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Approximate Solutions of Bratu-Type Boundary Value Problems

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Abstract

The Bratu's equation is a strongly nonlinear second-order ordinary differential equation that arises in electrospinning process and models temperature distribution within a flame in combustion theory. Bratu-type equations are used to simulate the ignition of flammable gases and flame propagation. In this paper, two methods are proposed to obtain highly accurate and reliable approximate solutions of Bratu-type boundary value problems. The first technique is a power series method which is based on the generalised Cauchy product that simplifies the difficulty associated with the nonlinear terms. Subsequently, explicit recurrence relations for the expansion coefficients of the series solutions are obtained. The second approach uses a twelfth-order second derivative backward differentiation formula that is implemented as a boundary value method. This numerical method is referred to as second derivative backward differentiation boundary value method. Three examples are given to illustrate the effectiveness, reliability, and accuracy of the proposed methods. The results obtained from both methods are in excellent agreement with the known exact solution. Comparison of the approximate and exact solutions shows that the proposed methods are reliable and accurate in solving a class of strongly nonlinear boundary value problems of Bratu-type.

Keywords: Boundary value problem, Bratu-type equation, Power series method, Boundary value method, Backward differentiation formula.
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1. Introduction

The famous Bratu's equation ([13]) is a class of strongly nonlinear second-order ordinary differential equations with the nonlinear term given in terms of the exponential function. This equation arises in combustion theory, electrospinning, vibration-electrospinning, bubble-electrospinning, and magneto-electrospinning processes ([20], [36], [41], [59], [60]) for the fabrication of nano-fibres ([57]). Though there is a lot of research works on the theoretical estimation of electrospinning; however, a well-defined mathematical formulation of electrospinning was first established by Wan et al. [61] and models various physical phenomena such as radioactive heat transfer, solid fuel ignition of the thermal

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combustion theory, thermal reaction theory, chemical reaction theory, nanotechnology, and Chandrasekhar model of the expansion of the Universe ([13], [25], [34], [35]).

Bratu-type equations arise in several physical and engineering phenomena. Bratu-type equations have been used for the simulation of the ignition of flammable gases and flame propagation in combustion theory. For instance, in [27], the Bratu-type equation was used for the investigation of the propagation of premixed and non-premixed flames, laminar and turbulent flames, and flame extinction in different environments and conditions. Since Bratu-type equations are characterised by strong nonlinearities, the exact solutions are difficult to obtain. Hence, the need to apply approximate solutions by analytical and numerical methods becomes imperative. Towards this end, a variety of these techniques have been proposed by several authors to obtain approximate analytical and numerical solutions of Bratu-type initial and boundary value problems. For instance, the author in [37] obtained the solution of the Bratu's equation using pseudo-arclength continuation analysis. In [31], Khuri applied a technique based on Laplace transform decomposition algorithm to obtain approximate solutions to the Bratu problem. Buckmire in [15] used Mickens finite-difference scheme to obtain numerical solutions of Bratu-Gelfand problem. In [62], the author examined analytical solutions of Bratu-type equations with initial and boundary conditions using the Adomian decomposition method (ADM). Syam and Hamdan in [56] employed Laplace Adomain decomposition method to solve Bratu-type equations.

The fitted method was used in [41] to approximate solution of the Liouville-Bratu-Gelfand problem. Jalilian in [26] applied a non-polynomial spline method to get approximate solution of the Bratu problem. The authors in [5] used perturbation method to approximate the solutions of Bratu-type problems, while the Lie-group shooting method was implemented in [1]. Wazwaz in [63] used ADM coupled with Padé approximation to solve Bratu-type equations with boundary conditions. The authors in [43], [64] implemented the Chebyshev wavelet method to obtain the numerical solution of the Bratu's equation with initial conditions. In [22], the authors presented the homotopy analysis method (HAM) to obtain approximate solutions of the Bratu problem, while the variational iteration method (VIM) was applied in [23] to solve Bratu-type equations. The implementation of continuous genetic algorithm for the solution of the Bratu problem was presented in [4]. In [39], the author employed standard and non-standard finite difference methods to obtain the solutions of the one-dimensional and two-dimensional Bratu-type problems. Abd-Elhameed et al. in [3] investigated singularly perturbed and Bratu-type equations using operational matrix method based on shifted Legendre polynomials. Singh et al. in [53] examined the ADM with Green's function to obtain series solutions of Bratu-type problems. In [28], Kafri and Khuri implemented fixed-point iterations and Green's functions to approximate solutions of Bratu-type problems. Numerical solutions of Bratu-type equations using variational iteration technique were presented in [18].

The authors in [58] presented iterative finite difference method for approximating solutions of Bratu-type problems, while the use of ADM was considered in [38]. In [29], Kafri et al. implemented Green's functions and Picard's fixed-point iteration scheme for Liu–Wang version of Bratu-type problems. Kashkari and Abbas in [30] applied a modified homotopy perturbation method for the Bratu problem with initial conditions, while

Muzara et al. [42] used bivariate spectral quasi-linearization method for solving Bratu-type problems. In [35], Koudahoun et al. used Sundman Transformation for generalized Bratu equations. In [54], Singh implemented optimal iterative method for nonlocal elliptic Bratu-type boundary value problems, while Singh et al. [55] employed Legendre spectral method for solving the fractional Bratu problem. In [6], the authors implemented Lie symmetry analysis for Bratu-Gelfand problem. Abd-Elhakem and Youssri in [2] used two spectral Legendre's derivative algorithms to obtain approximate solutions of Bratu-type equations. In [57], Swaminathan et al. presented spectral collocation method based on Genocchi polynomials for solving Bratu-type equations with initial and boundary conditions. In [52], Shahni and Singh implemented Bernstein and Gegenbauer-wavelet collocation methods for Bratu-type boundary value problems. Salem and Thanoon [51] used perturbation method to solve Bratu's type equation. Recently, in [32], the author applied successive differentiation and integration techniques to obtain approximate solution of Bratu boundary value problem. Very recently, Motsa et al. [40] employed rational hybrid block method for solving Bratu-type boundary value problems.

To the best of our knowledge, a power series method based on the generalised Cauchy product, and a linear multistep method, based on backward differentiation formula, for the approximate solutions of Bratu-type boundary value problems have not appeared in the literature. In this paper, approximate solutions of Bratu-type boundary value problems using analytical and numerical techniques are proposed. The first technique is a power series method (PSM) that is based on the generalised Cauchy product ([9], [10]). It is assumed that the nonlinear term is analytic and thus admits a power series expansion in the dependent variable. Upon substituting the assumed power series solution into the nonlinear term, the difficulty associated with the resulting term is subsequently simplified using the generalised Cauchy product of power series. A central result in the proposed power series method is the recurrence relation for the expansion coefficients of the series solution. The second approach which is a numerical method is based on the generalised backward differentiation formula (GBDF) proposed in [14], [44]. The method is derived by Taylor series comprising of initial methods, main method, and final method. These methods are implemented as simultaneous numerical integrators. Three illustrative examples are given to implement the reliability, accuracy, effectiveness, and convenience of the proposed methods. The results of the two methods are tabulated and plotted. Comparisons of results obtained from the two methods show that both methods are in excellent agreement with each other as well as the exact solutions (where they exist). Hence, the proposed methods are both reliable and accurate in solving a class of strongly nonlinear boundary value problems of Bratu-type.

2. Bratu-Type Boundary Value Problems

This section presents a class of boundary value problems with strong nonlinearities. We introduce the proposed problems as well as the nature of their associated nonlinear terms.

Indeed, consider a generalised boundary value problem

$$\begin{aligned} &y''(x) + f(y(x)) = 0, \quad 0 < x < 1, \\ &y(0) = \phi_1, \quad y(1) = \varphi, \end{aligned} \tag{2.1}$$

where $f(y(x))$ is a continuous real-valued nonlinear function of $y(x)$. Apart from giving a unified result (according to whether the method is analytical or numerical) for the generalised boundary value problem (2.1), we also consider important special cases in which the nonlinear functions $f(y(x))$ are given in terms of exponential functions. The resulting problems involving exponential functions are called Bratu-type boundary value problems. To be more precise, we shall discuss approximate solutions of strongly nonlinear boundary value problems (2.1) with the functions $f(y(x))$ and the constants ϕ_1, φ described as follows:

(a) $f(y(x)) = \mu e^{y(x)}, \phi_1 = \varphi = 0$ (see, e.g., [3], [52], [53], [54], [57], [62]).
Here x is axial coordinate, $y = -6 \ln u$, where u is the axial velocity of the jet and e^y models the temperature dependent resistance. Upon solving the mass, linear momentum, and electric charge balance equations, Wan et al. [61] obtained the explicit value

$$\mu = \frac{18E^2}{\rho^2 r^4} \left(I - r^2 KE\right), \tag{2.2}$$

where the parameters I, K, E, ρ, and r represent the current, conductivity, applied voltage, density, and radius of the jet, respectively ([29]).

(b) $f(y(x)) = \mu e^{-y(x)}, \phi_1 = \varphi = 0$ (see, e.g., [3], [52], [53], [54], [57], [62]).
Here x, y, μ are as illustrated in (a).

(c) $f(y(x)) = \frac{18(E/Q)^2}{e^{y(x)/3}} \left(I - \frac{QKE}{\rho} e^{y(x)/6}\right)^2 e^{y(x)}, \phi_1 = \varphi = 0$ (see [23], [29], [52]).
However, Liu and Wang [34] opined that the parameter μ must not be constant for electrospinning, vibration-electrospinning, or bubble-electrospinning. As a result, they derived a different value for μ as a function given by

$$\mu = \mu(y(x)) = \frac{18\,(E/Q)^2}{e^{y(x)/3}} \left(I - \frac{QKE}{\rho} e^{y(x)/6}\right)^2, \tag{2.3}$$

where Q is the flow rate. The Bratu-type boundary value problem (2.1) with $f(y(x))$ given in (c) is also referred to as Liu-Wang equation, which is a development of Wan-Guo-Pan equation (Wan et al. [61]).

The Bratu-type problems (2.1) with the functions $f(y(x))$ given in (a)-(c) were recently considered by Shahni and Singh [52] using Bernstein and Gegenbauer wavelet collocation methods. In this paper, we consider same Bratu-type boundary value problems using a power series method and a numerical method as illustrated in Sects. 3 and 4, respectively.

For further discussions on models involving boundary value problems, see [7], [8], [12].

3. Power Series Method of Solution

This section illustrates the proposed power series method for solving the boundary value problem (2.1). We begin by establishing a result on the generalised Cauchy product before presenting the first main result of this paper.

To this end, assume that the solution $y(x)$ of the boundary value problem (2.1) can be represented by a power series in x. Assume further that the nonlinear function $f(y(x))$ admits a power series representation

$$f(y(x)) = \sum_{p=0}^{\infty} a_p y^p(x), \tag{3.1}$$

where a_p is the expansion coefficient, which in this case, Maclaurin coefficient. Upon substituting the series solution $y(x)$ into (3.1), one is required to apply the generalised Cauchy product. The result of this product expansion is given as follows:

Proposition 3.1 ([9], [10]). *Let the function $y(x)$ be given by a convergent power series about $x = 0$:*

$$y(x) = \sum_{k=0}^{\infty} A_k x^k, \qquad 0 < x < 1, \tag{3.2}$$

where $A_k \in \mathbb{R}$ is the expansion coefficient to be determined. Suppose that the function $f(y(x))$ admits the power series representation (3.1). Then we have

$$f(y(x)) = \sum_{\ell=0}^{\infty} C_\ell x^\ell, \tag{3.3}$$

where the coefficients C_ℓ are a class of generalised expansion coefficients given by

$$C_\ell = \sum_{p=0}^{\infty} a_p B_{\ell,p}. \tag{3.4}$$

Here, a_p, $p = 0, 1, 2, \ldots$, are the Maclaurin coefficients; and for $p = 0, 1, 2$, the numbers $B_{\ell,p}$ are defined by

$$B_{\ell,0} = \begin{cases} 1, & \ell = 0, \\ 0, & \ell = 1, 2, \cdots, \end{cases} \qquad B_{\ell,1} = A_\ell, \quad B_{\ell,2} = \sum_{k_1=0}^{\ell} A_{k_1} A_{\ell-k_1}. \tag{3.5}$$

For $p = 3, 4, 5, \ldots$, we have

$$B_{\ell,p} = \sum_{k_{p-1}=0}^{\ell} \sum_{k_{p-2}=0}^{k_{p-1}} \cdots \sum_{k_1=0}^{k_2} A_{\ell-k_{p-1}} A_{k_{p-1}-k_{p-2}} \cdots A_{k_2-k_1} A_{k_1}. \tag{3.6}$$

In particular, $B_{0,p} = A_0^p$ $(p = 0, 1, 2, \ldots)$.

We now proceed to obtaining the analytical solution of a class of generalised strongly nonlinear boundary value problem (2.1) using a power series method. In this case, the required initial condition, that is not present, is introduced as a dummy constant at the beginning of the process of the series solution. The optimal value of this dummy constant is then obtained using the right boundary condition. The result is given as follows:

Theorem 3.1. *Consider a generalised boundary value problem*

$$\begin{aligned} &y''(x) + f(y(x)) = 0, \quad 0 < x < 1, \\ &y(0) = \phi_1, \quad y(1) = \varphi, \end{aligned} \tag{3.7}$$

where $f(y(x))$ *is a continuous real-valued nonlinear function, and* ϕ_1, φ *are real constants. Then one has an analytical solution given by the power series*

$$y(x) = \phi_1 + \phi_2 x + \sum_{\ell=0}^{\infty} A_{\ell+2} x^{\ell+2}, \tag{3.8}$$

where ϕ_2 *is a constant coefficient to be determined in terms of the number* φ*, and the expansion coefficient* $A_{\ell+2}$ *is given explicitly by a recurrence relation*

$$A_{\ell+2} = -\frac{C_\ell}{(\ell+1)(\ell+2)}, \qquad \ell \geqslant 0. \tag{3.9}$$

The coefficients $C_\ell, \ell \geqslant 0$*, are as given in Proposition 3.1, with the following special values:*

$$C_0 = a_0 + \sum_{p=1}^{\infty} a_p B_{0,p}, \qquad B_{0,p} = (A_0)^p = \phi_1^p. \tag{3.10}$$

In particular, for $p = 1, 2, 3, \ldots$,

$$B_{0,p} = \begin{cases} 1, & \phi_1 = 1, \\ 0, & \phi_1 = 0; \end{cases} \qquad C_0 = \begin{cases} a_0 + a_1 + a_2 + \cdots, & \phi_1 = 1, \\ a_0, & \phi_1 = 0. \end{cases} \tag{3.11}$$

Proof. The starting point is to substitute the power series (3.2) into equation (3.7), and then use Proposition 3.1 to see that

$$A_0 = \phi_1, \qquad A_1 = \phi_2, \qquad \sum_{k=2}^{\infty} k(k-1) A_k x^{k-2} + \sum_{\ell=0}^{\infty} C_\ell x^\ell = 0. \tag{3.12}$$

Introducing an appropriate change in index in equation (3.12) so that all terms in the equation have the same form, we have

$$\sum_{\ell=0}^{\infty} \left((\ell+2)(\ell+1) A_{\ell+2} + C_\ell \right) x^\ell = 0. \tag{3.13}$$

Equating the coefficients of each power $x^\ell, \ell = 0, 1, 2, \ldots$, to zero, we obtain the recurrence relation

$$A_{\ell+2} = -\frac{C_\ell}{(\ell+2)(\ell+1)}, \qquad \ell = 0, 1, 2, 3, \ldots, \tag{3.14}$$

where the coefficients C_ℓ are as given in Proposition 3.1. Using the boundary condition $y(1) = \varphi$ on the solution (3.8)-(3.9), one obtains a nonlinear algebraic equation

$$\phi_1 + \phi_2 + A_2 + A_3 + A_4 + \cdots = \varphi, \tag{3.15}$$

from which one computes the optimal value of the assumed initial condition ϕ_2 as a solution to the emanating polynomial equation, and this completes the proof of the theorem. □

Special cases of Theorem 3.1 will be given as examples in the sequel. A more explicit version of Theorem 3.1 is given in the following result as a special case.

Corollary 3.1. *The generalised boundary value problem*

$$\begin{aligned} &y''(x) + f(y(x)) = 0, \qquad 0 < x < 1, \\ &y(0) = 0, \quad y(1) = \varphi, \end{aligned} \tag{3.16}$$

has an approximate analytical solution given by the power series

$$\begin{aligned} y_{12}(x) =& \phi_2 x - \frac{a_0}{2}x^2 - \frac{a_1\phi_2}{3!}x^3 - \frac{2a_2\phi_2^2 - a_0a_1}{4!}x^4 - \frac{6a_3\phi_2^3 - \left(a_1^2 + 6a_0a_2\right)\phi_2}{5!}x^5 \\ &+ \frac{A_6}{6!}x^6 + \frac{A_7}{7!}x^7 + \frac{A_8}{8!}x^8 + \frac{A_9}{9!}x^9 + \frac{A_{10}}{10!}x^{10} + \frac{A_{11}}{11!}x^{11} + \frac{A_{12}}{12!}x^{12}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} A_6 =& -24a_4\phi_2^4 + 2\left(5a_1a_2 + 18a_0a_3\right)\phi_2^2 - a_0\left(a_1^2 + 6a_0a_2\right) \\ A_7 =& -120a_5\phi_2^5 + 2\left(10a_2^2 + 33a_1a_3 + 120a_0a_4\right)\phi_2^3 - \left(a_1^3 + 36a_0a_2a_1 + 90a_0^2a_3\right)\phi_2 \\ A_8 =& -720a_6\phi_2^6 + 36\left(7a_2a_3 + 14a_1a_4 + 50a_0a_5\right)\phi_2^4 \\ &- 6\left(180a_4a_0^2 + 22a_2^2a_0 + 81a_1a_3a_0 + 7a_1^2a_2\right)\phi_2^2 + a_0\left(a_1^3 + 36a_0a_2a_1 + 90a_0^2a_3\right) \\ A_9 =& -5040a_7\phi_2^7 + 36\left(21a_3^2 + 56a_2a_4 + 120a_1a_5 + 420a_0a_6\right)\phi_2^5 \\ &- 12\left(1050a_5a_0^2 + 231a_2a_3a_0 + 510a_1a_4a_0 + 25a_1a_2^2 + 51a_1^2a_3\right)\phi_2^3 \\ &+ \left(a_1^4 + 162a_0a_2a_1^2 + 1350a_0^2a_3a_1 + 336a_0^2a_2^2 + 2520a_0^3a_4\right)\phi_2 \\ A_{10} =& -40320a_8\phi_2^8 + 144\left(84a_3a_4 + 130a_2a_5 + 285a_1a_6 + 980a_0a_7\right)\phi_2^6 \\ &- 12\left(50a_2^3 + 585a_1a_3a_2 + 2784a_0a_4a_2 + 1008a_0a_3^2 + 714a_1^2a_4 + 6450a_0a_1a_5\right. \\ &\left.+12600a_0^2a_6\right)\phi_2^4 + 2\left(25200a_5a_0^3 + 6516a_2a_3a_0^2 + 15660a_1a_4a_0^2 + 1110a_1a_2^2a_0\right. \\ &\left.+2511a_1^2a_3a_0 + 85a_1^3a_2\right)\phi_2^2 - a_0\left(a_1^4 + 162a_0a_2a_1^2 + 1350a_0^2a_3a_1 + 336a_0^2a_2^2 + 2520a_0^3a_4\right). \end{aligned}$$

$$
\begin{aligned}
\mathtt{A}_{11} =& -362880a_9\phi_2^9 + 144\left(336a_4^2 + 810a_3a_5 + 1350a_2a_6 + 2975a_1a_7 + 10080a_0a_8\right)\phi_2^7 \\
& -36\left(52920a_7a_0^2 + 7488a_3a_4a_0 + 12060a_2a_5a_0 + 28140a_1a_6a_0 + 921a_1a_3^2 + 540a_2^2a_3\right. \\
& \left.+2660a_1a_2a_4 + 3340a_1^2a_5\right)\phi_2^5 + 12\left(75600a_6a_0^3 + 7290a_3^2a_0^2 + 20700a_2a_4a_0^2\right. \\
& \left.+51450a_1a_5a_0^2 + 570a_2^3a_0 + 7296a_1a_2a_3a_0 + 9750a_1^2a_4a_0 + 270a_1^2a_2^2 + 461a_1^3a_3\right)\phi_2^3 \\
& -\left(a_1^5 + 672a_0a_2a_1^3 + 14580a_0^2a_3a_1^2 + 6096a_0^2a_2^2a_1 + 78120a_0^3a_4a_1 + 30780a_0^3a_2a_3\right. \\
& \left.+113400a_0^4a_5\right)\phi_2 \\
\mathtt{A}_{12} =& -3628800a_{10}\phi_2^{10} + 1440\left(660a_4a_5 + 891a_3a_6 + 1540a_2a_7 + 3388a_1a_8 + 11340a_0a_9\right)\phi_2^8 \\
& -72\left(352800a_8a_0^2 + 19680a_4^2a_0 + 48150a_3a_5a_0 + 83220a_2a_6a_0 + 193060a_1a_7a_0\right. \\
& \left.+2541a_2a_3^2 + 3740a_2^2a_4 + 11418a_1a_3a_4 + 19360a_1a_2a_5 + 24090a_1^2a_6\right)\phi_2^6 \\
& +12\left(1323000a_7a_0^3 + 232740a_3a_4a_0^2 + 387300a_2a_5a_0^2 + 957600a_1a_6a_0^2 + 50283a_1a_3^2a_0\right. \\
& +27822a_2^2a_3a_0 + 149484a_1a_2a_4a_0 + 201750a_1^2a_5a_0 + 1650a_1a_2^3 + 11682a_1^2a_2a_3 \\
& \left.+11594a_1^3a_4\right)\phi_2^4 - 2\left(1701000a_6a_0^4 + 177390a_3^2a_0^3 + 512280a_2a_4a_0^3 + 1348200a_1a_5a_0^3\right. \\
& +16356a_2^3a_0^2 + 224946a_1a_2a_3a_0^2 + 321840a_1^2a_4a_0^2 + 12972a_1^2a_2^2a_0 + 23886a_1^3a_3a_0 \\
& \left.+341a_1^4a_2\right)\phi_2^2 + a_0\left(a_1^5 + 672a_0a_2a_1^3 + 14580a_0^2a_3a_1^2 + 6096a_0^2a_2^2a_1 + 78120a_0^3a_4a_1\right. \\
& \left.+30780a_0^3a_2a_3 + 113400a_0^4a_5\right).
\end{aligned}
$$

Here ϕ_2 is the optimal solution of the nonlinear algebraic equation

$$
\mathcal{A}_{10}\phi_2^{10} + \mathcal{A}_9\phi_2^9 + \mathcal{A}_8\phi_2^8 + \mathcal{A}_7\phi_2^7 + \mathcal{A}_6\phi_2^6 + \mathcal{A}_5\phi_2^5 + \mathcal{A}_4\phi_2^4 + \mathcal{A}_3\phi_2^3 + \mathcal{A}_2\phi_2^2 + \mathcal{A}_1\phi_2 + \mathcal{A}_0 = \varphi, \tag{3.18}
$$

where

$$
\begin{aligned}
\mathcal{A}_{10} =& -\frac{a_{10}}{132}, \qquad \mathcal{A}_9 = -\frac{a_9}{110} \\
\mathcal{A}_8 =& \frac{660a_4a_5 + 891a_3a_6 + 1540a_2a_7 + 3388a_1a_8 - 3696a_8 + 11340a_0a_9}{332640} \\
\mathcal{A}_7 =& \frac{336a_4^2 + 810a_3a_5 + 1350a_2a_6 + 2975a_1a_7 - 3850a_7 + 10080a_0a_8}{277200} \\
\mathcal{A}_6 =& -\frac{352800a_8a_0^2 + 19680a_4^2a_0 + 48150a_3a_5a_0 + 83220a_2a_6a_0 + 193060a_1a_7a_0}{6652800} \\
& + \frac{258720a_7a_0 - 2541a_2a_3^2 - 3740a_2^2a_4 - 11418a_1a_3a_4 + 22176a_3a_4 - 19360a_1a_2a_5}{6652800} \\
& + \frac{34320a_2a_5 - 24090a_1^2a_6 + 75240a_1a_6 - 118800a_6}{6652800} \\
\mathcal{A}_5 =& -\frac{52920a_7a_0^2 + 7488a_3a_4a_0 + 12060a_2a_5a_0 + 28140a_1a_6a_0 - 46200a_6a_0 + 921a_1a_3^2}{1108800} \\
& + \frac{2310a_3^2 - 540a_2^2a_3 - 2660a_1a_2a_4 + 6160a_2a_4 - 3340a_1^2a_5 + 13200a_1a_5 - 26400a_5}{1108800}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_4 =&\frac{1323000a_7a_0^3+232740a_3a_4a_0^2+387300a_2a_5a_0^2+957600a_1a_6a_0^2-1663200a_6a_0^2}{39916800}\\
&+\frac{50283a_1a_3^2a_0-133056a_3^2a_0+27822a_2^2a_3a_0+149484a_1a_2a_4a_0-367488a_2a_4a_0}{39916800}\\
&+\frac{201750a_1^2a_5a_0-851400a_1a_5a_0+1782000a_5a_0+1650a_1a_2^3-6600a_2^3+11682a_1^2a_2a_3}{39916800}\\
&-\frac{77220a_1a_2a_3-249480a_2a_3-11594a_1^3a_4+94248a_1^2a_4-498960a_1a_4+1330560a_4}{39916800}\\
\mathcal{A}_3 =&\frac{75600a_6a_0^3+7290a_3^2a_0^2+20700a_2a_4a_0^2+51450a_1a_5a_0^2-115500a_5a_0^2+570a_2^3a_0}{3326400}\\
&+\frac{7296a_1a_2a_3a_0-25410a_2a_3a_0+9750a_1^2a_4a_0-56100a_1a_4a_0+158400a_4a_0+270a_1^2a_2^2}{3326400}\\
&-\frac{2750a_1a_2^2-13200a_2^2-461a_1^3a_3+5610a_1^2a_3-43560a_1a_3+166320a_3}{3326400}\\
\mathcal{A}_2 =&-\frac{1701000a_6a_0^4+177390a_3^2a_0^3+512280a_2a_4a_0^3+1348200a_1a_5a_0^3-3326400a_5a_0^3}{239500800}\\
&-\frac{16356a_2^3a_0^2+224946a_1a_2a_3a_0^2-860112a_2a_3a_0^2+321840a_1^2a_4a_0^2-2067120a_1a_4a_0^2}{239500800}\\
&-\frac{6415200a_4a_0^2+12972a_1^2a_2^2a_0-146520a_1a_2^2a_0+784080a_2^2a_0+23886a_1^3a_3a_0}{239500800}\\
&+\frac{331452a_1^2a_3a_0-2886840a_1a_3a_0+11975040a_3a_0-341a_1^4a_2+11220a_1^3a_2}{239500800}\\
&-\frac{249480a_1^2a_2-3326400a_1a_2+19958400a_2}{239500800}\\
\mathcal{A}_1 =&-\frac{a_1^5-110a_1^4+672a_0a_2a_1^3+7920a_1^3-17820a_0a_2a_1^2+14580a_0^2a_3a_1^2-332640a_1^2}{39916800}\\
&-\frac{6096a_0^2a_2^2a_1+285120a_0a_2a_1-148500a_0^2a_3a_1+78120a_0^3a_4a_1+6652800a_1}{39916800}\\
&+\frac{36960a_0^2a_2^2+1995840a_0a_2-712800a_0^2a_3-30780a_0^3a_2a_3+277200a_0^3a_4}{39916800}\\
&-\frac{113400a_0^4a_5-39916800}{39916800}\\
\mathcal{A}_0 =&\frac{a_0\left(a_1^5-132a_1^4+672a_0a_2a_1^3+11880a_1^3-21384a_0a_2a_1^2+14580a_0^2a_3a_1^2-665280a_1^2\right)}{479001600}\\
&+\frac{a_0\left(6096a_0^2a_2^2a_1+427680a_0a_2a_1-178200a_0^2a_3a_1+78120a_0^3a_4a_1+19958400a_1\right)}{479001600}\\
&-\frac{a_0\left(44352a_0^2a_2^2+3991680a_0a_2-1069200a_0^2a_3-30780a_0^3a_2a_3+332640a_0^3a_4\right)}{479001600}\\
&+\frac{a_0\left(113400a_0^4a_5-239500800\right)}{479001600}.
\end{aligned}
$$

The expansion coefficients a_p $(p=0,1,2,3,\dots)$ *are as defined in* (3.1).

Proof. One has, from Theorem 3.1, the series solution

$$\begin{aligned} y_{12}(x) =& \phi_2 x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} \\ & + A_{11}x^{11} + A_{12}x^{12}, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} & A_2 = -\frac{C_0}{2} = -\frac{a_0}{2}, \quad A_3 = -\frac{C_1}{6}, \quad A_4 = -\frac{C_2}{12}, \quad A_5 = -\frac{C_3}{20}, \quad A_6 = -\frac{C_4}{30}, \quad A_7^8 = -\frac{C_5}{42} \\ & A_8 = -\frac{C_6}{56}, \quad A_9 = -\frac{C_7}{72}, \quad A_{10} = -\frac{C_8}{90}, \quad A_{11} = -\frac{C_9}{110}, \quad A_{12} = -\frac{C_{10}}{132}. \end{aligned} \tag{3.20}$$

To calculate C_ℓ $(\ell = 1, 2, 3, \dots, 10)$ explicitly, we use the formula

$$C_\ell = a_1A_\ell + a_2B_{\ell,2} + a_3B_{\ell,3} + a_4B_{\ell,4} + a_5B_{\ell,5} + a_6B_{\ell,6} + a_7B_{\ell,7} + a_8B_{\ell,8} + a_9B_{\ell,9} + a_{10}B_{\ell,10}. \tag{3.21}$$

Upon noting from (3.5)-(3.6) that

$$\begin{aligned} & B_{2,2} = \phi_2^2, \quad B_{2,p} = 0, \quad p = 3, 4, 5, \dots \\ & B_{3,2} = -a_0\phi_2, \quad B_{3,3} = \phi_2^3, \quad B_{3,p} = 0, \quad p = 4, 5, 6, \dots \\ & B_{4,2} = -\frac{1}{3}a_1\phi_2^2 + \frac{a_0^2}{4}, \quad B_{4,3} = -\frac{3}{2}a_0\phi_2^2, \quad B_{4,4} = \phi_2^4, \quad B_{4,p} = 0, \quad p = 5, 6, 7, \dots \\ & B_{5,2} = -\frac{1}{6}a_2\phi_2^3 + \frac{1}{4}a_0a_1\phi_2, \quad B_{5,3} = -\frac{1}{2}a_1\phi_2^3 + \frac{3}{4}a_0^2\phi_2, \quad B_{5,4} = -2a_0\phi_2^3 \\ & B_{5,5} = \phi_2^5, \quad B_{5,p} = 0, \quad p = 6, 7, 8, \dots \\ & B_{6,2} = -\frac{1}{10}a_3\phi_2^4 + \frac{1}{180}\left(8a_1^2 + 33a_0a_2\right)\phi_2^2 - \frac{1}{24}a_0^2a_1, \; B_{6,3} = -\frac{1}{4}a_2\phi_2^4 + \frac{5}{8}a_0a_1\phi_2^2 - \frac{a_0^3}{8} \\ & B_{6,4} = -\frac{2}{3}a_1\phi_2^4 + \frac{3}{2}a_0^2\phi_2^2, \; B_{6,5} = -\frac{5}{2}a_0\phi_2^4, \quad B_{6,6} = \phi_2^6, \quad B_{6,p} = 0, \quad p = 7, 8, 9, \dots \end{aligned}$$

$$\begin{aligned} B_{7,2} =& -\frac{1}{15}a_4\phi_2^5 + \frac{1}{180}\left(10a_1a_2 + 27a_0a_3\right)\phi_2^3 - \frac{1}{120}a_0\left(3a_1^2 + 8a_0a_2\right)\phi_2 \\ B_{7,3} =& -\frac{1}{20}3a_3\phi_2^5 + \frac{1}{120}\left(13a_1^2 + 48a_0a_2\right)\phi_2^3 - \frac{1}{4}a_0^2a_1\phi_2 \\ B_{7,4} =& -\frac{1}{3}a_2\phi_2^5 + \frac{7}{6}a_0a_1\phi_2^3 - \frac{1}{2}a_0^3\phi_2 \\ B_{7,5} =& -\frac{5}{6}a_1\phi_2^5 + \frac{5}{2}a_0^2\phi_2^3, \quad B_{7,6} = -3a_0\phi_2^5, \quad B_{7,7} = \phi_2^7, \quad B_{7,p} = 0, \quad p = 8, 9, 10, \dots \end{aligned}$$

$$B_{8,2} = -\frac{1}{21}a_5\phi_2^6 + \frac{\left(25a_2^2+72a_1a_3+216a_0a_4\right)\phi_2^4}{1680} - \frac{\left(16a_1^3+261a_0a_2a_1+432a_0^2a_3\right)\phi_2^2}{5040} + \frac{1}{960}a_0^2\left(3a_1^2+8a_0a_2\right)$$

$$B_{8,3} = -\frac{1}{10}a_4\phi_2^6 + \frac{1}{40}\left(5a_1a_2+12a_0a_3\right)\phi_2^4 - \frac{1}{80}a_0\left(9a_1^2+19a_0a_2\right)\phi_2^2 + \frac{1}{32}a_0^3a_1$$

$$B_{8,4} = -\frac{1}{5}a_3\phi_2^6 + \frac{1}{10}\left(2a_1^2+7a_0a_2\right)\phi_2^4 - \frac{3}{4}a_0^2a_1\phi_2^2 + \frac{a_0^4}{16}$$

$$B_{8,5} = -\frac{1}{12}5a_2\phi_2^6 + \frac{15}{8}a_0a_1\phi_2^4 - \frac{5}{4}a_0^3\phi_2^2,\ B_{8,6} = -a_1\phi_2^6 + \frac{15}{4}a_0^2\phi_2^4,\ B_{8,7} = -\frac{7}{2}a_0\phi_2^6$$

$$B_{8,8} = \phi_2^8, \quad B_{8,p} = 0, \quad p = 9, 10, 11, \ldots$$

$$B_{9,2} = \frac{a_0\left(17a_1^3+192a_0a_2a_1+270a_0^2a_3\right)\phi_2}{12096} - \frac{\left(612a_4a_0^2+114a_2^2a_0+351a_1a_3a_0+49a_1^2a_2\right)\phi_2^3}{6048} + \frac{\left(105a_2a_3+182a_1a_4+570a_0a_5\right)\phi_2^5}{5040} - \frac{1}{28}a_6\phi_2^7$$

$$B_{9,3} = \frac{a_0^2\left(7a_1^2+12a_0a_2\right)\phi_2}{192} - \frac{\left(41a_1^3+531a_0a_2a_1+729a_0^2a_3\right)\phi_2^3}{3024} + \frac{\left(55a_2^2+150a_1a_3+408a_0a_4\right)\phi_2^5}{1680} - \frac{1}{14}a_5\phi_2^7$$

$$B_{9,4} = \frac{5}{24}a_0^3a_1\phi_2 - \frac{1}{36}a_0\left(11a_1^2+21a_0a_2\right)\phi_2^3 + \frac{1}{18}\left(4a_1a_2+9a_0a_3\right)\phi_2^5 - \frac{2}{15}a_4\phi_2^7$$

$$B_{9,5} = \frac{5}{16}a_0^4\phi_2 - \frac{5}{3}a_0^2a_1\phi_2^3 + \frac{1}{72}\left(23a_1^2+78a_0a_2\right)\phi_2^5 - \frac{1}{4}a_3\phi_2^7$$

$$B_{9,6} = -\frac{5}{2}a_0^3\phi_2^3 + \frac{11}{4}a_0a_1\phi_2^5 - \frac{1}{2}a_2\phi_2^7$$

$$B_{9,7} = \frac{21}{4}a_0^2\phi_2^5 - \frac{7}{6}a_1\phi_2^7, \quad B_{9,8} = -4a_0\phi_2^7, \quad B_{9,9} = \phi_2^9, \quad B_{9,p} = 0, \quad p = 10, 11, 12, \ldots$$

$$
\begin{aligned}
B_{10,2} =& -\frac{a_0^2\left(17a_1^3+192a_0a_2a_1+270a_0^2a_3\right)}{120960} \\
&+\frac{\left(128a_1^4+5931a_0a_2a_1^2+26865a_0^2a_3a_1+8178a_0^2a_2^2+36900a_0^3a_4\right)\phi_2^2}{907200} \\
&-\frac{\left(17250a_5a_0^2+5271a_2a_3a_0+9810a_1a_4a_0+800a_1a_2^2+1296a_1^2a_3\right)\phi_2^4}{151200} \\
&+\frac{\left(84a_3^2+210a_2a_4+400a_1a_5+1275a_0a_6\right)\phi_2^6}{12600}-\frac{1}{36}a_7\phi_2^8 \\
B_{10,3} =& -\frac{a_0^3\left(7a_1^2+12a_0a_2\right)}{1920}+\frac{a_0\left(475a_1^3+3660a_0a_2a_1+3942a_0^2a_3\right)\phi_2^2}{40320} \\
&-\frac{\left(556a_4a_0^2+128a_2^2a_0+365a_1a_3a_0+63a_1^2a_2\right)\phi_2^4}{2240} \\
&+\frac{\left(147a_2a_3+238a_1a_4+690a_0a_5\right)\phi_2^6}{3360}-\frac{3}{56}a_6\phi_2^8
\end{aligned}
$$

$$
\begin{aligned}
B_{10,4} =& -\frac{1}{48}a_0^4a_1+\frac{1}{480}a_0^2\left(81a_1^2+116a_0a_2\right)\phi_2^2-\frac{\left(272a_1^3+3177a_0a_2a_1+3942a_0^2a_3\right)\phi_2^4}{7560} \\
&+\frac{\left(145a_2^2+384a_1a_3+984a_0a_4\right)\phi_2^6}{2520}-\frac{2}{21}a_5\phi_2^8 \\
B_{10,5} =& -\frac{a_0^5}{32}+\frac{35}{48}a_0^3a_1\phi_2^2-\frac{1}{48}a_0\left(31a_1^2+56a_0a_2\right)\phi_2^4+\frac{1}{72}\left(25a_1a_2+54a_0a_3\right)\phi_2^6-\frac{1}{6}a_4\phi_2^8 \\
B_{10,6} =& \frac{15}{16}a_0^4\phi_2^2-\frac{25}{8}a_0^2a_1\phi_2^4+\frac{1}{60}\left(28a_1^2+93a_0a_2\right)\phi_2^6-\frac{3}{10}a_3\phi_2^8 \\
B_{10,7} =& \frac{1}{8}(-35)a_0^3\phi_2^4+\frac{91}{24}a_0a_1\phi_2^6-\frac{7}{12}a_2\phi_2^8, \quad B_{10,8}=7a_0^2\phi_2^6-\frac{4}{3}a_1\phi_2^8, \quad B_{10,9}=\frac{1}{2}(-9)a_0\phi_2^8 \\
B_{10,10} =& \phi_2^{10}, \quad B_{10,p}=0, \quad p=11,12,13,\dots,
\end{aligned}
$$

we obtain recursively, the following values:

$$
\begin{aligned}
C_1 =& a_1A_1 = a_1\phi_2 \qquad (B_{1,p}=0, p=2,3,4,\dots) \\
C_2 =& a_2\phi_2^2-\frac{a_0a_1}{2}, \qquad C_3=a_3\phi_2^3-\frac{1}{6}\left(a_1^2+6a_0a_2\right)\phi_2 \\
C_4 =& a_4\phi_2^4-\frac{1}{12}\left(5a_1a_2+18a_0a_3\right)\phi_2^2+\frac{1}{24}a_0\left(a_1^2+6a_0a_2\right) \\
C_5 =& a_5\phi_2^5-\frac{1}{60}\left(10a_2^2+33a_1a_3+120a_0a_4\right)\phi_2^3+\frac{1}{120}\left(a_1^3+36a_0a_2a_1+90a_0^2a_3\right)\phi_2 \\
C_6 =& a_6\phi_2^6-\frac{1}{20}\left(7a_2a_3+14a_1a_4+50a_0a_5\right)\phi_2^4 \\
&+\frac{1}{120}\left(180a_4a_0^2+22a_2^2a_0+81a_1a_3a_0+7a_1^2a_2\right)\phi_2^2-\frac{1}{720}a_0\left(a_1^3+36a_0a_2a_1+90a_0^2a_3\right) \\
C_7 =& a_7\phi_2^7-\frac{1}{140}\left(21a_3^2+56a_2a_4+120a_1a_5+420a_0a_6\right)\phi_2^5 \\
&+\frac{1}{420}\left(1050a_5a_0^2+231a_2a_3a_0+510a_1a_4a_0+25a_1a_2^2+51a_1^2a_3\right)\phi_2^3 \qquad (3.22) \\
&-\frac{\left(a_1^4+162a_0a_2a_1^2+1350a_0^2a_3a_1+336a_0^2a_2^2+2520a_0^3a_4\right)\phi_2}{5040}
\end{aligned}
$$

$$
\begin{aligned}
C_8 =& a_8\phi_2^8 - \frac{84a_3a_4 + 130a_2a_5 + 285a_1a_6 + 980a_0a_7}{280}\phi_2^6 \\
&+ \frac{50a_2^3 + 585a_1a_3a_2 + 2784a_0a_4a_2 + 1008a_0a_3^2 + 714a_1^2a_4}{3360}\phi_2^4 \\
&+ \frac{6450a_0a_1a_5 + 12600a_0^2a_6}{3360}\phi_2^4 \\
&- \frac{25200a_5a_0^3 + 6516a_2a_3a_0^2 + 15660a_1a_4a_0^2 + 1110a_1a_2^2a_0 + 2511a_1^2a_3a_0 + 85a_1^3a_2}{20160}\phi_2^2 \\
&+ \frac{a_0\left(a_1^4 + 162a_0a_2a_1^2 + 1350a_0^2a_3a_1 + 336a_0^2a_2^2 + 2520a_0^3a_4\right)}{40320} \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
C_9 =& a_9\phi_2^9 - \frac{\left(336a_4^2 + 810a_3a_5 + 1350a_2a_6 + 2975a_1a_7 + 10080a_0a_8\right)\phi_2^7}{2520} \\
&+ \frac{\left(52920a_7a_0^2 + 7488a_3a_4a_0 + 12060a_2a_5a_0 + 28140a_1a_6a_0 + 921a_1a_3^2 + 540a_2^2a_3\right)\phi_2^5}{10080} \\
&+ \frac{\left(2660a_1a_2a_4 + 3340a_1^2a_5\right)\phi_2^5}{10080} \\
&- \frac{\left(75600a_6a_0^3 + 7290a_3^2a_0^2 + 20700a_2a_4a_0^2 + 51450a_1a_5a_0^2 + 570a_2^3a_0\right)\phi_2^3}{30240} \\
&- \frac{\left(7296a_1a_2a_3a_0 + 9750a_1^2a_4a_0 + 270a_1^2a_2^2 + 461a_1^3a_3\right)\phi_2^3}{30240} \\
&+ \frac{\left(a_1^5 + 672a_0a_2a_1^3 + 14580a_0^2a_3a_1^2 + 6096a_0^2a_2^2a_1 + 78120a_0^3a_4a_1\right)\phi_2}{362880} \\
&+ \frac{\left(30780a_0^3a_2a_3 + 113400a_0^4a_5\right)\phi_2}{362880} \\
C_{10} =& a_{10}\phi_2^{10} - \frac{\left(660a_4a_5 + 891a_3a_6 + 1540a_2a_7 + 3388a_1a_8 + 11340a_0a_9\right)\phi_2^8}{2520} \\
&+ \frac{\left(352800a_8a_0^2 + 19680a_4^2a_0 + 48150a_3a_5a_0 + 83220a_2a_6a_0 + 193060a_1a_7a_0\right)\phi_2^6}{50400} \\
&+ \frac{\left(2541a_2a_3^2 + 3740a_2^2a_4 + 11418a_1a_3a_4 + 19360a_1a_2a_5 + 24090a_1^2a_6\right)\phi_2^6}{50400} \\
&- \frac{\left(1323000a_7a_0^3 + 232740a_3a_4a_0^2 + 387300a_2a_5a_0^2 + 957600a_1a_6a_0^2 + 50283a_1a_3^2a_0\right)\phi_2^4}{302400} \\
&- \frac{\left(27822a_2^2a_3a_0 + 149484a_1a_2a_4a_0 + 201750a_1^2a_5a_0 + 1650a_1a_2^3\right)\phi_2^4}{302400} \\
&- \frac{\left(11682a_1^2a_2a_3 + 11594a_1^3a_4\right)\phi_2^4}{302400} \\
&+ \frac{\left(1701000a_6a_0^4 + 177390a_3^2a_0^3 + 512280a_2a_4a_0^3 + 1348200a_1a_5a_0^3 + 16356a_2^3a_0^2\right)\phi_2^2}{1814400} \\
&+ \frac{\left(224946a_1a_2a_3a_0^2 + 321840a_1^2a_4a_0^2 + 12972a_1^2a_2^2a_0 + 23886a_1^3a_3a_0 + 341a_1^4a_2\right)\phi_2^2}{1814400} \\
&- \frac{a_0\left(a_1^5 + 672a_0a_2a_1^3 + 14580a_0^2a_3a_1^2 + 6096a_0^2a_2^2a_1 + 78120a_0^3a_4a_1\right)}{3628800} \\
&- \frac{a_0\left(30780a_0^3a_2a_3 + 113400a_0^4a_5\right)}{3628800}.
\end{aligned} \tag{3.24}
$$

Using (3.22)-(3.24) in (3.20), we obtain the result as required. □

4. Second Derivative Backward Differentiation Boundary Value Method

This section presents a numerical method for finding the solution of the boundary value problem (2.1). Linear multistep methods (LMMs) for the numerical integration of ordinary differential equations have gained popular interests. The backward differentiation formula (BDF), a subclass of LMMs was proposed by [17, 21, 16], and extended to second derivative backward differentiation formula (SDBDF) by [19, 47, 48], just to mention a few. Here, in the spirit of [14, 44], we implement a twelfth order SDBDF as a boundary value method (BVM) with some sets of boundary conditions. This method would be referred to as second derivative backward differentiation boundary value method (SDBDBVM). The proposed SDBDBVM has the general form given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_\nu f_{n+\nu} + h^2 f'_{n+\nu} \tag{4.1}$$

together with $(\nu, k-\nu)$ - boundary conditions, where

$$\nu = \begin{cases} \frac{k+2}{2} & \text{for even } k, \\ \frac{k+1}{2} & \text{for odd } k, \end{cases}$$

and h is the stepsize defined as $h = x_n - x_{n-1}$, α_j, $j = 0(1)k$ and β_ν are constant coefficients, $y_{n+j} \approx y(x_n + jh)$, $f_{n+\nu} \approx y'(x_n + \nu h)$ and $f'_{n+\nu} \approx y''(x_n + \nu h)$. The method (4.1) has maximum order $p = k+1$, and to implement (4.1) requires $(\nu, k-\nu)$-boundary conditions. The boundary conditions are derived using the initial methods

$$\sum_{j=0}^{k} \alpha_j^{(i)} y_{n+j} = h\beta_i^{(i)} f_i + h^2 f'_i, \qquad i = 1, 2, 3, \ldots, \nu - 1 \tag{4.2}$$

and final methods

$$\sum_{j=0}^{k} \alpha_j^{(f)} y_{n+j} = h\beta_i^{(f)} f_i + h^2 f'_i, \qquad i = \nu + 1, \ldots, N. \tag{4.3}$$

To derive the twelfth order SDBDBVM, we set $k = 11$ and $\nu = 6$ in (4.1), giving

$$\begin{aligned} \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5} + \alpha_6 y_{n+6} + \alpha_7 y_{n+7} \\ + \alpha_8 y_{n+8} + \alpha_9 y_{n+9} + \alpha_{10} y_{n+10} + \alpha_{11} y_{n+11} = h\beta_6 f_{n+6} + h^2 f'_{n+6}. \end{aligned} \tag{4.4}$$

Expanding (4.4) by Taylor series about x_n and solving the resulting system of equations generated by equating the coefficients of each $y^{(i)} = \frac{d^{(i)}y}{dx^{(i)}}$, $i = 0(1)12$ gives the twelfth order SDBDBVM

$$\begin{aligned} -\frac{1}{8316} y_n + \frac{1}{525} y_{n+1} - \frac{5}{336} y_{n+2} + \frac{5}{63} y_{n+3} - \frac{5}{14} y_{n+4} + 2y_{n+5} - \frac{5369}{1800} y_{n+6} + \frac{10}{7} y_{n+7} \\ -\frac{5}{28} y_{n+8} + \frac{5}{189} y_{n+9} - \frac{1}{336} y_{n+10} + \frac{1}{5775} y_{n+11} = -\frac{h}{3} f_{n+6} + h^2 f'_{n+6}, \end{aligned} \tag{4.5}$$

for $n = 6, \ldots, N-1$. To implement (4.5), a $(6,5)$-boundary conditions are derived using (4.2), and (4.3) alongside (4.5). The procedure for deriving these initial and final methods is the same as the one discussed earlier. The initial methods are

$$\begin{aligned} \frac{2}{11} y_0 - \frac{19910683}{3175200} y_1 + 10y_2 - \frac{15}{2} y_3 + \frac{20}{3} y_4 - \frac{21}{4} y_5 + \frac{84}{25} y_6 - \frac{5}{3} y_7 + \frac{30}{49} y_8 - \frac{5}{32} y_9 \\ + \frac{2}{81} y_{10} - \frac{1}{550} y_{11} = h\frac{4861}{1260} f_1 + h^2 f'_1, \end{aligned} \tag{4.6}$$

$$-\frac{1}{110}y_0+\frac{2}{5}y_1-\frac{14465971}{3175200}y_2+6y_3-3y_4+\frac{28}{15}y_5-\frac{21}{20}y_6+\frac{12}{25}y_7-\frac{1}{6}y_8+\frac{2}{49}y_9 -\frac{1}{160}y_{10}+\frac{2}{4455}y_{11}=h\frac{3349}{1260}f_2+h^2f_2', \tag{4.7}$$

$$\frac{2}{1485}y_0-\frac{1}{30}y_1+\frac{2}{3}y_2-\frac{1295099}{352800}2y_3+4y_4-\frac{7}{5}y_5+\frac{28}{45}y_6-\frac{1}{4}y_7+\frac{2}{25}y_8-\frac{1}{54}y_9 +\frac{2}{735}y_{10}-\frac{1}{5280}y_{11}=h\frac{743}{420}f_3+h^2f_3', \tag{4.8}$$

$$-\frac{1}{2640}y_0+\frac{1}{135}y_1-\frac{1}{12}y_2+y_3-\frac{281801}{88200}y_4+\frac{14}{5}y_5-\frac{7}{10}y_6+\frac{2}{9}y_7-\frac{1}{16}y_8+\frac{1}{75}y_9 -\frac{1}{540}y_{10}+\frac{1}{8085}y_{11}=h\frac{107}{105}f_4+h^2f_4', \tag{4.9}$$

$$\frac{1}{5775}y_0-\frac{1}{336}y_1+\frac{5}{189}y_2-\frac{5}{28}y_3+\frac{10}{7}y_4-\frac{5369}{1800}y_5+2y_6-\frac{5}{14}y_7+\frac{5}{63}y_8-\frac{5}{336}y_9 +\frac{1}{525}y_{10}-\frac{1}{8316}y_{11}=\frac{h}{3}f_5+h^2f_5', \tag{4.10}$$

and the final method is

$$\frac{2}{121}y_{N-11}-\frac{11}{50}y_{N-10}+\frac{110}{81}y_{N-9}-\frac{165}{32}y_{N-8}+\frac{660}{49}y_{N-7}-\frac{77}{3}y_{N-6}+\frac{924}{25}y_{N-5} -\frac{165}{4}y_{N-4}+\frac{110}{3}y_{N-3}-\frac{55}{2}y_{N-2}+22y_{N-1}-\frac{4102360483}{384199200}y_N=-h\frac{83711}{13860}f_N+h^2f_N'. \tag{4.11}$$

To verify the order p of (4.5), SDBDBVM (4.5) can be written as

$$-\frac{1}{8316}y(x_n)+\frac{1}{525}y(x_n+h)-\frac{5}{336}y(x_n+2h)+\frac{5}{63}y(x_n+3h)-\frac{5}{14}y(x_n+4h) +2y(x_n+5h)-\frac{5369}{1800}y(x_n+6h)+\frac{10}{7}y(x_n+7h)-\frac{5}{28}y(x_n+8h)+\frac{5}{189}y(x_n+9h) -\frac{1}{336}y(x_n+10h)+\frac{1}{5775}y(x_n+11h)+\frac{h}{3}y'(x_n+6h)-h^2y''(x_n+6h)=O(h^{p+1}). \tag{4.12}$$

Expanding (4.12) by Taylor series about x_n gives

$$-\frac{1}{8316}y(x_n)+\frac{1}{525}y(x_n+h)-\frac{5}{336}y(x_n+2h)+\frac{5}{63}y(x_n+3h)-\frac{5}{14}y(x_n+4h) +2y(x_n+5h)-\frac{5369}{1800}y(x_n+6h)+\frac{10}{7}y(x_n+7h)-\frac{5}{28}y(x_n+8h)+\frac{5}{189}y(x_n+9h) -\frac{1}{336}y(x_n+10h)+\frac{1}{5775}y(x_n+11h)+\frac{h}{3}y'(x_n+6h)-h^2y''(x_n+6h) =\frac{h^{13}y^{(13)}(x)}{36036}+O\left(h^{14}\right). \tag{4.13}$$

Thus, the SDBDBVM (4.5) has order $p = 12$ with error constant $\frac{1}{36036}$. The SDBDBVM (4.5) has a region of absolute stability in the interval $[-\infty, 0]$ (see Figure 1 for the boundary locus plot) making it suitable to solve problems having rapidly and slowly decaying transients in their solution ([11], [24], [50]).

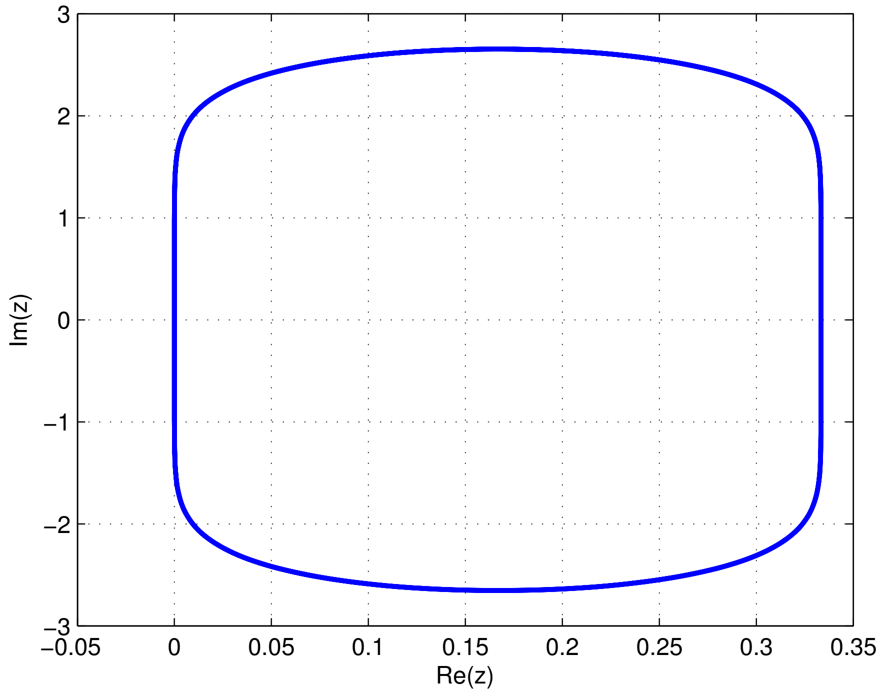

Figure 1: Stability region of the SDBDBVM (4.5): Points outside the boundary are stable region of absolute stability.

5. Illustrative Examples

To make our calculations more explicit and computationally interesting, we consider those nonlinear functions whose expansion coefficients are explicitly known in the case of a power series method. Coincidentally, the special cases in which the nonlinear functions are given in terms of exponential functions are the Bratu-type equations. Specifically, we consider special cases of Corollary 3.1 with the nonlinear terms given by the exponential functions enumerated in Section 2 (see (a)-(c)). The results obtained from the numerical implementation of SDBDBVM (4.5) for the proposed examples are shown in Tables 1, 2, and 3; and Figures 2, 3, and 4.

The absolute error is given by $|y_{\text{exact}}(x) - y_{\text{approx.}}(x)|$, where $y_{\text{exact}}(x)$ is the exact solution and $y_{\text{approx.}}(x)$ is the approximate solution $(0 < x \leqslant 1)$.

Example 5.1. *Consider a strongly nonlinear Bratu-type boundary value problem*

$$\begin{aligned} &y''(x) + \mu e^{y(x)} = 0, \quad 0 < x < 1, \\ &y(0) = 0, \;\; y(1) = 0. \end{aligned} \tag{5.1}$$

The exact solution is

$$y(x) = -2\ln\left(\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{u}{2}\right)}{\cosh\frac{u}{4}}\right), \tag{5.2}$$

where u *is a solution of the equation* $u-\sqrt{2\mu}\cosh\left(\frac{u}{4}\right)=0$*. One observes that the nonlinear function* $f(y(x))$ *in this case takes the series expansion*

$$f(y(x)) = \mu e^{y(x)} = \sum_{p=0}^{\infty} a_p y^p(x) \qquad (a_p = \mu/p!). \tag{5.3}$$

Thus the Bratu-type boundary value problem (5.1) *admits an analytical solution given by*

$$\begin{aligned} y(x) =& \phi_2 x - \frac{\mu}{2}x^2 - \frac{\mu}{6}\phi_2 x^3 + \frac{\mu\left(\mu-\phi_2^2\right)}{24}x^4 + \frac{\mu\phi_2\left(4\mu-\phi_2^2\right)}{120}x^5 \\ &- \frac{\mu\left(4\mu^2-11\mu\phi_2^2+\phi_2^4\right)}{720}x^6 - \frac{\mu\phi_2\left(34\mu^2-26\mu\phi_2^2+\phi_2^4\right)}{5040}x^7 \\ &+ \frac{\mu\left(34\mu^3-180\mu^2\phi_2^2+57\mu\phi_2^4-\phi_2^6\right)x^8}{40320} + \frac{\mu\phi_2\left(496\mu^3-768\mu^2\phi_2^2+120\mu\phi_2^4-\phi_2^6\right)}{362880}x^9 \\ &- \frac{\mu\left(496\mu^4-4288\mu^3\phi_2^2+2904\mu^2\phi_2^4-247\mu\phi_2^6+\phi_2^8\right)}{3628800}x^{10} \\ &- \frac{\mu\phi_2\left(11056\mu^4-28768\mu^3\phi_2^2+10194\mu^2\phi_2^4-502\mu\phi_2^6+\phi_2^8\right)}{39916800}x^{11} \\ &+ \frac{\mu\left(11056\mu^5-141584\mu^4\phi_2^2+166042\mu^3\phi_2^4-34096\mu^2\phi_2^6+1013\mu\phi_2^8-\phi_2^{10}\right)x^{12}}{479001600} + \cdots, \end{aligned} \tag{5.4}$$

where ϕ_2 *is to be determined using the right boundary condition. To see the effectiveness and reliability of the present method, we consider the special case* $\mu=-\pi^2$ *([62]). Indeed one has from* (5.4) *the series solution*

$$\begin{aligned} y(x) =& \phi_2 x + \frac{\pi^2 x^2}{2!} + \frac{\pi^2\phi_2}{3!}x^3 + \frac{\pi^2\left(\phi_2^2+\pi^2\right)}{4!}x^4 + \frac{\pi^2\phi_2\left(\phi_2^2+4\pi^2\right)}{5!}x^5 \\ &+ \frac{\pi^2\left(\phi_2^4+11\pi^2\phi_2^2+4\pi^4\right)}{6!}x^6 + \frac{\pi^2\phi_2\left(\phi_2^4+26\pi^2\phi_2^2+34\pi^4\right)}{7!}x^7 \\ &+ \frac{\pi^2\left(\phi_2^6+57\pi^2\phi_2^4+180\pi^4\phi_2^2+34\pi^6\right)}{8!}x^8 \\ &+ \frac{\pi^2\phi_2\left(\phi_2^6+120\pi^2\phi_2^4+768\pi^4\phi_2^2+496\pi^6\right)}{9!}x^9 \\ &+ \frac{\pi^2\left(\phi_2^8+247\pi^2\phi_2^6+2904\pi^4\phi_2^4+4288\pi^6\phi_2^2+496\pi^8\right)}{10!}x^{10} \\ &+ \frac{\pi^2\phi_2\left(\phi_2^8+502\pi^2\phi_2^6+10194\pi^4\phi_2^4+28768\pi^6\phi_2^2+11056\pi^8\right)}{11!}x^{11} \\ &+ \frac{\pi^2\left(\phi_2^{10}+1013\pi^2\phi_2^8+34096\pi^4\phi_2^6+166042\pi^6\phi_2^4+141584\pi^8\phi_2^2+11056\pi^{10}\right)}{12!}x^{12} \\ &+ \cdots, \end{aligned} \tag{5.5}$$

which is in excellent agreement with [62, Eq. (23)]. To obtain the exact solution, we apply the boundary condition $y(1) = 0$ *to get* $\phi_2 = \pi$ *([62]). As a result, the solution* (5.5) *reduces to*

$$\begin{aligned} y(x) =& \pi x + \frac{\pi^2 x^2}{2} + \frac{\pi^3 x^3}{6} + \frac{\pi^4 x^4}{12} + \frac{\pi^5 x^5}{24} + \frac{\pi^6 x^6}{45} + \frac{61\pi^7 x^7}{5040} + \frac{17\pi^8 x^8}{2520} + \frac{277\pi^9 x^9}{72576} \\ &+ \frac{31\pi^{10} x^{10}}{14175} + \frac{50521\pi^{11} x^{11}}{39916800} + \frac{691\pi^{12} x^{12}}{935550} + \cdots, \end{aligned} \tag{5.6}$$

which converges to the exact solution

$$y(x) = -\ln\left(1 + \cos\left(\left(x + \frac{1}{2}\right)\pi\right)\right). \tag{5.7}$$

Furthermore, for numerical comparison purposes, we consider the special values $\mu = 1, 1.5, 2$. *Thus we are interested in the approximate solution, say,* $y_{12}(x)$. *Hence,* ϕ_2 *is the optimal solution of the algebraic equation*

$$\begin{aligned} &+ \mu\phi_2^{10} + 12\mu\phi_2^9 - \mu(1013\mu - 132)\phi_2^8 - 24\mu(251\mu - 55)\phi_2^7 + 4\mu\left(8524\mu^2 - 8151\mu + 2970\right)\phi_2^6 \\ &+ 72\mu\left(1699\mu^2 - 2200\mu + 1320\right)\phi_2^5 - 2\mu\left(83021\mu^3 - 191664\mu^2 + 338580\mu - 332640\right)\phi_2^4 \\ &- 384\mu\left(899\mu^3 - 2640\mu^2 + 6435\mu - 10395\right)\phi_2^3 \\ &+ 16\mu\left(8849\mu^4 - 35376\mu^3 + 133650\mu^2 - 457380\mu + 1247400\right)\phi_2^2 \\ &+ 192\left(691\mu^5 - 3410\mu^4 + 16830\mu^3 - 83160\mu^2 + 415800\mu - 2494800\right)\phi_2 \\ &- 16\mu\left(691\mu^5 - 4092\mu^4 + 25245\mu^3 - 166320\mu^2 + 1247400\mu - 14968800\right) = 0. \end{aligned} \tag{5.8}$$

(i) $\mu = 1$: *It is understood here that*

$$\begin{aligned} &\phi_2^{10} + 12\phi_2^9 - 881\phi_2^8 - 4704\phi_2^7 + 13372\phi_2^6 + 58968\phi_2^5 + 205406\phi_2^4 + 2189184\phi_2^3 \\ &+ 14354288\phi_2^2 - 412425408\phi_2 + 221854016 = 0, \end{aligned}$$

with the optimal solution given by $\phi_2 = 0.549362$. *As a result, we have the series solution*

$$\begin{aligned} y(x) =& 0.549362x - 0.5x^2 - 0.0915604x^3 + 0.0290917x^4 + 0.0169304x^5 - 0.00107124x^6 \\ &- 0.00286064x^7 - 0.000375983x^8 + 0.000416504x^9 + 0.000148918x^{10} \\ &- 0.0000452593x^{11} - 0.000036491x^{12}. \end{aligned} \tag{5.9}$$

(ii) $\mu = 1.5$: *Here we have the equation*

$$\begin{aligned} &\phi_2^{10} + 12\phi_2^9 - 1387.5\phi_2^8 - 7716\phi_2^7 + 39690\phi_2^6 + 132678\phi_2^5 - 48363.7\phi_2^4 \\ &+ 1.40098 \times 10^6\phi_2^3 + 1.25991 \times 10^7\phi_2^2 - 2.57718 \times 10^8\phi_2 + 2.14435 \times 10^8 = 0 \end{aligned}$$

whose optimal solution is given by $\phi_2 = 0.873148$. *Hence, we obtain the series solution*

$$\begin{aligned} y(x) =& 0.873148x - 0.75x^2 - 0.218287x^3 + 0.0461007x^4 + 0.0571652x^5 + 0.00624619x^6 \\ &- 0.0123041x^7 - 0.00538559x^8 + 0.00166304x^9 + 0.00202064x^{10} \\ &+ 0.000165783x^{11} - 0.000533177x^{12}. \end{aligned} \tag{5.10}$$

(iii) $\mu = 2$: *Here one has the polynomial equation*

$$\phi_2^{10} + 12\phi_2^9 - 1894\phi_2^8 - 10728\phi_2^7 + 83056\phi_2^6 + 267552\phi_2^5 - 484064\phi_2^4 + 342912\phi_2^3 + 11613056\phi_2^2 - 181790208\phi_2 + 207690880 = 0,$$

which upon solving gives the optimal solution $\phi_2 = 1.24441$. *Hence, we have the series solution*

$$\begin{aligned} y(x) =& 1.24441x - 1.x^2 - 0.414802x^3 + 0.0376213x^4 + 0.133804x^5 + 0.0435277x^6 \\ &- 0.0285787x^7 - 0.0284372x^8 - 0.00149051x^9 + 0.0105593x^{10} \\ &+ 0.00532682x^{11} - 0.00193596x^{12}. \end{aligned} \tag{5.11}$$

Table 1 presents the comparison between PSM and SDBDBVM. Subsequently, the SDBDBVM will be simply referred to as BVM.

Remark 5.1 (Bratu-type initial value problem). It can be easily deduced that upon setting $\mu = -2$ and $\phi_2 = 0$ in the generalised series solution (5.4), one sees that a series solution to the Bratu-type initial value problem

$$\begin{aligned} &y''(x) - 2e^{y(x)} = 0, \quad 0 < x < 1, \\ &y(0) = 0, \ \ y'(0) = 0, \end{aligned} \tag{5.12}$$

is given by ([57], [62])

$$\begin{aligned} y(x) =& x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \frac{62x^{10}}{14175} + \frac{691x^{12}}{467775} + \cdots \\ =& -2\left(-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \frac{31x^{10}}{14175} - \frac{691x^{12}}{935550} - \cdots\right). \end{aligned} \tag{5.13}$$

The series solution (5.13) interestingly converges to the exact solution

$$y(x) = -\ln(\cos(x)). \tag{5.14}$$

Example 5.2. *Consider another strongly nonlinear Bratu-type boundary value problem*

$$\begin{aligned} &y''(x) + \mu e^{-y(x)} = 0, \quad 0 < x < 1, \\ &y(0) = 0, \ \ y(1) = 0. \end{aligned} \tag{5.15}$$

Similarly, one has the following series expansion for $f(y(x))$*:*

$$f(y(x)) = \mu e^{-y(x)} = \sum_{p=0}^{\infty} a_p y^p(x) \qquad (a_p = (-1)^p \mu / p!) . \tag{5.16}$$

Hence, the Bratu-type boundary value problem (5.15) *admits an analytical solution given by*

$$\begin{aligned} y(x) =& \phi_2 x - \frac{\mu}{2}x^2 + \frac{\mu\phi_2}{6}x^3 - \frac{\mu\left(\mu+\phi_2^2\right)}{24}x^4 + \frac{\mu\phi_2\left(4\mu+\phi_2^2\right)}{120}x^5 \\ &- \frac{\mu\left(4\mu^2+11\mu\phi_2^2+\phi_2^4\right)}{720}x^6 + \frac{\mu\phi_2\left(34\mu^2+26\mu\phi_2^2+\phi_2^4\right)x^7}{5040} \\ &- \frac{\mu\left(34\mu^3+180\mu^2\phi_2^2+57\mu\phi_2^4+\phi_2^6\right)x^8}{40320} + \frac{\mu\phi_2\left(496\mu^3+768\mu^2\phi_2^2+120\mu\phi_2^4+\phi_2^6\right)}{362880}x^9 \\ &- \frac{\mu\left(496\mu^4+4288\mu^3\phi_2^2+2904\mu^2\phi_2^4+247\mu\phi_2^6+\phi_2^8\right)}{3628800}x^{10} \\ &+ \frac{\mu\phi_2\left(11056\mu^4+28768\mu^3\phi_2^2+10194\mu^2\phi_2^4+502\mu\phi_2^6+\phi_2^8\right)}{39916800}x^{11} \\ &- \frac{\mu x^{12}\left(11056\mu^5+141584\mu^4\phi_2^2+166042\mu^3\phi_2^4+34096\mu^2\phi_2^6+1013\mu\phi_2^8+\phi_2^{10}\right)}{479001600} + \cdots, \end{aligned} \tag{5.17}$$

where ϕ_2 *is to be determined using the right boundary condition. One may see the effectiveness and reliability of the present method by setting* $\mu = \pi^2$ *([62]). Indeed one has from* (5.17) *the series solution*

$$\begin{aligned} y(x) =& \phi_2 x - \frac{\pi^2 x^2}{2!} + \frac{\pi^2\phi_2}{3!}x^3 - \frac{\pi^2\left(\phi_2^2+\pi^2\right)}{4!}x^4 + \frac{\pi^2\phi_2\left(\phi_2^2+4\pi^2\right)}{5!}x^5 \\ &- \frac{\pi^2\left(\phi_2^4+11\pi^2\phi_2^2+4\pi^4\right)}{6!}x^6 + \frac{\pi^2\phi_2\left(\phi_2^4+26\pi^2\phi_2^2+34\pi^4\right)}{7!}x^7 \\ &- \frac{\pi^2\left(\phi_2^6+57\pi^2\phi_2^4+180\pi^4\phi_2^2+34\pi^6\right)}{8!}x^8 \\ &+ \frac{\pi^2\phi_2\left(\phi_2^6+120\pi^2\phi_2^4+768\pi^4\phi_2^2+496\pi^6\right)}{9!}x^9 \\ &- \frac{\pi^2\left(\phi_2^8+247\pi^2\phi_2^6+2904\pi^4\phi_2^4+4288\pi^6\phi_2^2+496\pi^8\right)}{10!}x^{10} \\ &+ \frac{\pi^2\phi_2\left(\phi_2^8+502\pi^2\phi_2^6+10194\pi^4\phi_2^4+28768\pi^6\phi_2^2+11056\pi^8\right)}{11!}x^{11} \\ &- \frac{\pi^2\left(\phi_2^{10}+1013\pi^2\phi_2^8+34096\pi^4\phi_2^6+166042\pi^6\phi_2^4+141584\pi^8\phi_2^2+11056\pi^{10}\right)}{12!}x^{12} \\ &+ \cdots, \end{aligned} \tag{5.18}$$

The solution (5.18) *agrees perfectly with [62, Eq. (35)].The exact solution is obtained upon applying the boundary condition* $y(1) = 0$ *to have* $\phi_2 = \pi$ *([62]). As a result, the solution* (5.18) *reduces to*

$$\begin{aligned} y(x) =& \pi x - \frac{\pi^2x^2}{2} + \frac{\pi^3x^3}{6} - \frac{\pi^4x^4}{12} + \frac{\pi^5x^5}{24} - \frac{\pi^6x^6}{45} + \frac{61\pi^7x^7}{5040} - \frac{17\pi^8x^8}{2520} + \frac{277\pi^9x^9}{72576} \\ &- \frac{31\pi^{10}x^{10}}{14175} + \frac{50521\pi^{11}x^{11}}{39916800} - \frac{691\pi^{12}x^{12}}{935550} + \cdots, \end{aligned} \tag{5.19}$$

which converges to the exact solution

$$y(x) = \ln\left(1+\sin(\pi x+1)\right). \tag{5.20}$$

Moreover, for numerical comparisons of results, we consider the special values $\mu = 1, 1.5, 2$. *Similarly, in this case, we consider the approximate solution* $y_{12}(x)$. *Thus* ϕ_2 *is the optimal solution of the algebraic equation*

$$\begin{aligned}
&\mu\phi_2^{10} - 12\mu\phi_2^9 + \mu(1013\mu + 132)\phi_2^8 - 24\mu(251\mu + 55)\phi_2^7 + 4\mu\left(8524\mu^2 + 8151\mu + 2970\right)\phi_2^6 \\
&- 72\mu\left(1699\mu^2 + 2200\mu + 1320\right)\phi_2^5 + 2\mu\left(83021\mu^3 + 191664\mu^2 + 338580\mu + 332640\right)\phi_2^4 \\
&- 384\mu\left(899\mu^3 + 2640\mu^2 + 6435\mu + 10395\right)\phi_2^3 \\
&+ 16\mu\left(8849\mu^4 + 35376\mu^3 + 133650\mu^2 + 457380\mu + 1247400\right)\phi_2^2 \\
&- 192\left(691\mu^5 + 3410\mu^4 + 16830\mu^3 + 83160\mu^2 + 415800\mu + 2494800\right)\phi_2 \\
&+ 16\mu\left(691\mu^5 + 4092\mu^4 + 25245\mu^3 + 166320\mu^2 + 1247400\mu + 14968800\right) = 0. \qquad (5.21)
\end{aligned}$$

(i) $\mu = 1$: *Here one sees that*

$$\begin{aligned}
&\phi_2^{10} - 12\phi_2^9 + 1145\phi_2^8 - 7344\phi_2^7 + 78580\phi_2^6 - 375768\phi_2^5 + 1891810\phi_2^4 - 7821696\phi_2^3 \\
&+ 30122480\phi_2^2 - 578820672\phi_2 + 262600768 = 0
\end{aligned}$$

with the optimal solution given by $\phi_2 = 0.463662$. *It follows that the associated series solution is given by*

$$\begin{aligned}
y(x) =&0.463662x - 0.500000x^2 + 0.077277x^3 - 0.0506243x^4 + 0.0162861x^5 \\
&- 0.0089042x^6 + 0.00364635x^7 - 0.00186858x^8 + 0.000851813x^9 \\
&- 0.000428383x^{10} + 0.000205792x^{11} - 0.000103359x^{12}. \qquad (5.22)
\end{aligned}$$

(ii) $\mu = 1.5$: *Here we have the equation*

$$\begin{aligned}
&\phi_2^{10} - 12\phi_2^9 + 1651.5\phi_2^8 - 10356\phi_2^7 + 137502\phi_2^6 - 607878\phi_2^5 + 3.1039 \times 10^6\phi_2^4 \\
&- 1.11443 \times 10^7\phi_2^3 + 3.8374 \times 10^7\phi_2^2 - 4.3327 \times 10^8\phi_2 + 2.77205 \times 10^8 = 0,
\end{aligned}$$

with the optimal solution given by $\phi_2 = 0.673413$. *Hence, we have the series solution*

$$\begin{aligned}
y(x) =&0.673413x - 0.750000x^2 + 0.168353x^3 - 0.122093x^4 + 0.0543233x^5 \\
&- 0.034767x^6 + 0.018918x^7 - 0.0117592x^8 + 0.00694439x^9 - 0.0043205x^{10} \\
&+ 0.00265172x^{11} - 0.00166428x^{12}. \qquad (5.23)
\end{aligned}$$

(iii) $\mu = 2$: *Here one has the polynomial equation*

$$\begin{aligned}
&\phi_2^{10} - 12\phi_2^9 + 2158\phi_2^8 - 13368\phi_2^7 + 213472\phi_2^6 - 901152\phi_2^5 + 4.88125 \times 10^6\phi_2^4 \\
&- 1.57505 \times 10^7\phi_2^3 + 4.99416 \times 10^7\phi_2^2 - 3.71554 \times 10^8\phi_2 + 2.94695 \times 10^8 = 0,
\end{aligned}$$

which upon solving gives the optimal solution $\phi_2 = 0.874226$. *Hence, one obtains the series solution*

$$\begin{aligned}
y(x) =&0.874226x - 1.000000x^2 + 0.291409x^3 - 0.230356x^4 + 0.127699x^5 \\
&- 0.0927724x^6 + 0.0611702x^7 - 0.0441126x^8 + 0.031109x^9 - 0.0226849x^{10} \\
&+ 0.0165159x^{11} - 0.0122029x^{12}. \qquad (5.24)
\end{aligned}$$

Table 2 presents the comparison between PSM and BVM.

Example 5.3. *Consider the Liu-Wang version of Bratu-type boundary value problem given by*

$$y''(x) + \frac{18\,(E/Q)^2}{e^{y(x)/3}}\left(I - \frac{QKE}{\rho}e^{y(x)/6}\right)^2 e^{y(x)} = 0, \quad 0 < x < 1, \qquad (5.25)$$
$$y(0) = 0, \;\; y(1) = 0,$$

where the parameters E, Q, I, K, ρ *are as described in Section 2. One observes that the nonlinear function* $f(y(x))$ *in this case takes the series expansion*

$$f(y(x)) = \alpha e^{y(x)} + \beta e^{5y(x)/6} + \gamma e^{2y(x)/3} = \sum_{p=0}^{\infty} a_p y^p(x), \qquad (5.26)$$

where

$$a_p = \frac{\alpha}{p!} + \frac{5^p\beta}{6^p p!} + \frac{2^p\gamma}{3^p p!}, \qquad \alpha = \frac{18E^4K^2}{\rho^2}, \beta = -\frac{36E^3KI}{\rho Q}, \gamma = \frac{18E^2I^2}{Q^2}. \qquad (5.27)$$

For numerical comparison purposes, we consider the special values $E = 1.23, Q = 5, K = 1, \rho = 1.5, I = 4$*;* $E = 1, Q = 5, K = 1, \rho = 2.5, I = 3$*; and* $E = 1.23, Q = 8, K = 1, \rho = 2.5, I = 4$*. Thus we look for the approximate solution* $y_{12}(x)$*.*

(i) $E = 1.23, Q = 5, K = 1, \rho = 1.5, I = 4$: *Indeed one has from Corollary 3.1 the series solution*

$$\begin{aligned} y(x) =& \phi_2 x - 0.00544644x^2 - 0.0260219\phi_2 x^3 + \frac{0.00170072 - 1.24542\phi_2^2}{24}x^4 \\ &+ \frac{\phi_2\left(0.0650756 - 2.79866\phi_2^2\right)}{120}x^5 - \frac{4.52337\phi_2^4 - 1.15516\phi_2^2 + 0.000708861}{720}x^6 \\ &- \frac{\phi_2\left(6.24751\phi_2^4 - 13.0546\phi_2^2 + 0.0469131\right)}{5040}x^7 \\ &- \frac{7.87556\phi_2^6 - 89.0475\phi_2^4 + 1.6048\phi_2^2 + 0.000511019}{40320}x^8 \\ &- \frac{\phi_2\left(9.35978\phi_2^6 - 438.007\phi_2^4 + 35.9315\phi_2^2 + 0.0551202\right)}{362880}x^9 \\ &- \frac{10.6816\phi_2^8 - 1743.23\phi_2^6 + 581.027\phi_2^4 - 3.05086\phi_2^2 + 0.000600418}{3628800}x^{10} \\ &- \frac{\phi_2\left(11.8398\phi_2^8 - 6003.77\phi_2^6 + 6980.5\phi_2^4 - 113.869\phi_2^2 + 0.0955136\right)}{39916800}x^{11} \\ &- \frac{12.8428\phi_2^{10} - 18644\phi_2^8 + 65223.7\phi_2^6 - 3160.8\phi_2^4 + 7.84056\phi_2^2 - 0.00104042}{479001600}x^{12}. \end{aligned} \qquad (5.28)$$

Upon applying the right boundary condition, we have

$$\begin{aligned} &\phi_2^{10} + 11.0629\phi_2^9 - 1341.92\phi_2^8 - 4647.77\phi_2^7 - 5553.41\phi_2^6 + 7736.72\phi_2^5 + 157673\phi_2^4 \\ &+ 776834\phi_2^3 + 1.87706 \times 10^6\phi_2^2 - 3.63466 \times 10^7\phi_2 + 200531 = 0, \end{aligned}$$

with the optimal solution given by $\phi_2 = 0.00551875$*. As a result, we have the series solution*

$$\begin{aligned} y(x) =& 0.00551875x - 0.00544644x^2 - 0.000143608x^3 + 0.0000692828x^4 \\ &+ 2.9888795423469765 \times 10^{-6}x^5 - 9.356702036789943 \times 10^{-7}x^6 \\ &- 5.0934013623794574 \times 10^{-8}x^7 + 1.1463902381384029 \times 10^{-8}x^8 \\ &+ 8.216421463230975 \times 10^{-10}x^9 - 1.4000162229976573 \times 10^{-10}x^{10} \\ &- 1.2726773685304435 \times 10^{-11}x^{11} + 1.679642411906002 \times 10^{-12}x^{12}. \end{aligned} \quad (5.29)$$

(ii) $E = 1, Q = 5, K = 1, \rho = 2.5, I = 3$: *Here also we obtain from Corollary 3.1 a series solution*

$$\begin{aligned} y(x) =& \phi_2 x - 0.36x^2 + 0.01\phi_2^2x^4 + \frac{0.2\phi_2^2 - 0.5184}{120}\phi_2 x^5 \\ &+ \frac{0.00666667\phi_2^4 - 0.864\phi_2^2 + 0.373248}{720}x^6 - \frac{0.261111\phi_2^4 - 0.24\phi_2^2 - 1.5552}{5040}\phi_2 x^7 \\ &- \frac{0.55537\phi_2^6 - 3.828\phi_2^4 + 1.21306\phi_2^2 + 1.11974}{40320}x^8 \\ &- \frac{0.847994\phi_2^6 - 9.3044\phi_2^4 + 22.1962\phi_2^2 - 2.24695}{362880}\phi_2 x^9 \\ &- \frac{1.12345\phi_2^8 - 12.3196\phi_2^6 + 69.9016\phi_2^4 - 67.956\phi_2^2 + 1.61781}{3628800}x^{10} \\ &- \frac{1.37407\phi_2^8 - 2.6696\phi_2^6 + 77.0126\phi_2^4 - 307.357\phi_2^2 + 112.266}{39916800}\phi_2 x^{11} \\ &- \frac{1.59696\phi_2^{10} + 33.8177\phi_2^8 - 238.849\phi_2^6 - 265.936\phi_2^4 + 835.782\phi_2^2 - 80.8312}{479001600}x^{12}. \end{aligned} \quad (5.30)$$

Applying the boundary condition, we see that

$$\begin{aligned} &\phi_2^{10} + 10.3251\phi_2^9 + 114.037\phi_2^8 + 680.864\phi_2^7 + 2963.6\phi_2^6 + 8427.45\phi_2^5 - 25642.9\phi_2^4 \\ &- 498154\phi_2^3 - 2.63559 \times 10^6\phi_2^2 - 2.98743 \times 10^8\phi_2 + 1.07833 \times 10^8 = 0, \end{aligned}$$

with the optimal solution given by $\phi_2 = 0.359736$*. Hence, we have the series solution*

$$\begin{aligned} y(x) =& 0.359736x - 0.36x^2 + 0.0012941x^4 - 0.00147647x^5 + 0.000363263x^6 \\ &+ 0.000112909x^7 - 0.0000301047x^8 - 4.6738179469380473 \times 10^{-7}x^9 \\ &+ 1.6622906757240492 \times 10^{-6}x^{10} - 6.64868824358838 \times 10^{-7}x^{11} \\ &- 4.669196561353817 \times 10^{-8}x^{12}. \end{aligned} \quad (5.31)$$

(iii) $E = 1.23, Q = 8, K = 1, \rho = 2.5, I = 4$: *Indeed one obtains from Corollary 3.1 an* *analytical solution*

$$
\begin{aligned}
y(x) =&\phi_2 x - 0.00087143x^2 + 0.00576112\phi_2 x^3 - \frac{0.3134\phi_2^2 + 0.000060245}{24}x^4 \\
&+ \frac{0.0028335 - 0.855523\phi_2^2}{120}\phi_2 x^5 \\
&- \frac{1.47539\phi_2^4 + 0.0452198\phi_2^2 + 4.938393543891888 \times 10^{-6}}{720}x^6 \\
&- \frac{2.10402\phi_2^4 - 0.191514\phi_2^2 - 0.000342176}{5040}\phi_2 x^7 \\
&- \frac{2.70258\phi_2^6 - 4.61456\phi_2^4 + 0.00953976\phi_2^2 + 5.9636562705634 \times 10^{-7}}{40320}x^8 \\
&- \frac{3.25119\phi_2^6 - 32.2713\phi_2^4 - 0.0644166\phi_2^2 - 0.0000599621}{362880}\phi_2 x^9 \qquad (5.32) \\
&- \frac{3.74157\phi_2^8 - 152.294\phi_2^6 - 1.57434\phi_2^4 + 0.002206\phi_2^2 + 1.045056 \times 10^{-7}}{3628800}x^{10} \\
&- \frac{4.17236\phi_2^8 - 579.481\phi_2^6 + 21.3666\phi_2^4 - 0.0170301\phi_2^2 - 0.0000130206}{39916800}\phi_2 x^{11} \\
&- \frac{4.54611\phi_2^{10} - 1919.94\phi_2^8 + 645.697\phi_2^6 - 1.21004\phi_2^4}{479001600}x^{12} \\
&- \frac{0.000541282\phi_2^2 + 2.26931682 \times 10^{-8}}{479001600}x^{12}.
\end{aligned}
$$

Upon applying the boundary condition, one obtains the algebraic equation

$$
\begin{aligned}
&\phi_2^{10} + 11.0134\phi_2^9 - 313.687\phi_2^8 - 585.599\phi_2^7 + 2782.49\phi_2^6 + 34672.3\phi_2^5 + 203805\phi_2^4 \\
&+ 747164\phi_2^3 + 1.38254 \times 10^6\phi_2^2 - 1.05975 \times 10^8\phi_2 + 92083.7 = 0,
\end{aligned}
$$

whose optimal solution is given by $\phi_2 = 0.000868931$. *Hence, we have the series solution*

$$
\begin{aligned}
y(x) =&0.000868931x - 0.00087143x^2 + 5.00601 \times 10^{-6}x^3 - 2.5200 \times 10^{-6}x^4 \\
&+ 2.051294 \times 10^{-8}x^5 - 6.90630 \times 10^{-9}x^6 + 5.9018 \times 10^{-11}x^7 \\
&- 1.4969 \times 10^{-11}x^8 + 1.436982 \times 10^{-13}x^9 - 2.92577 \times 10^{-14}x^{10} \\
&+ 2.8372014 \times 10^{-16}x^{11} - 4.82277 \times 10^{-17}x^{12}. \qquad (5.33)
\end{aligned}
$$

Table 3 presents the comparison between PSM and BVM.

6. Numerical Results and Discussion

Numerical solutions of the Bratu-type problems discussed in section 5 are obtained by PSM and BVM. The results are presented and compared in Tables 1, 2, and 3; and plotted in Figures 2, 3, and 4. To be more detailed, Table 1 shows the results of Example 5.1 obtained by PSM and BVM, and their comparison with the exact solution. The graphical description of the results is presented in Figure 2. It can be seen in Table 1 and Figure 2

Table 1: Comparison between the present methods with $\mu = 1, 1.5, 2$, for Example 5.1.

$x \backslash y(x)$	$y_{exact}(x)$ $\mu = 1$	$y_{PSM}(x)$ $\mu = 1$	$y_{BVM}(x)$ $\mu = 1$	$y_{exact}(x)$ $\mu = 1.5$	$y_{PSM}(x)$ $\mu = 1.5$	$y_{BVM}(x)$ $\mu = 1.5$	$y_{exact}(x)$ $\mu = 2$	$y_{PSM}(x)$ $\mu = 2$	$y_{BVM}(x)$ $\mu = 2$
0.1	0.049847	0.049847	0.049846	0.079610	0.079602	0.076605	0.114411	0.114031	0.114416
0.2	0.089190	0.089191	0.089188	0.142993	0.142976	0.142987	0.206419	0.205668	0.206624
0.3	0.117609	0.117612	0.117615	0.189090	0.189065	0.189042	0.273879	0.272775	0.273891
0.4	0.134790	0.134794	0.134785	0.217090	0.217057	0.217102	0.315089	0.313662	0.315554
0.5	0.140539	0.140544	0.140550	0.226482	0.226442	0.226475	0.328952	0.327240	0.328841
0.6	0.134790	0.134796	0.134725	0.217090	0.217044	0.217110	0.315089	0.313143	0.315123
0.7	0.117609	0.117615	0.117609	0.189090	0.189039	0.188999	0.273879	0.271773	0.273901
0.8	0.089190	0.0891964	0.089221	0.142993	0.142942	0.142987	0.206419	0.204305	0.206520
0.9	0.049846	0.0498524	0.049869	0.079610	0.079571	0.079857	0.114411	0.112718	0.113545

Table 2: Comparison between the present methods with $\mu = 1, 1.5, 2$, for Example 5.2.

$x \backslash y(x)$	$y_{PSM}(x)$ $\mu = 1$	$y_{BVM}(x)$ $\mu = 1$	$y_{PSM}(x)$ $\mu = 1.5$	$y_{BVM}(x)$ $\mu = 1.5$	$y_{PSM}(x)$ $\mu = 2$	$y_{BVM}(x)$ $\mu = 2$
0.1	0.041439	0.0415865	0.059998	0.059887	0.077692	0.077754
0.2	0.073274	0.072545	0.105849	0.112544	0.136843	0.137519
0.3	0.095809	0.0957562	0.138191	0.136542	0.178524	0.178321
0.4	0.109250	0.109301	0.157453	0.157922	0.203449	0.202895
0.5	0.113719	0.115621	0.163886	0.164211	0.212032	0.218064
0.6	0.109256	0.109344	0.157573	0.157602	0.204408	0.212001
0.7	0.095822	0.092576	0.138427	0.137930	0.180417	0.181145
0.8	0.073292	0.073405	0.106181	0.106578	0.139500	0.138297
0.9	0.041456	0.041498	0.060329	0.060411	0.080353	0.081274

that the results obtained from the PSM and the BVM are in excellent agreement with the exact solution. Table 2 shows the results of Example 5.2 obtained by PSM and BVM. The graphical illustration of the results is presented in Figure 3. Table 3 shows the results of Example 5.3 obtained by PSM and BVM. The graphical description of the results is shown in Figure 4. It follows, therefore, that the results in Tables 1, 2, 3; and Figures 2, 3, and 4 indicate that excellent agreements exist between the results obtained from the PSM and those obtained from the SDBDBVM. It is also worth noting that the approximate solutions $y_{PSM}(x)$ illustrated in Table 3 are in excellent agreement with those computed in [28, Tables 2, 3, and 4] using a Green's function fixed-point iteration approach. Our results $y_{PSM}(x)$ in Table 3 also agree perfectly with those given in [28, Table 5] using variational iteration method.

7. Concluding Remarks

In this paper, we have used the power series method and the twelfth-order second derivative backward differentiation boundary value method to, respectively, find analytical and numerical solutions of a class of generalised strongly nonlinear boundary value problems. Three special cases of the nonlinear terms that reduced the generalised problem to the Bratu-type were considered as illustrative examples. The results using PSM

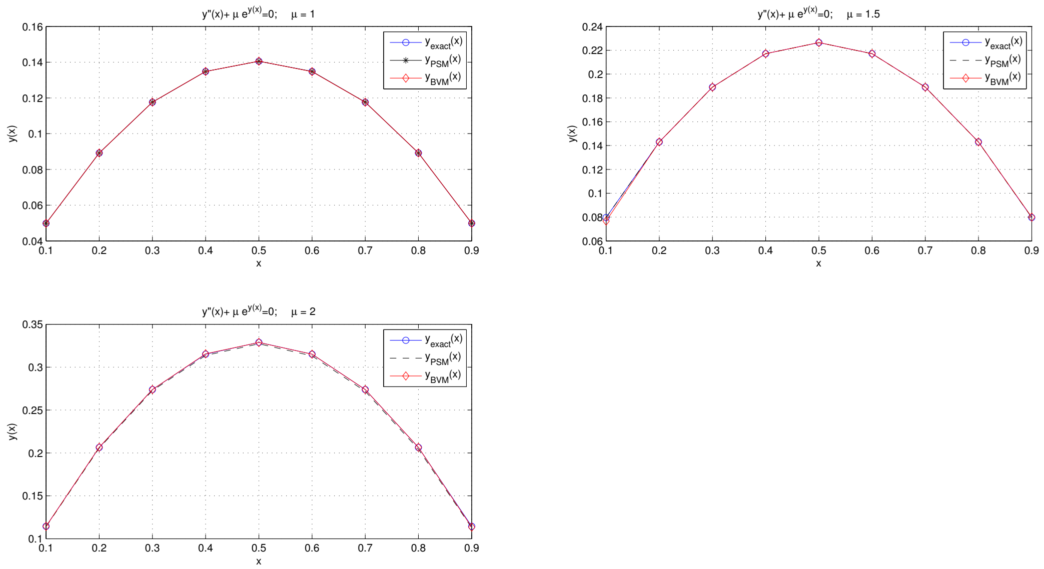

Figure 2: Graphical comparison of numerical solutions of the Bratu-type problem (5.1) using the PSM and the SDBDBVM.

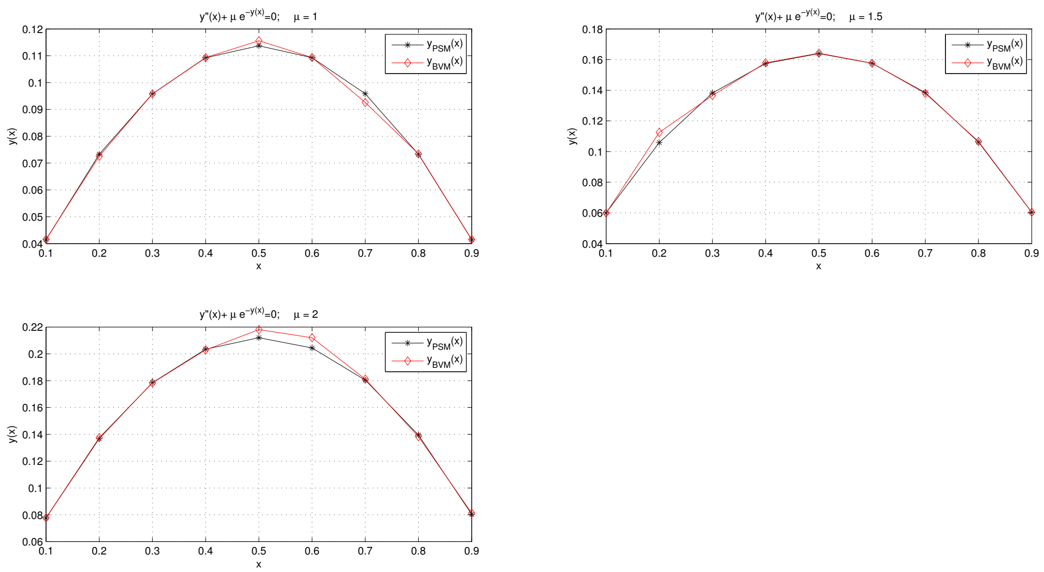

Figure 3: Graphical comparison of numerical solutions of the Bratu-type problem (5.15) using the PSM and SDBDBVM.

Table 3: Comparison between the present methods with $E = 1.23, Q = 5, K = 1, \rho = 1.5, I = 4$; $E = 1, Q = 5, K = 1, \rho = 2.5, I = 3$; and $E = 1.23, Q = 8, K = 1, \rho = 2.5, I = 4$, for Example 5.3.

$x \backslash y(x)$	$y_{PSM}(x)$ $E = 1.23, Q = 5$ $\rho = 1.5, I = 4$	$y_{BVM}(x)$ $E=1.23, Q=5$ $\rho = 1.5, I = 4$	$y_{PSM}(x)$ $E = 1, Q = 5$ $\rho = 2.5, I = 3$	$y_{BVM}(x)$ $E = 1, Q = 5$ $\rho = 2.5, I = 3$	$y_{PSM}(x)$ $E = 1.23, Q = 8$ $\rho = 2.5, I = 4$	$y_{BVM}(x)$ $E = 1.23, Q = 8$ $\rho = 2.5, I = 4$
0.1	0.000497274	0.000489570	0.0323737	0.0324216	0.0000781835	0.0000782764
0.2	0.000884855	0.000882544	0.0575488	0.0575329	0.000138965	0.000138837
0.3	0.00116214	0.00116327	0.0755279	0.0754872	0.000182365	0.000182520
0.4	0.00132868	0.00132741	0.0863140	0.0863195	0.000208400	0.000211312
0.5	0.00138422	0.00138586	0.0899091	0.0898110	0.000217077	0.000216421
0.6	0.00132868	0.00131672	0.0863140	0.0864520	0.000208400	0.000212432
0.7	0.00116214	0.00117170	0.0755280	0.0755437	0.000182365	0.000182421
0.8	0.000884855	0.000882674	0.0575488	0.0574975	0.000138965	0.000139002
0.9	0.000497274	0.000487210	0.0323737	0.0333104	0.0000781835	0.0000782041

and SDBDBVM were presented. Tabular and graphical illustrations of the obtained results were presented for comparison purposes. The elementary computations involved in the numerical algorithms need a proper care to avoid errors that can throw off the subsequent computations. Using Wolfram MATHEMATICA and MATLAB software, both methods generated highly accurate results for a moderate computational cost and implementation time.

Given a power series of a function $y(x)$ about the point $x = 0$, there exists a number $0 \leqslant R \leqslant \infty$ for which the power series converges for $|x| < R$ and diverges for $|x| > R$. The number R is called the radius of convergence. In order for a series solution to a differential equation to exist at a particular x, it needs to be convergent at that x. The power series method converges faster for values of x in the interval $(0, 1)$. The SDBDBVM, on the other hand, was implemented using a fixed step size of $h = 0.01$. Given the high order of the SDBDBVM, the numerical solution is relatively insensitive to moderate variations in the step size. Thus, even with a step size of $h = 0.01$, the SDBDBVM maintains accuracy and stability when solving the nonlinear Bratu-type problem demonstrating the robustness and efficiency of the SDBDBVM for boundary value problems. The unified results presented in this paper can be applied to other nonlinear initial and boundary value problems.

Conflicts of Interest:. The authors declare that they have no conflicts of interest.

Data Availability Statements:. Not applicable.

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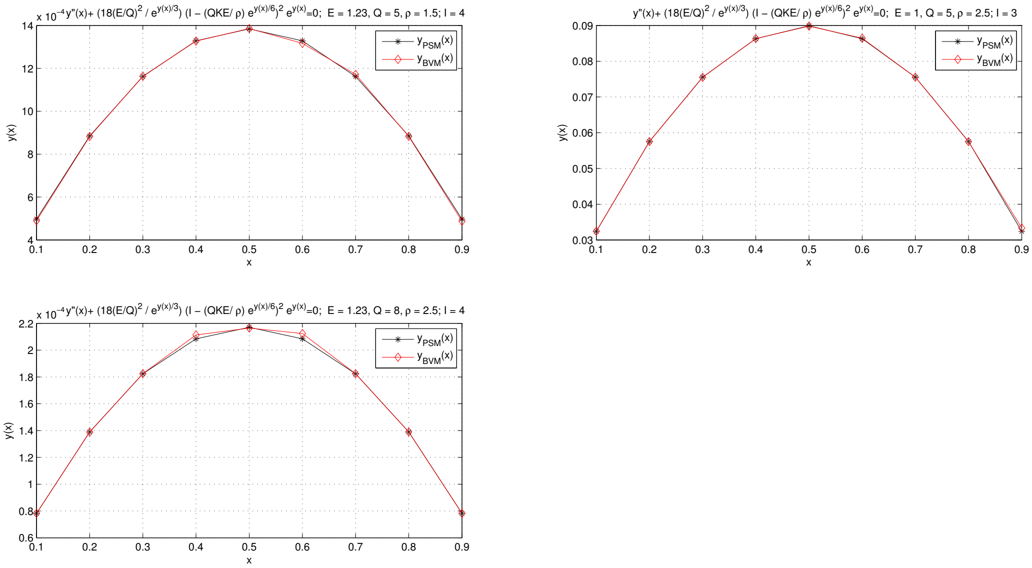

Figure 4: Graphical comparison of numerical solutions of the Bratu-type problem (5.25) using the PSM and SDBDBVM.

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