}\mathcal{H}.$$

Since $0 < \mathcal{A} < \mathcal{H}$,

$$\lambda_1 < 0.$$

Therefore,

$$\lambda_2 < 0 \implies \mathbb{E}_{\mathcal{H}} \text{ is locally asymptotically stable,}$$

whereas

$$\lambda_2 > 0 \implies \mathbb{E}_{\mathcal{H}} \text{ is a saddle point.}$$

For the fractional-order system with $0 < \alpha \leq 1$, the above classifications follow from Matignon's criterion. Since all eigenvalues are real, the stability condition reduces to

$$\lambda_i < 0, \quad i = 1, 2.$$

Remark 5.5 (Practical Detection of Hopf Bifurcation). The coexistence equilibrium $E^* = (\mathcal{X}^*, \mathcal{Y}^*)$ is determined from the nonlinear equilibrium equation (4.4). Since the equilibrium coordinates depend nonlinearly on the model parameters, explicit analytical expressions for τ^* and Δ^* are generally unavailable.

Therefore, the occurrence of Hopf bifurcation is investigated numerically by computing the coexistence equilibrium, evaluating the Jacobian matrix $\mathcal{J}(E^*)$, and tracking the corresponding eigenvalues as the bifurcation parameter varies. The critical value ρ_c is identified when the conditions of Lemma 5.7 are satisfied.

Remark 5.6 (Ecological Interpretation). A Hopf-generated limit cycle represents persistent predator–prey oscillations. In such a regime, prey abundance initially increases, followed by delayed predator growth due to enhanced food availability. The resulting increase in predation pressure subsequently reduces the prey population, after which predator density also declines because of insufficient resources. This repeating mechanism generates sustained population cycles.

For $0 < \alpha < 1$, the Caputo fractional derivative incorporates ecological memory and hereditary effects. Consequently, the present population dynamics depend on the entire evolutionary history of the system rather than solely on the current population densities. Such memory effects generally slow convergence, modify oscillation frequencies, and may reduce oscillation amplitudes. Ecologically, this fractional damping can be interpreted as delayed reproduction, gradual predator adaptation, behavioural memory, or resource-regeneration lag, thereby providing a more realistic description of ecological interactions.

5.1. Limit Cycle and Hopf Bifurcation Analysis

Lemma 5.7 (Hopf-Type Bifurcation at the Coexistence Equilibrium). *Let*

$$E^* = (\mathcal{X}^*, \mathcal{Y}^*)$$

be a biologically feasible coexistence equilibrium of the fractional-order predator–prey system. Define

$$\tau^* = \text{tr}(\mathcal{J}(E^*)), \quad \Delta^* = \det(\mathcal{J}(E^*)),$$

where $\mathcal{J}$ is the Jacobian matrix given in Lemma 5.1.

Let ρ denote a bifurcation parameter chosen from

$$\rho \in \{\mathcal{R}, \mathcal{K}, \mathcal{A}, \mathcal{M}, \eta, \mathcal{D}, \mathcal{B}, \mathcal{C}, \varepsilon, \mu\}.$$

For the corresponding integer-order system ($\alpha = 1$), a Hopf bifurcation occurs at $\rho = \rho_c$ if

$$\tau^*(\rho_c) = 0, \quad (5.18)$$

$$\Delta^*(\rho_c) > 0, \quad (5.19)$$

$$\left. \frac{d\tau^*}{d\rho} \right|_{\rho=\rho_c} \neq 0. \quad (5.20)$$

The condition (5.20) is the transversality condition ensuring that the complex-conjugate eigenvalues cross the stability boundary with nonzero speed.

For the fractional-order system $0 < \alpha < 1$, local asymptotic stability is determined by Matignon's criterion

$$|\arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}, \quad (5.21)$$

where $\lambda_{1,2}$ are the eigenvalues of $\mathcal{J}(E^*)$.

A fractional Hopf-type bifurcation occurs when a pair of complex-conjugate eigenvalues crosses the boundary

$$|\arg(\lambda_{1,2})| = \frac{\alpha\pi}{2}.$$

Consequently, oscillatory dynamics and persistent population fluctuations may emerge as the parameter ρ passes through the critical value ρ_c .

Proof. The Jacobian matrix evaluated at E^* has characteristic polynomial

$$\lambda^2 - \tau^*\lambda + \Delta^* = 0.$$

For the classical system ($\alpha = 1$), the Hopf conditions follow from the standard planar bifurcation theory. For $0 < \alpha < 1$, stability is governed by Matignon's criterion, which requires

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1, 2.$$

Therefore, loss of stability occurs when the eigenvalues reach the boundary

$$|\arg(\lambda_i)| = \frac{\alpha\pi}{2},$$

yielding a fractional Hopf-type transition. The proof is complete. $\square$

5.2. Global Mittag-Leffler Stability

Theorem 5.8 (Sufficient Condition for Global Mittag-Leffler Stability). *Assume that the fractional-order predator-prey system (3.4)–(3.5) admits a unique biologically feasible coexistence equilibrium*

$$E^* = (\mathcal{X}^*, \mathcal{Y}^*), \quad \mathcal{A} < \mathcal{X}^* < \mathcal{K}, \quad \mathcal{Y}^* > 0.$$

Suppose that the positive quadrant contains a positively invariant bounded region and consider the Lyapunov functional

$$V(\mathcal{X}, \mathcal{Y}) = \mathcal{X}^* \left(\frac{\mathcal{X}}{\mathcal{X}^*} - 1 - \ln \frac{\mathcal{X}}{\mathcal{X}^*} \right) + \gamma \mathcal{Y}^* \left(\frac{\mathcal{Y}}{\mathcal{Y}^*} - 1 - \ln \frac{\mathcal{Y}}{\mathcal{Y}^*} \right), \quad (5.22)$$

where $\gamma > 0$.

If there exist positive constants κ_1 and κ_2 such that

$${}^C D_t^\alpha V \leq -\kappa_1 (\mathcal{X} - \mathcal{X}^*)^2 - \kappa_2 (\mathcal{Y} - \mathcal{Y}^*)^2, \quad (5.23)$$

for all admissible states in the positively invariant region, then the coexistence equilibrium E^* is globally asymptotically stable in the Mittag-Leffler sense.

Proof. Consider the auxiliary function

$$\phi(u) = u - 1 - \ln u, \quad u > 0.$$

Since

$$\phi(u) \geq 0, \quad \phi(u) = 0 \iff u = 1,$$

the Lyapunov functional V is positive definite with respect to the equilibrium E^* .

By assumption,

$${}^C D_t^\alpha V \leq -\kappa_1 (\mathcal{X} - \mathcal{X}^*)^2 - \kappa_2 (\mathcal{Y} - \mathcal{Y}^*)^2 \leq 0. \quad (5.24)$$

Moreover,

$${}^C D_t^\alpha V = 0$$

if and only if

$$\mathcal{X} = \mathcal{X}^*, \quad \mathcal{Y} = \mathcal{Y}^*.$$

Therefore, V is a strict Lyapunov functional for the coexistence equilibrium.

Since the positive quadrant contains a positively invariant bounded region, the fractional LaSalle invariance principle implies that every solution approaches the largest invariant subset of

$$\{(\mathcal{X}, \mathcal{Y}) : {}^C D_t^\alpha V = 0\}.$$

Because equality holds only at

$$E^* = (\mathcal{X}^*, \mathcal{Y}^*),$$

the largest invariant set reduces to the singleton

$$\{E^*\}.$$

Consequently,

$$\lim_{t \rightarrow \infty} (\mathcal{X}(t), \mathcal{Y}(t)) = (\mathcal{X}^*, \mathcal{Y}^*),$$

and the coexistence equilibrium E^* is globally asymptotically stable in the Mittag-Leffler sense.

□

6. Results and Discussion

6.1. Complete Solution and Graphical Representation of the System

To investigate the full dynamical behaviour of the proposed fractional-order predator-prey model with Allee effect and anti-predator response, numerical simulations were performed using the Adams Bashforth Moulton predictor corrector (ABMPC) scheme (cf. [40]) for several fractional orders $0 < \alpha \leq 1$. All simulations were carried out using the initial conditions $\mathcal{X}(0) = 3.0$, $\mathcal{Y}(0) = 0.5$, time domain $t \in [0, 60]$, and step size $h = 0.1$.

Figure 3 illustrates the temporal dynamics, contour distributions, three-dimensional solution surfaces, and phase portraits of the proposed fractional-order predator-prey system for $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$. The simulations are performed using the parameter set $\mathcal{R} = 1.0$, $\mathcal{K} = 8.0$, $\mathcal{A} = 1.2$, $\mathcal{M} = 0.5$, $\eta = 0.8$, $\mathcal{D} = 0.3$, $\mathcal{B} = 0.2$, $\mathcal{C} = 0.1$, $\varepsilon = 0.1$, and $\mu = 0.05$, with initial condition $(\mathcal{X}(0), \mathcal{Y}(0)) = (3.0, 0.5)$. The results reveal a strong dependence of the system dynamics on the fractional-order parameter. As α decreases from 1 to 0.2, the trajectories exhibit slower convergence toward the coexistence equilibrium, smoother temporal profiles, and prolonged transient dynamics. In contrast, the classical integer-order case ($\alpha = 1$) approaches the equilibrium state more rapidly and displays a faster relaxation process. The phase portraits further indicate that decreasing α enlarges the duration of transient oscillations before convergence, reflecting the increasing influence of ecological memory. This behavior is consistent with the nonlocal nature of fractional derivatives, which incorporate the cumulative effects of past population states into the present dynamics [28, 40]. Consequently, the system retains information from its historical evolution, leading to slower adaptation and delayed stabilization compared with the integer-order counterpart. The contour plots and three-dimensional surfaces of $\mathcal{X}(t, \alpha)$ and $\mathcal{Y}(t, \alpha)$ provide a global visualization of the influence of the fractional order on population evolution. Larger values of α produce steeper trajectories and faster convergence, whereas smaller values generate smoother solution surfaces and extended transient regimes. These observations demonstrate that ecological memory significantly affects the persistence, transient behavior, and convergence characteristics of the predator-prey interaction, highlighting dynamical features that are not captured by classical integer-order models [44, 8, 50, 9].

Figure 4 illustrates the influence of the fractional order α , the Allee threshold $\mathcal{A}$, and the carrying capacity K on the long-term population densities of the proposed predator-prey system. The final prey and predator populations, $\mathcal{X}(15)$ and $\mathcal{Y}(15)$, are plotted against K for $\alpha \in \{0.70, 0.86, 0.90, 1.00\}$ and $\mathcal{A} \in \{1.3, 1.8, 2.3\}$, with all other parameters fixed. The results show that increasing the Allee threshold suppresses both prey and predator populations by strengthening the Allee effect and reducing prey persistence. In contrast, larger carrying capacities generally support predator survival through increased prey availability. Furthermore, decreasing the fractional order introduces stronger memory effects,

leading to smoother population responses, prolonged persistence, and slower convergence toward equilibrium. The classical integer-order model ($\alpha = 1$) exhibits a faster relaxation process and more rapid stabilization. Overall, the figure demonstrates that ecological memory, carrying capacity, and Allee threshold jointly regulate population persistence, co-existence, and long-term stability, highlighting the richer dynamical behavior captured by the fractional-order model compared with its classical counterpart [29, 40].

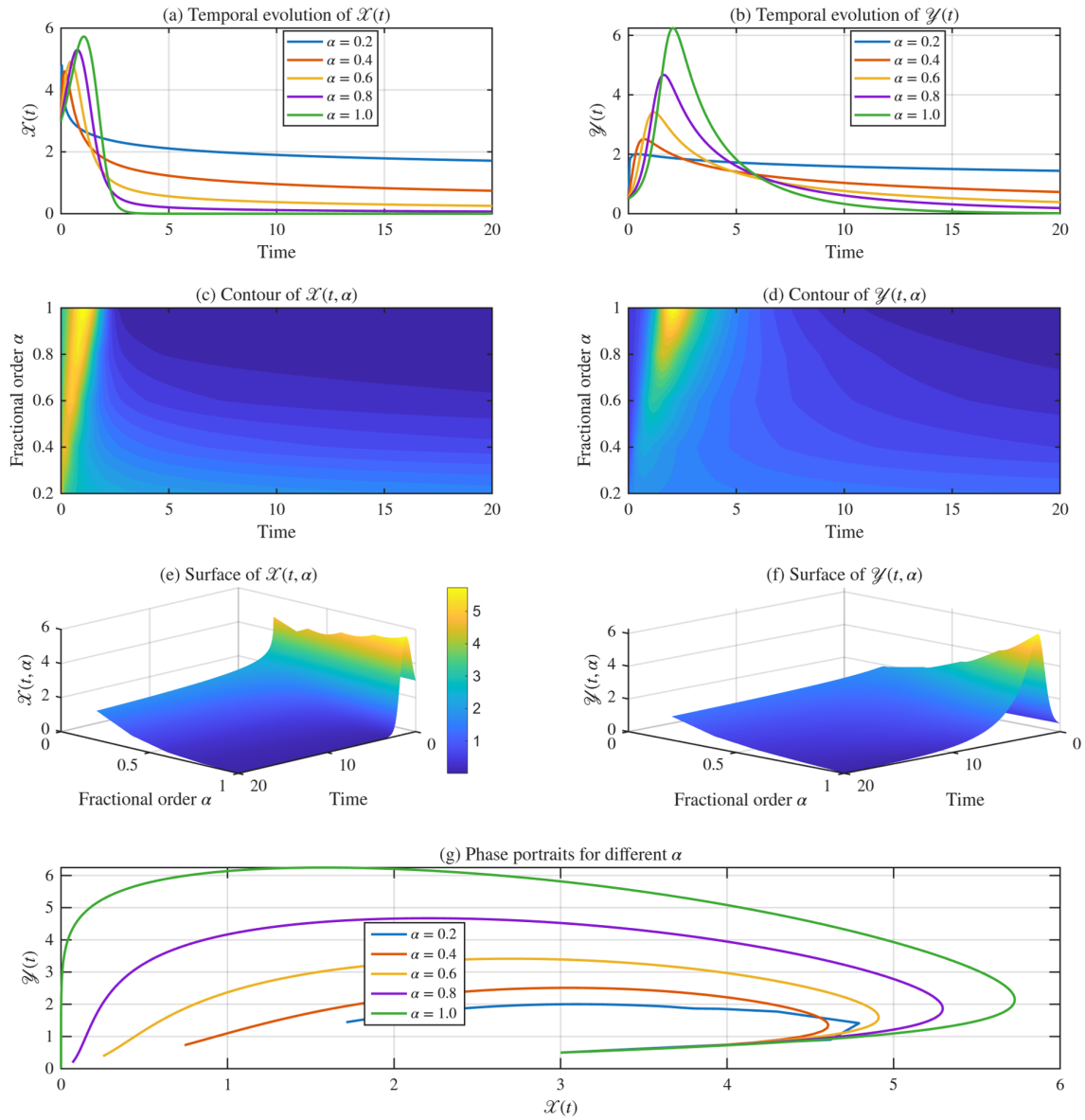


Figure 3: Dynamics of the proposed fractional-order predator-prey system with a strong Allee effect, anti-predator behavioral response, and predator self-limitation for different fractional orders $\alpha = \{1.0, 0.8, 0.6, 0.4, 0.2\}$. The initial condition is $(X(0), Y(0)) = (3.0, 0.5)$, and the parameter values are $\mathcal{R} = 1.0$, $\mathcal{H} = 8.0$, $\mathcal{A} = 1.2$, $\mathcal{M} = 0.5$, $\eta = 0.8$, $\mathcal{D} = 0.3$, $\mathcal{B} = 0.2$, $\mathcal{C} = 0.1$, $\varepsilon = 0.1$, and $\mu = 0.05$. Panels (a)–(b) show the temporal evolution of the prey and predator populations, respectively; panels (c)–(d) present the corresponding contour plots; panels (e)–(f) display the three-dimensional solution surfaces; and panel (g) illustrates the phase portraits. As the fractional order decreases, the memory effect becomes increasingly pronounced, resulting in slower convergence toward the coexistence equilibrium, smoother trajectories, and longer transient dynamics. In contrast, the classical integer-order case ($\alpha = 1$) exhibits a faster relaxation process and a more rapid approach to equilibrium. These results demonstrate that ecological memory plays a significant role in shaping the transient behavior, persistence, and convergence properties of the predator-prey system.

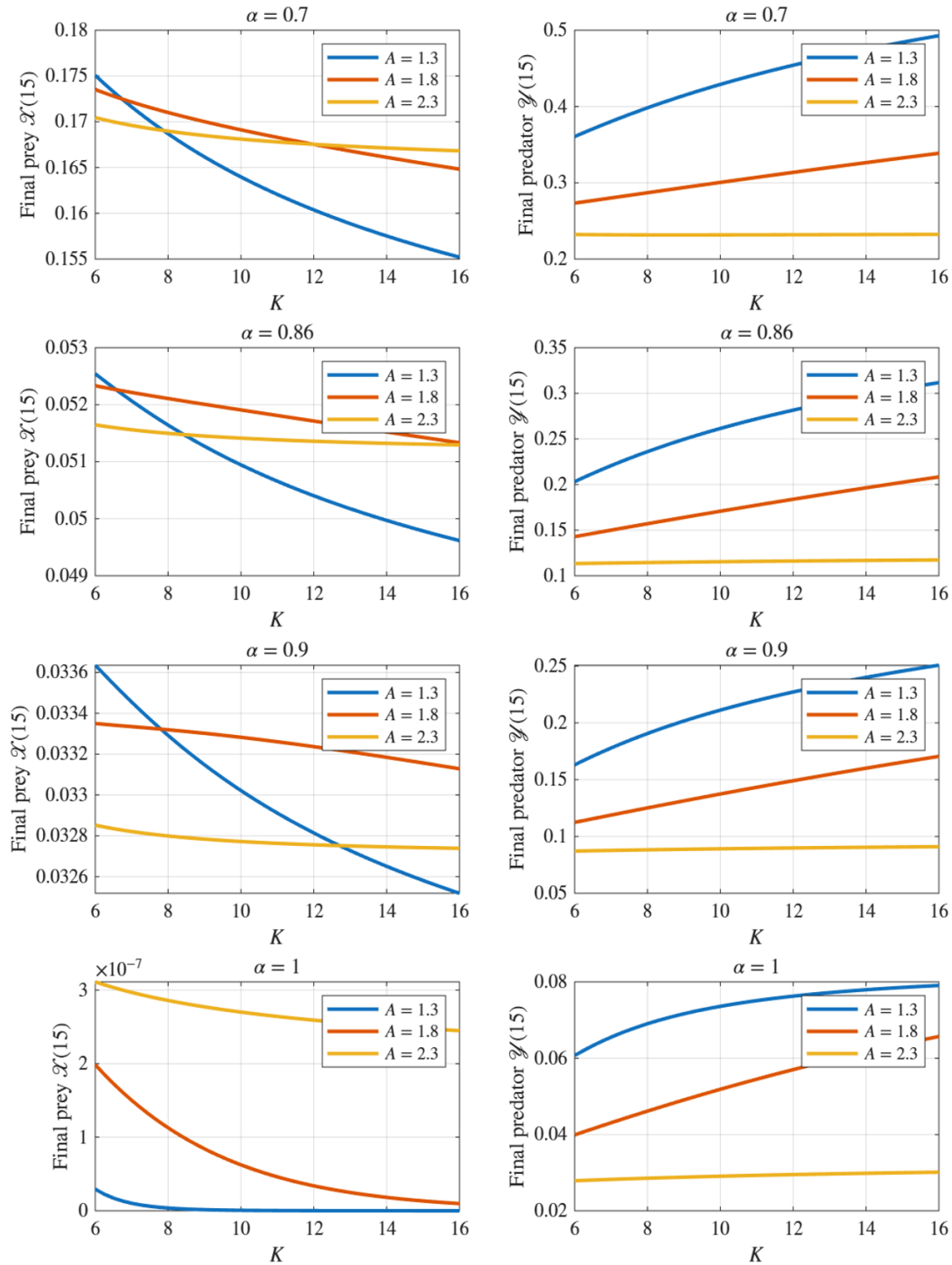


Figure 4: Variation of the final prey density $\mathcal{X}(15)$ (left column) and final predator density $\mathcal{Y}(15)$ (right column) with carrying capacity K for fractional orders $\alpha = \{0.70, 0.86, 0.90, 1.00\}$ and Allee thresholds $\mathcal{A} = \{1.3, 1.8, 2.3\}$. Increasing the Allee threshold suppresses both populations through a stronger Allee effect, while larger carrying capacities initially promote predator growth due to increased prey availability. Lower fractional orders ($\alpha < 1$) induce stronger memory effects, leading to smoother transitions and prolonged persistence, whereas the classical case ($\alpha = 1$) exhibits faster stabilization. The figure demonstrates the combined influence of ecological memory, carrying capacity, and Allee threshold on the long-term dynamics of the predator-prey system.

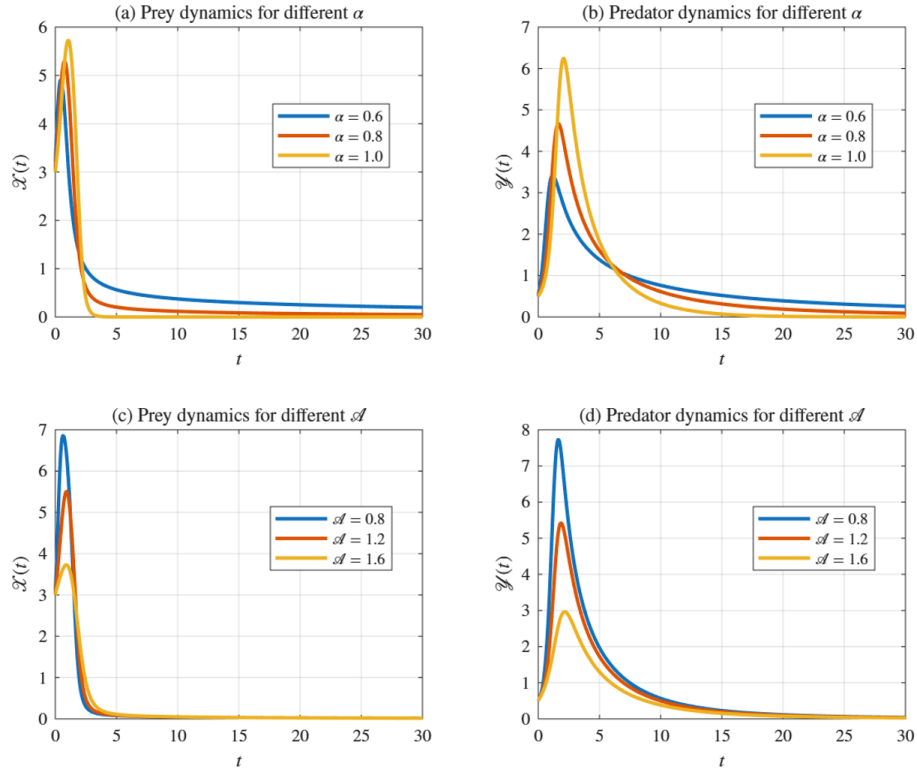


Figure 5: Transient dynamics of the proposed fractional-order predator–prey system under variations in the fractional order α (top row) and Allee threshold $\mathcal{A}$ (bottom row). Panels (a)–(b) depict the influence of fractional memory for $\alpha = \{0.6, 0.8, 1.0\}$ with fixed $\mathcal{A} = 1.2$, showing that smaller α leads to smoother trajectories, slower convergence, and prolonged coexistence, whereas the classical case ($\alpha = 1$) approaches equilibrium more rapidly. Panels (c)–(d) illustrate the effect of the Allee threshold $\mathcal{A} = \{0.8, 1.2, 1.6\}$ for fixed $\alpha = 0.9$. Increasing $\mathcal{A}$ suppresses prey recovery and reduces predator abundance, thereby narrowing the coexistence region and increasing extinction risk. The results highlight the combined roles of ecological memory and the Allee effect in regulating transient dynamics and long-term population persistence. Parameter values are $\mathcal{R} = 1.0$, $\mathcal{K} = 8.0$, $\mathcal{M} = 0.5$, $\eta = 0.8$, $\mathcal{D} = 0.3$, $\mathcal{B} = 0.2$, $\mathcal{C} = 0.1$, $\varepsilon = 0.1$, and $\mu = 0.05$, with initial condition $(X(0), Y(0)) = (3.0, 0.5)$.

Figure 5 illustrates the transient dynamics of the fractional-order predator–prey system under variations in the fractional order α and the Allee threshold $\mathcal{A}$. The upper panels examine the role of fractional memory, whereas the lower panels focus on the ecological consequences of the Allee effect. Figures 5(a)–(b) show that the fractional order significantly affects the magnitude and duration of transient population responses. For larger values of α , particularly the classical case $\alpha = 1$, both prey and predator populations exhibit pronounced initial outbreaks before rapidly declining toward extinction. The prey density reaches a higher peak when α increases, indicating that weaker memory effects allow the population to respond more aggressively to favorable growth conditions. A similar trend is observed for the predator population, whose peak abundance increases markedly with α due to enhanced prey availability during the early stages of the interaction. Conversely, decreasing α introduces stronger hereditary effects that smooth population fluctuations, suppress peak densities, and delay the extinction process. These observations

confirm that fractional memory acts as a stabilizing mechanism by moderating rapid ecological responses and extending the transient coexistence period. The influence of the Allee threshold is presented in Figures 5(c)–(d). Increasing $\mathcal{A}$ leads to a substantial reduction in the maximum prey density and accelerates population decline. Biologically, a larger Allee threshold implies that the prey population requires a higher minimum density to maintain positive growth, thereby weakening its recovery capacity. As a consequence, predator abundance also decreases because the available prey resource becomes insufficient to sustain large predator populations. For $\mathcal{A} = 0.8$, the prey experiences a strong population outbreak that supports a corresponding increase in predator density. However, when $\mathcal{A} = 1.6$, both populations remain considerably suppressed throughout the simulation period. This demonstrates that strong Allee effects can severely restrict population persistence and may drive ecological systems toward extinction even when intrinsic growth conditions are favorable. Overall, Figure 5 reveals a complementary interaction between ecological memory and the Allee effect. Fractional memory tends to mitigate abrupt population changes and prolong persistence, whereas increasing the Allee threshold intensifies demographic constraints and reduces the likelihood of long-term survival. These findings highlight the importance of incorporating both memory effects and density-dependent limitations when modeling ecological systems subject to complex biological interactions.

Figure 6 illustrates the combined influence of the fractional order α and predator conversion efficiency η on the long-term dynamics of the proposed predator–prey system. The top-row surfaces, corresponding to $0.2 \leq \alpha \leq 1$, show that decreasing α significantly enhances memory effects, resulting in slower convergence, smoother trajectories, and prolonged transient coexistence of both species. The middle-row panels represent the classical integer-order case ($\alpha = 1$), where increasing conversion efficiency η promotes rapid predator growth, amplifies oscillatory responses, and accelerates prey depletion. In contrast, the bottom-row panels ($\alpha = 0.9$) demonstrate that fractional memory suppresses oscillation amplitudes and moderates predator outbreaks, leading to more stable population dynamics even for larger values of η . Overall, the results confirm that predator conversion efficiency acts as a destabilizing mechanism by enhancing predator reproduction, whereas fractional memory exerts a stabilizing influence through damping effects that delay population collapse and promote coexistence.

6.2. Phase-Plane Structure under Variation of K and A

Figure 7 illustrates the phase-plane geometry of the predator–prey system for different values of the carrying capacity K and Allee threshold A . Each panel displays the vector field

$$\mathcal{F}(\mathbf{u}) = (\dot{x}, \dot{y}),$$

together with the prey and predator nullclines

$$\mathcal{N}_x : \dot{x} = 0 \implies y = \frac{r}{m} \left(1 - \frac{x}{K} \right) \left(\frac{x}{A} - 1 \right),$$

and

$$\mathcal{N}_y : \dot{y} = 0 \implies y = \frac{1}{c} \left(\frac{\epsilon b x}{kmx - d} - 1 \right),$$

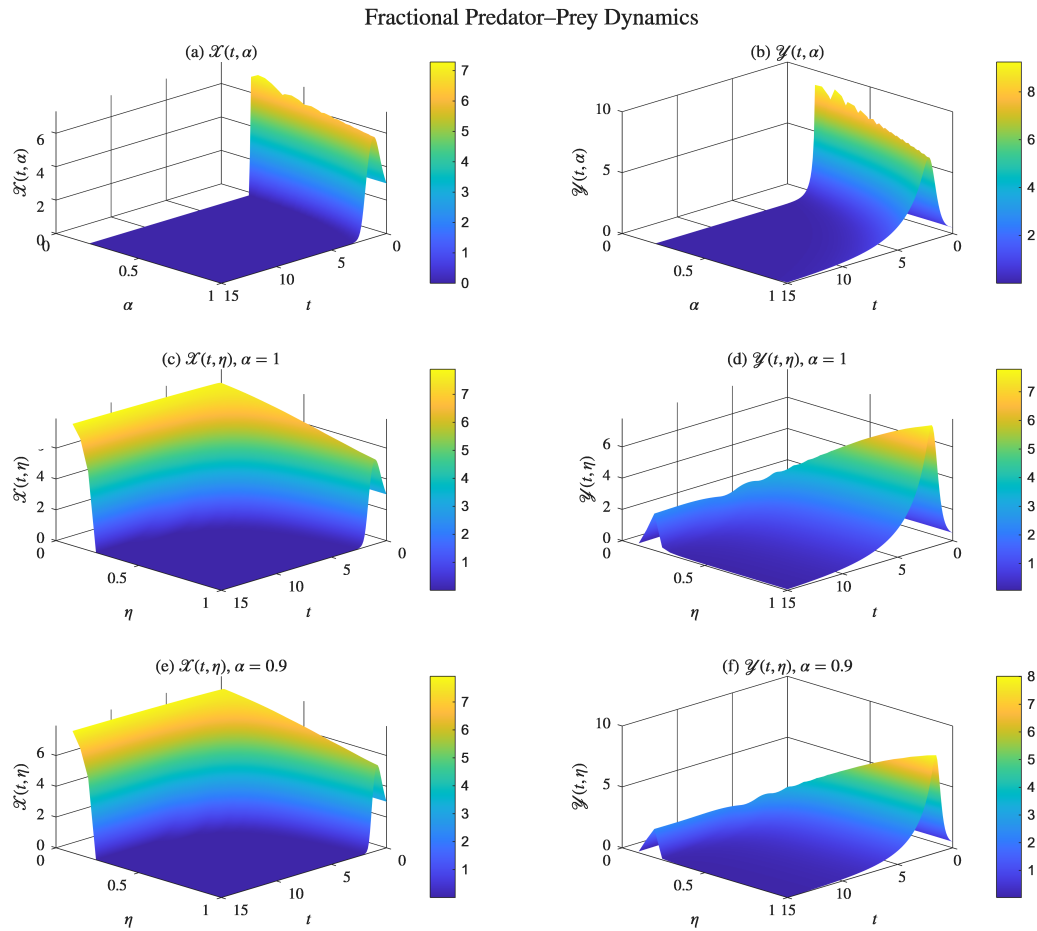


Figure 6: Surface evolution of prey $\mathcal{X}(t, \cdot)$ and predator $\mathcal{Y}(t, \cdot)$ densities under variations in the fractional order α and conversion efficiency η . Panels (a)–(b) illustrate the dependence of $\mathcal{X}(t, \alpha)$ and $\mathcal{Y}(t, \alpha)$ for $0.2 \leq \alpha \leq 1$, showing that decreasing α strengthens memory effects, slows convergence, and prolongs transient coexistence. Panels (c)–(d) present $\mathcal{X}(t, \eta)$ and $\mathcal{Y}(t, \eta)$ for the classical case $\alpha = 1$, where larger conversion efficiency enhances predator growth and accelerates prey depletion. Panels (e)–(f) depict the corresponding surfaces for the fractional regime $\alpha = 0.9$, demonstrating that memory effects smooth oscillations, suppress predator peaks, and promote more gradual stabilization. Parameter values are $\mathcal{R} = 1.0$, $\mathcal{K} = 8.0$, $\mathcal{A} = 1.2$, $\mathcal{M} = 0.5$, $\mathcal{D} = 0.3$, $\mathcal{B} = 0.2$, $\mathcal{C} = 0.1$, $\varepsilon = 0.1$, and $\mu = 0.05$, with initial condition $(\mathcal{X}(0), \mathcal{Y}(0)) = (3.0, 0.5)$.

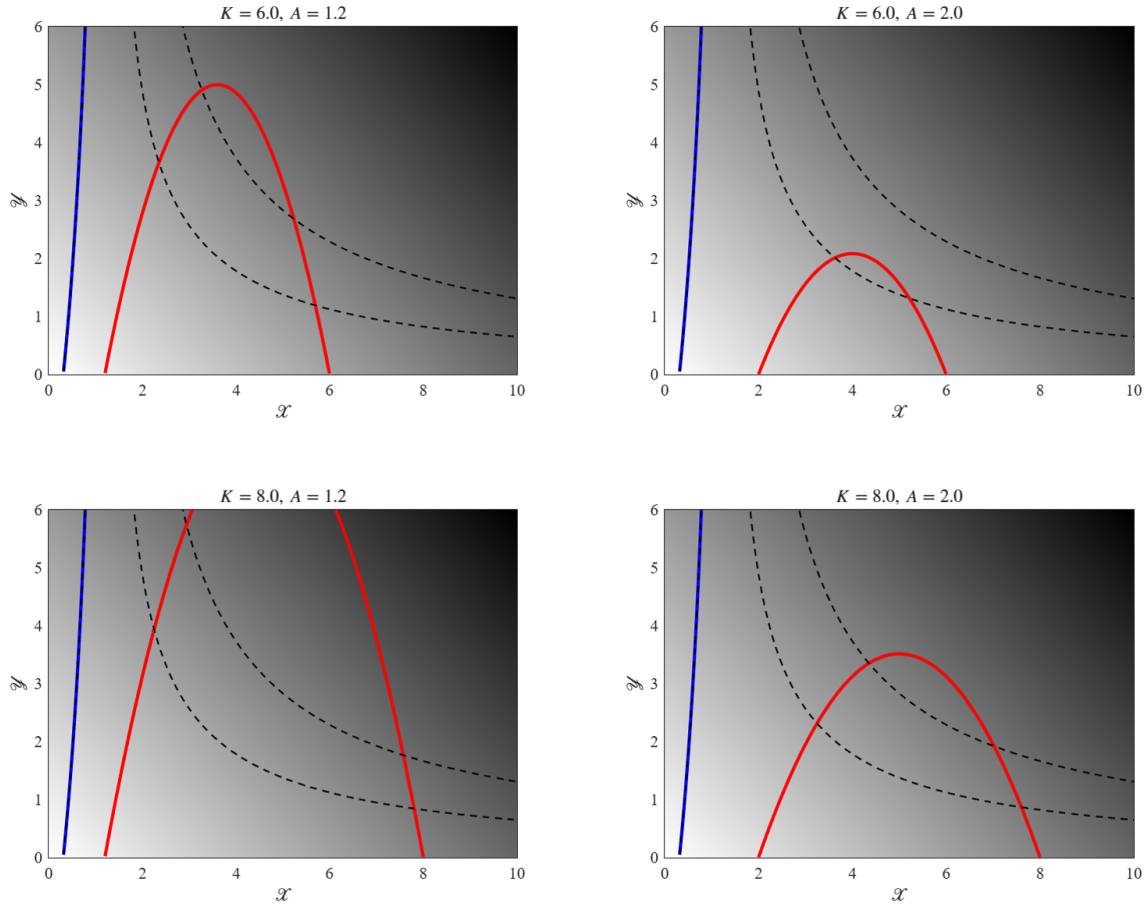


Figure 7: Phase-plane structure of the predator-prey system under variations in the carrying capacity K and Allee threshold A . Each panel depicts the phase portrait together with the prey nullcline N_X (red), predator nullcline N_Y (blue), and the coexistence equilibrium point (X^*, Y^*) (yellow). The remaining parameters are fixed at $r = 1.0$, $m = 0.5$, $\eta = 0.8$, $d = 0.3$, $b = 0.2$, $c = 0.1$, and $\varepsilon = 0.1$. Increasing the carrying capacity K shifts the prey nullcline rightward, enlarges the biologically feasible region, and supports higher equilibrium predator densities. In contrast, increasing the Allee threshold A strengthens the Allee effect, elevates the critical prey-density barrier, and reduces the potential for species coexistence. Across all parameter regimes, trajectories converge toward a unique positive equilibrium whenever persistence conditions are satisfied, illustrating the influence of (K, A) on feasibility, basin geometry, and long-term population dynamics. Overall, the results demonstrate that enhanced habitat productivity (K) promotes population persistence and stable coexistence, whereas strong Allee effects (A large) diminish ecological resilience and may drive the system toward extinction or collapse.

respectively. The prey nullcline $\mathcal{N}_x$ exhibits the characteristic hump-shaped profile generated by the strong Allee effect, whereas the predator nullcline $\mathcal{N}_y$ reflects the balance between predator growth and mortality under the nonlinear functional response. Their intersection,

$$\mathcal{N}_x \cap \mathcal{N}_y = \{E^*\},$$

determines the biologically feasible coexistence equilibrium. The numerical trajectories converge toward the coexistence equilibrium according to

$$\|(\mathcal{X}(t), \mathcal{Y}(t)) - E^*\| \rightarrow 0, \quad t \rightarrow \infty,$$

indicating asymptotic stabilization whenever the coexistence state exists and remains locally stable. Increasing the carrying capacity K shifts the prey nullcline rightward, enlarges the biologically feasible region, and supports higher equilibrium population densities. In contrast, increasing the Allee threshold A raises the critical prey-density threshold required for persistence, thereby reducing the size of the coexistence region. Consequently, larger values of K promote species persistence and ecological resilience, whereas stronger Allee effects hinder recovery from low population densities and increase the risk of extinction. Overall, the phase portraits reveal a dissipative ecological structure in which prey dynamics are jointly regulated by environmental carrying capacity and Allee constraints, while predator persistence is governed by mortality and saturation effects. Across all illustrated parameter regimes, trajectories approach a unique biologically meaningful coexistence state, in agreement with the theoretical and numerical findings reported in [44, 8, 50, 9].

Figure 8 further compares the phase-plane dynamics of the predator–prey system under classical ($\alpha = 1$) and fractional-order ($0 < \alpha < 1$) formulations while keeping all ecological parameters fixed. In the classical case, trajectories converge rapidly toward the coexistence equilibrium E^* and may exhibit pronounced transient oscillations resulting from strong predator–prey feedback interactions. In contrast, the fractional-order system displays smoother and more gradually damped trajectories that approach E^* at a slower rate due to memory-induced effects. The observed relaxation follows the characteristic Mittag–Leffler decay behavior associated with fractional dynamical systems, reflecting the nonlocal influence of past population states on current dynamics. The incorporation of memory delays predator response and moderates prey depletion, thereby reducing overshoot and suppressing large-amplitude oscillations. Consequently, fractional dynamics promote a more balanced adjustment toward equilibrium and enhance the persistence of both species. As the fractional order decreases, the damping effect becomes increasingly pronounced, enlarging the stability region, attenuating oscillatory behavior, and extending coexistence times. These results demonstrate that ecological memory acts as an important stabilizing mechanism, yielding more realistic and resilient population dynamics than the corresponding classical integer-order model. Furthermore, the combined influence of carrying capacity K , Allee threshold A , and fractional memory α governs the overall geometry of the phase space, determining both the rate of convergence and the long-term persistence of the predator–prey community.

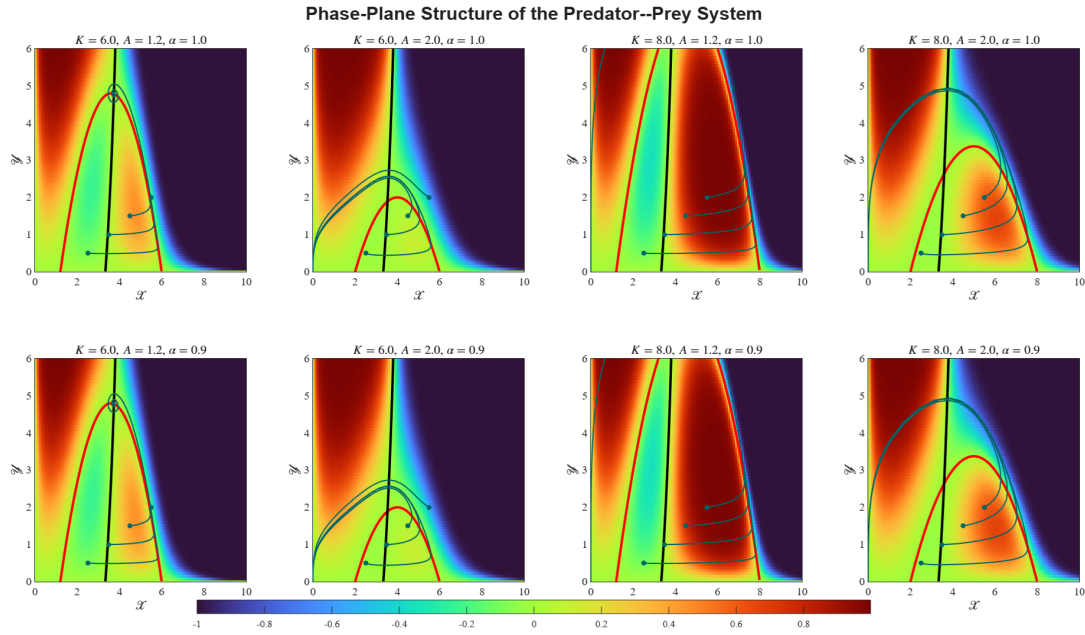


Figure 8: **Classical and fractional-order phase-plane dynamics of the predator-prey system for different values of the carrying capacity K and Allee threshold A .** The upper row corresponds to the classical model ($\alpha = 1$), while the lower row represents the fractional-order model ($\alpha = 0.9$). Trajectories, nullclines, and coexistence equilibria are shown for $(K, A) \in (6, 1.2), (6, 2.0), (8, 1.2), (8, 2.0)$ with initial condition $(\mathcal{X}(0), \mathcal{Y}(0)) = (3.0, 0.5)$. Fractional memory reduces oscillations, delays convergence, and promotes smoother long-term coexistence compared with the classical case. Fixed parameters are $r = 1.2$, $m = 0.2$, $k = 0.6$, $d = 0.4$, $b = 0.3$, $c = 0.1$, and $\varepsilon = 0.05$.

6.3. Eigenvalue Results and Stability Analysis

Table 3 summarises, for each value of the carrying capacity K , the discriminant Δ of the cubic equilibrium equation $P(\mathcal{X}) = 0$, the number of real algebraic roots, the number of biologically feasible coexistence states, and the eigenvalue-based stability classification of each feasible equilibrium. For a cubic with real coefficients, the sign of the discriminant Δ determines the algebraic root structure: $\Delta > 0$ implies three distinct real roots, $\Delta = 0$ a multiple real root, and $\Delta < 0$ exactly one real and two complex conjugate roots. However, not every real root corresponds to a biologically meaningful equilibrium. A coexistence state must satisfy

$$\mathcal{X}^* > 0, \quad \mathcal{Y}^* = \frac{r}{m} \left(1 - \frac{\mathcal{X}^*}{K} \right) \left(\frac{\mathcal{X}^*}{A} - 1 \right) > 0, \quad km\mathcal{X}^* - d > 0.$$

Therefore, the column 'feasible' in Table 3 counts only those real roots that satisfy all three feasibility conditions. For each feasible root $\mathcal{X}_j^*$, the associated predator density $\mathcal{Y}_j^*$ is computed from the prey nullcline, and the two eigenvalues $\lambda_{1,j}, \lambda_{2,j}$ of the Jacobian $J(E_j^*)$ are evaluated. If only one feasible root exists, the system possesses a single coexistence equilibrium with two eigenvalues. If two (or three) feasible roots exist, the model admits two (or three) distinct coexistence equilibria, giving rise to four (or six) corresponding eigenvalues in total.

Table 3: Equilibrium coordinates and eigenvalue-based stability classification under variation of the carrying capacity K . The table reports the equilibrium point $(\mathcal{X}^*, \mathcal{Y}^*)$, the real parts of the Jacobian eigenvalues, and the corresponding stability type. The results indicate that the equilibrium remains unstable throughout the investigated range of K , as both eigenvalues possess positive real parts. Fixed parameters are $\mathcal{R} = 1.0$, $\mathcal{A} = 1.2$, $\eta = 0.8$, $\mathcal{M} = 0.2$, $\mu = 0.05$, $\mathcal{B} = 0.2$, $\mathcal{C} = 0.1$, and $\varepsilon = 0.1$.

K	$\mathcal{X}^*$	$\mathcal{Y}^*$	$\Re(\lambda_1)$	$\Re(\lambda_2)$	Stability
6.5	1.2000	0.0000	0.8154	0.1420	Unstable
7.0	1.2000	0.0000	0.8286	0.1420	Unstable
7.5	1.2000	0.0000	0.8400	0.1420	Unstable
8.0	1.2000	0.0000	0.8500	0.1420	Unstable
8.5	1.2000	0.0000	0.8588	0.1420	Unstable
9.0	1.2000	0.0000	0.8667	0.1420	Unstable
9.5	1.2000	0.0000	0.8737	0.1420	Unstable
10.0	1.2000	0.0000	0.8800	0.1420	Unstable
10.5	1.2000	0.0000	0.8857	0.1420	Unstable
11.0	1.2000	0.0000	0.8909	0.1420	Unstable
11.5	1.2000	0.0000	0.8957	0.1420	Unstable
12.0	1.2000	0.0000	0.9000	0.1420	Unstable

Table 4 summarizes the influence of the Allee threshold A on the equilibrium structure and local stability of the predator–prey system. For each value of A , the equilibrium coordinates $(\mathcal{X}^*, \mathcal{Y}^*)$ and the corresponding eigenvalues of the Jacobian matrix are computed numerically. The results reveal that the equilibrium remains at the predator-free boundary state

$$(\mathcal{X}^*, \mathcal{Y}^*) = (A, 0),$$

throughout the investigated parameter range. As A increases, the equilibrium prey density increases proportionally, while the predator population remains extinct. The first eigenvalue decreases gradually from 0.8750 to 0.6750, whereas the second eigenvalue increases from 0.1100 to 0.3660. Nevertheless, both eigenvalues retain positive real parts for all considered values of A , indicating that the predator-free equilibrium is locally unstable throughout the entire parameter interval. Biologically, this implies that small perturbations away from the boundary state grow over time, preventing the system from remaining near this equilibrium. Although increasing the Allee threshold modifies the local growth characteristics and reduces the dominant instability, it is insufficient to stabilize the equilibrium. Consequently, the predator-free state does not represent a sustainable long-term ecological configuration, and the system is expected to evolve toward alternative dynamical behaviors or biologically feasible coexistence states whenever such states exist.

7. Conclusion and Future Directions

In this study, a fractional-order predator–prey model incorporating a strong Allee effect in the prey population and nonlinear anti-predator feedback in the predator population has been proposed and analyzed. The model extends classical ecological systems by incorporating fractional-order dynamics, thereby accounting for memory and hereditary effects that are frequently observed in real ecological processes. Fundamental mathematical properties of the system, including existence, uniqueness, boundedness, and positivity of

Table 4: Equilibrium coordinates and eigenvalue-based stability classification under variation of the Allee threshold A . The table reports the predator-free equilibrium $(\mathcal{X}^*, \mathcal{Y}^*)$, the real parts of the Jacobian eigenvalues, and the corresponding stability type. The results indicate that the boundary equilibrium remains locally unstable throughout the investigated range of A , since both eigenvalues possess positive real parts. Fixed parameters are $\mathcal{R} = 1.0$, $\mathcal{K} = 8.0$, $\eta = 0.8$, $\mathcal{M} = 0.2$, $\mu = 0.05$, $\mathcal{B} = 0.2$, $\mathcal{C} = 0.1$, and $\varepsilon = 0.1$.

A	$\mathcal{X}^*$	$\mathcal{Y}^*$	$\Re(\lambda_1)$	$\Re(\lambda_2)$	Stability
1.00	1.0000	0.0000	0.8750	0.1100	Unstable
1.05	1.0500	0.0000	0.8688	0.1180	Unstable
1.10	1.1000	0.0000	0.8625	0.1260	Unstable
1.15	1.1500	0.0000	0.8563	0.1340	Unstable
1.20	1.2000	0.0000	0.8500	0.1420	Unstable
1.25	1.2500	0.0000	0.8438	0.1500	Unstable
1.30	1.3000	0.0000	0.8375	0.1580	Unstable
1.35	1.3500	0.0000	0.8312	0.1660	Unstable
1.40	1.4000	0.0000	0.8250	0.1740	Unstable
1.55	1.5500	0.0000	0.8062	0.1980	Unstable
1.60	1.6000	0.0000	0.8000	0.2060	Unstable
1.65	1.6500	0.0000	0.7937	0.2140	Unstable
1.70	1.7000	0.0000	0.7875	0.2220	Unstable
1.75	1.7500	0.0000	0.7812	0.2300	Unstable
1.80	1.8000	0.0000	0.7750	0.2380	Unstable
2.60	2.6000	0.0000	0.6750	0.3660	Unstable

solutions, were established to ensure the biological feasibility and well-posedness of the proposed model.

The qualitative behavior of the system was investigated through equilibrium analysis, Jacobian-based local stability theory, numerical simulations, and phase-plane investigations. The influence of key ecological parameters, including the carrying capacity K , Allee threshold A , predation intensity, and anti-predator feedback strength, was examined in detail. The obtained results demonstrate that the Allee effect plays a crucial role in determining prey persistence, while the nonlinear anti-predator feedback significantly influences predator survival and the overall population dynamics. The phase-plane analysis revealed that increasing the carrying capacity enlarges the biologically feasible region and promotes population persistence, whereas stronger Allee effects reduce the likelihood of long-term coexistence and increase the risk of population collapse.

A comparative study between the classical integer-order model ($\alpha = 1$) and the fractional-order model ($0 < \alpha < 1$) highlighted the significant influence of ecological memory on system behavior. Numerical simulations showed that fractional dynamics generate smoother trajectories, reduced oscillatory behavior, and slower convergence toward equilibrium states compared with the corresponding classical model. These observations indicate that memory effects act as a stabilizing mechanism by moderating rapid population fluctuations and promoting more realistic ecological responses. Furthermore, the two-parameter stability maps provided a comprehensive description of how variations in ecological parameters affect the local stability of equilibrium states and identified parameter regions associated with stable and unstable dynamics.

From an ecological perspective, the results demonstrate that population persistence is

governed by the combined influence of habitat capacity, Allee effects, predator–prey interactions, and memory-dependent processes. In particular, the Allee threshold introduces a critical population level below which prey extinction becomes increasingly likely, whereas fractional memory delays predator responses, smooths prey depletion, and mitigates abrupt population changes. Consequently, incorporating memory effects into ecological models offers a more realistic representation of population dynamics in environments where historical influences cannot be neglected.

Overall, the proposed fractional-order predator–prey model provides a flexible and biologically meaningful framework for investigating the interplay between memory effects, nonlinear regulation, and ecological stability. Future research may extend the present framework by incorporating spatial diffusion, time delays, environmental stochasticity, harvesting effects, optimal control strategies, and multi-species food-web interactions. In addition, parameter estimation and model validation using ecological field data would further enhance the practical applicability of fractional-order predator–prey models for ecosystem management, conservation planning, and sustainable resource utilization.

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