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Mpox transmission dynamics: A mathematical modeling approach with bifurcation analysis of control interventions

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Abstract

This study presents a new nonlinear mathematical model to investigate the transmission dynamics of Mpox. This model integrates vaccination, quarantine and hospitalization as key interventions to control the disease, with particular emphasis on the effectiveness of prompt quarantine strategies to limit its spread. A detailed qualitative analysis reveals three distinct epidemiological equilibria, as well as an Mpox-free equilibrium. The local and global stability of these steady states is investigated analytically by obtaining the basic reproduction rate ($\mathcal{R}_o$). Furthermore, the model shows backward bifurcation at a threshold parameter, which suggests that $\mathcal{R}_o$ may not be sufficient to eliminate the disease. To prevent this phenomenon, sufficient conditions were identified to increase the effectiveness of prevention and help eliminate the outbreak, which emphasizes the need for an integrated public health policy to limit disease outbreaks.

Keywords: Control interventions, Basic reproduction rate, Stability, Backward bifurcation, Mpox model.
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1. Introduction

Mpox (Monkeypox) is a viral zoonotic disease that has become a global public health concern. It can be transmitted through contact with infected animals, infected individuals, or handling materials contaminated with the virus [1, 9, 15, 34]. The incubation period for the Mpox disease ranges from approximately one to three weeks, during which the infection can be transmitted even before symptoms appear. Therefore, it is important to contain the Mpox disease and limit its spread by early detection and isolation of infected cases before symptoms appear or develop [33]. Most cases of the Mpox disease recovered due to the immune response, while some may require hospitalization. Moreover, many of infected cases acquire permanent immunity after recovery. However, some recovered

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patients may be exposed to the Mpox disease again due to various factors, including a weakened immune system [5]. In rare cases [6], death from the Mpox disease occurs.

Mathematical models play a crucial role in studying disease dynamics, providing valuable information for reducing the spread of diseases and understanding control strategies such as HIV [27], COVID-19 [13, 14, 24, 25], Mpox [2, 3, 10, 11, 12, 16, 19, 23, 26, 28, 29, 31, 35] and many others. One of the primary models used to study the dynamics of Mpox and understand control strategies is the SIR model [5]. Several studies have addressed the dynamics of the disease's spread, such as [21], which focused on the ways of Mpox spreads through the environment contaminated within viruses, taking into account the loss of immunity from the disease by some susceptible individuals due to the ineffectiveness of vaccination, as well as the possibility of reinfection of asymptomatic isolated individuals after an incubation period in susceptible populations. Another model for Mpox was developed in [20], which studied the possibility of immigrants increasing the number of susceptible humans and evaluated preventive measures such as quarantine and vaccination campaigns. The results showed that isolated individuals, who are no longer infected, can return to susceptible populations, and recovered individuals can also be vaccinated. Ashezua et al. [4] studied the effect of public awareness, vaccination, and treatment on controlling the spread of Mpox in the population. Their results indicated that using scenarios such as classifying susceptible individuals into two categories: symptomatic and asymptomatic, contributed to improving the accuracy of outbreak prediction. Soni et al. [30] also presented a mathematical model for Mpox that included a compartment for severe complications after infection, such as significant lung damage, along with control interventions of health education, vaccination, quarantine, and hospitalized. However, to the researchers' knowledge, models that include an asymptomatic category along with immigrants often overlooked in previous models.

This study aims at developing a new model focuses on the impact of prompt quarantine on Mpox dynamics under the presence of immigrants, asymptomatic who may develop symptoms during the incubation time, and recovered individuals without permanent immunity. The used model also includes vaccination and hospitalization as another control interventions on reducing disease spread in the population. Of course, the prompt quarantine strategy is to isolate uninfected exposed individuals or those undetected at diagnosis. The model also uses complex viral transmission interactions between humans and animals leading to multiple epidemic equilibria. The study relies on quantitative analytical techniques (such as the Routh-Hurwitz criterion and the Lyapunov-Lasalle theory) to examine the stability of equilibria affecting Mpox transmission by identifying a critical threshold ($\mathcal{R}_o$) that affects disease transmission [7], enhancing the scientific value of the study. The organization of the paper is set as follows: a new mathematical model for the spread of Mpox virus between twelve compartments is formulated in Section 2 including the dynamics of model structure and constructs of model dynamics. The model's schematic diagram is provided in Figure 1, while a brief description of the parameters is presented in Table 1. Section 3 is adapted to qualitative analysis of some basic properties of the model's solutions where the nonnegativity and boundedness of solutions are justified. Section 4 is devoted to dynamical analysis of the model where the existence of Mpox-free and Mpox-endemic equilibria points of the model are discussed in details. The basic reproduction rate of the model is addressed in Section 5 with elucidate its obligatory

connection with the uniqueness of Mpox-endemic equilibria. The steady state analysis for the model is investigated in Section 6 where the local and global stability of equilibria points are verified. Finally, the bifurcation analysis for the model is argued in Section 7 including its existence and direction.

2. Model formulation

Throughout this manuscript, the subscripts h and r denote respectively the *humans* and *animals* populations. We consider eight humans classes at time t including susceptible $S_h(t)$, vaccinated $V_h(t)$, exposed $E_h(t)$, quarantine $Q_h(t)$, infectious with asymptomatic $A_h(t)$, infectious with symptomatic $I_h(t)$, hospitalized $H_h(t)$ and recovered $R_h(t)$ such that

$$N_h(t) = S_h(t) + V_h(t) + E_h(t) + Q_h(t) + A_h(t) + I_h(t) + H_h(t) + R_h(t) \tag{2.1}$$

represents the humans population. Moreovere, we consider four animals classes at time t including susceptible $S_r(t)$, exposed $E_r(t)$, infectious $I_r(t)$, and recovered $R_r(t)$ such that

$$N_r(t) = S_r(t) + E_r(t) + I_r(t) + R_r(t) \tag{2.2}$$

represents the animals population. Thus, the total population is

$$N(t) = N_h(t) + N_r(t)\,. \tag{2.3}$$

2.1. Dynamics of model structure

In the humans submodel, the newly recruited people (e.g., birth or immigrants) increases S_h class over time at a constant rate ν_h. The size of S_h also replenishes via the progression of undetected cases after medical diagnosis from Q_h class who transition back into S_h at the proportion rate θ, as well as the loss of vaccine-acquired immunity by V_h class at a rate ρ_2. The size of V_h is generated vaccinating people with immunocompromised in both of S_h and R_h classes at effective vaccination rates ρ_1 and φ; respectively. Here, q indicates the fraction at which immigrants are vaccinated while the rest $1-q$ are not therefore lie in S_h class. The people in S_h class acquire infection from either infected humans or animals at a force of infection

$$\beta_h = \frac{\beta_1 A_h + \beta_2 I_h}{N_h} + \frac{\beta_3 I_r}{N_r}\,, \tag{2.4}$$

where β_1, β_2, and β_3 are probabilities of virus transmission respectively from infectious classes A_h, I_h, and I_r into S_h class. The size of E_h is formed by the infection of S_h at the effective contact rate β_h and then progresses to Q_h class at a rate η. The people in Q_h transition to either A_h or I_h at rate α while τ is the proportion of persons that progressed to A_h class. The persons in A_h either recover naturally and moved to R_h at a rate γ_1, or progress to I_h class after develop symptoms during the incubation period at a rate ψ, while those with symptoms in I_h are admitted to H_h class for hospitalization at a rate ω. The illness people in H_h are progressed to R_h class at a treatment recovery rate γ_2. As some of individuals in R_h are assumed never to have permanent immunity for the rest

of their lifetime, they return to V_h class due to the decay of temporary immunity that indirectly affect the size of S_h through unsuccessful vaccinations. In addition, the virus causes death for humans only in I_h and H_h at the mortality rate δ_h, while the natural mortality happens in all humans classes at a constant rate μ_h. In the animals submodel, the recruitment rate ν_r of new animals increases S_r class at any time. The animals in S_r catching the virus from infected animals at a force of infection

$$\beta_r = \frac{\beta_4 I_r}{N_r}, \tag{2.5}$$

where β_4 is the probability of virus transmission from infectious class I_r to S_r class. The E_r class is generated by the infection of S_r at the effective contact rate β_r. The animals in E_r moves to I_r class at a rate σ. The animals in I_r are dying due to the disease at a rate δ_h or moving to R_r class at a natural recovery rate γ_3 with permanent immunity. Moreover, μ_r is the rate at which a natural mortality happens in all the animal classes.

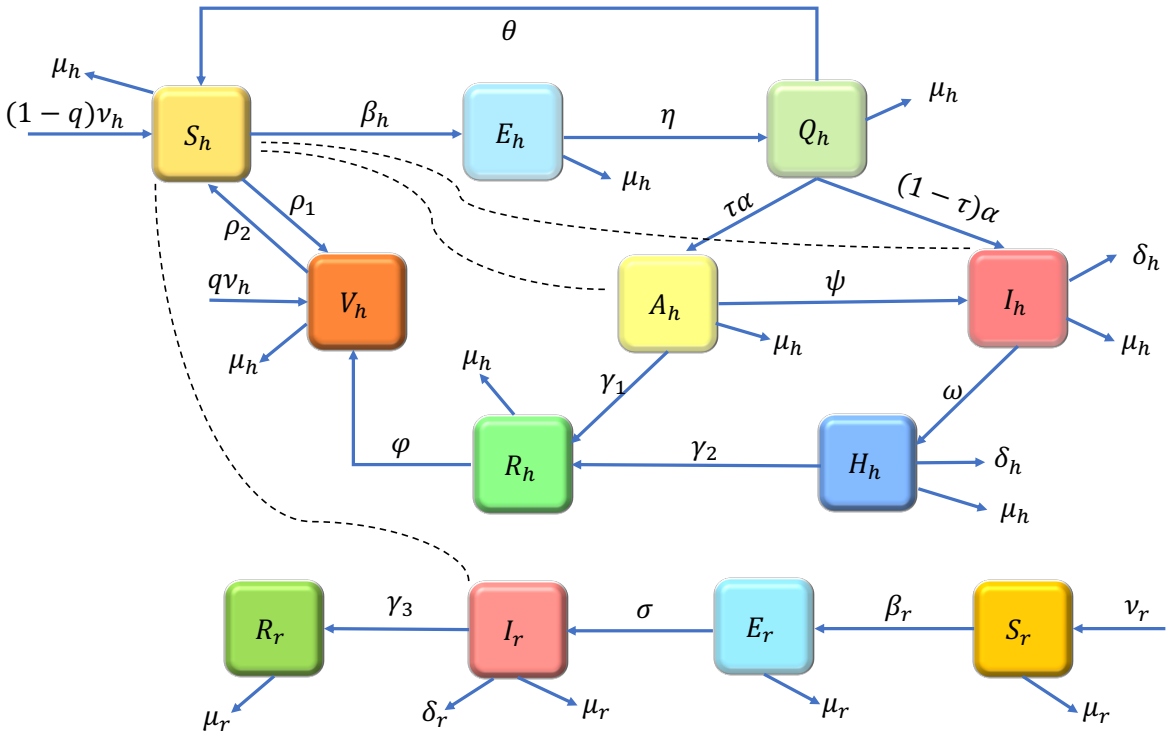

Figure 1: A flow diagram of transmission dynamics for the Mpox model.

2.2. *Constructs of model dynamics*

The dynamics of Mpox transmission in Figure 1 above are constructed by the system:

Table 1: Description of state variables over times and parameters for the (MPV) model

Variable	Description	Parameter	Description
S_h	Susceptible humans	ν_h, ν_r	Recruitment rates of susceptible humans and animals
V_h	Vaccinated humans	ρ_1	Effectiveness vaccination rate of susceptible humans
E_h	Exposed humans	ρ_2	Losing rate of effectiveness vaccination
Q_h	Quarantined humans	η	Transition rate from E_h to Q_h
A_h	Asymptomatic humans	α	Progression rate from Q_h to infectious compartments
I_h	Symptomatic humans	ψ	Progression from A_h to I_h due to immunocompromised
H_h	Hospitalized ill humans	ω	Hospitalization rate of symptomatic persons
R_h	Recovered humans	γ_1	Natural recovery rate of asymptomatic persons
S_r	Susceptible animals	γ_2	Recovery rate for the hospitalized ill humans
E_r	Exposed animals	σ	Progression rate from E_r to I_r
I_r	Infected animals	γ_3	Recovery rate of infected animals
R_r	Recovered animals	μ_h, μ_r	Natural mortality rates of humans and animals
		q	Fraction of vaccinated immigrants
		β_1	Human to humans transmission rate due to contact with A_h
		β_2	Human to humans transmission rate due to contact with I_h
		β_3	Animal to humans transmission rate due to contact with I_r
		θ	Progression rate of uninfected persons in Q_h moving to S_h
		τ	Proportion rate of infected persons in Q_h moving to A_h
		δ_h, δ_r	Mortality rates of humans and animals due to infections
		φ	Vaccination rate of recovered humans with immunity waning
		β_4	Animal to animals transmission rate due to contact with I_r

$$
\begin{aligned}
\dot{S}_h(t) &= (1-q)\nu_h - (\beta_h + \rho_1 + \mu_h)S_h(t) + \rho_2 V_h(t) + \theta Q_h(t)\,,\\
\dot{V}_h(t) &= q\nu_h - (\rho_2 + \mu_h)V_h(t) + \rho_1 S_h + \varphi R_h(t)\,,\\
\dot{E}_h(t) &= \beta_h\, S_h(t) - (\eta + \mu_h)\, E_h(t)\,,\\
\dot{Q}_h(t) &= \eta\, E_h(t) - [\tau\alpha + (1-\tau)\alpha + \theta + \mu_h]\, Q_h(t)\,,\\
\dot{A}_h(t) &= \tau\,\alpha\, Q_h(t) - (\gamma_1 + \psi + \mu_h)\, A_h(t)\,,\\
\dot{I}_h(t) &= (1-\tau)\,\alpha\, Q_h(t) + \psi\, A_h(t) - (\omega + \mu_h + \delta_h)\, I_h(t)\,,\\
\dot{H}_h(t) &= \omega\, I_h(t) - (\gamma_2 + \mu_h + \delta_h)\, H_h(t)\,,\\
\dot{R}_h(t) &= \gamma_1\, A_h(t) + \gamma_2\, H_h(t) - (\varphi + \mu_h)\, R_h(t)\,,\\
\dot{S}_r(t) &= \nu_r - (\beta_r + \mu_r)S_r(t)\,,\\
\dot{E}_r(t) &= \beta_r S_r(t) - (\sigma + \mu_r)E_r(t)\,,\\
\dot{I}_r(t) &= \sigma\, E_r(t) - (\gamma_3 + \mu_r + \delta_r)I_r(t)\,,\\
\dot{R}_r(t) &= \gamma_3\, I_r(t) - \mu_r\, R_r(t)\,,
\end{aligned}
\qquad \text{(MPV)}
$$

together, of course, with the prescribed initial conditions

$$
\begin{aligned}
&S_h(0) = S_h^o \geqslant 0\,,\ V_h(0) = V_h^o \geqslant 0\,,\ E_h(0) = E_h^o \geqslant 0\,,\ Q_h(0) = Q_h^o \geqslant 0\,,\\
&A_h(0) = A_h^o \geqslant 0\,,\ I_h(0) = I_h^o \geqslant 0\,,\ H_h(0) = H_h^o \geqslant 0\,,\ R_h(0) = R_h^o \geqslant 0\,,\\
&S_r(0) = S_r^o \geqslant 0\,,\ E_r(0) = E_r^o \geqslant 0\,,\ I_r(0) = I_r^o \geqslant 0\,,\ R_r(0) = R_r^o \geqslant 0\,,
\end{aligned}
\qquad \text{(IC)}
$$

where β_h and β_r are the associated forces of infection given respectively by (2.4) and (2.5).

3. Qualitative analysis of (MPV) system

This section highlights some basic features to ensure the epidemiological feasibility of (MPV) system.

3.1. *Nonnegatively-invariant for* (MPV) *solutions*

We consider the nonnegative solutions of (MPV) is contained for any $t \geqslant 0$ in the set

$$\mathcal{O}_{\geqslant 0} = \left\{ \left(S_h(t), V_h(t), E_h(t), Q_h(t), A_h(t), I_h(t), H_h(t), R_h(t), S_r(t), E_r(t), I_r(t), R_r(t)\right) \in \mathbb{R}^{12}_{\geqslant 0} \right\}.$$

The next theorem shows that all solutions will stay in $\mathcal{O}_{\geqslant 0}$ for all nonnegative times.

Theorem 3.1. *For the* (MPV) *system with* (IC)*, the set* $\mathcal{O}_{\geqslant 0}$ *is a nonnegatively-invariant.*

Proof. Set $t^* = \sup\left\{ t > 0 : S_h(t') \geqslant 0, V_h(t') \geqslant 0, E_h(t') \geqslant 0, Q_h(t') \geqslant 0, A_h(t') \geqslant 0, I_h(t') \geqslant 0, H_h(t') \geqslant 0, R_h(t') \geqslant 0, S_r(t') \geqslant 0, E_r(t') \geqslant 0, I_r(t') \geqslant 0, R_r(t') \geqslant 0 \text{ for all } 0 \leqslant t' \leqslant t \right\}$. Note that $0 < t^* < \infty$. From (MPV) we have

$$\dot{S}_h(t) + \left(\beta_h + \rho_1 + \mu_h\right) S_h(t) \geqslant \left(1 - q\right)\nu_h .$$

The solution procedure shows that

$$S_h(t^*) \geqslant \exp\left[-(\rho_1+\mu_h)t^* - \int_0^{t^*} \beta_h(z)\,dz\right]\left(S_h(0) + (1-q)\nu_h \int_0^{t^*} \exp\left[(\rho_1+\mu_h)t + \int_0^t \beta_h(z)\,dz\right] dt\right)$$

which implies that $S_h(t) \geqslant 0$ for every $t \geqslant 0$ (due to $S_h(0) \geqslant 0$ and $0 \leqslant q < 1$). Similarly, one can obtain that $V_h(t) \geqslant 0, E_h(t) \geqslant 0, Q_h(t) \geqslant 0, A_h(t) \geqslant 0, I_h(t) \geqslant 0, H_h(t) \geqslant 0, R_h(t) \geqslant 0, S_r(t) \geqslant 0, E_r(t) \geqslant 0, I_r(t) \geqslant 0, R_r(t) \geqslant 0$ for every $t \geqslant 0$. □

Notice that from the proof argument above, which is indeed classical in the modeling of ODE's (see for instance Theorem 3.2. of [24] or Lemma 3.2 of [25]), we deduce that S_h, V_h and S_r in $\mathcal{O}_{\geqslant 0}$ are positive for every $t > 0$ while the other state variables remaining nonnegative at all times. More precisely, all positive state variables of (MPV) will lie for all $t \geqslant 0$ in the subset

$$\mathcal{O}_+ = \left\{ \left(S_h(t), V_h(t), E_h(t), Q_h(t), A_h(t), I_h(t), H_h(t), R_h(t), S_r(t), E_r(t), I_r(t), R_r(t)\right) \in \mathcal{O}_{\geqslant 0} \right\}.$$

Corollary 3.2. *The set* $\mathcal{O}_+$ *is a positively-invariant for the* (MPV) *system with initial states:*

$$\begin{aligned} &S_h(0) > 0,\ V_h(0) > 0,\ E_h(0) > 0,\ Q_h(0) > 0,\ A_h(0) > 0,\ I_h(0) > 0,\\ &H_h(0) > 0,\ R_h(0) > 0,\ S_r(0) > 0,\ E_r(0) > 0,\ I_r(0) > 0,\ R_r(0) > 0. \end{aligned} \tag{PIC}$$

3.2. *Positivity-boundedness for* (MPV) *solutions*

We denote by $\mathcal{O} := \mathcal{O}_h \cup \mathcal{O}_r$ the closed set of all bounded solutions of (MPV) lie in $\mathcal{O}_+$ and start in $\mathbb{R}^{12}_+$ at $t = 0$ such that $0 \leqslant N(t) \leqslant \left(\nu_h/\mu_h\right) + \left(\nu_r/\mu_r\right)$ for any $0 \leqslant t < \infty$, where

$$\mathcal{O}_h = \left\{ \left(S_h(t), V_h(t), E_h(t), Q_h(t), A_h(t), I_h(t), H_h(t), R_h(t)\right) \in \mathbb{R}^8_+ \ :\ 0 \leqslant N_h(t) \leqslant \nu_h/\mu_h \right\}$$

with N_h satisfies (2.1), and

$$\mathcal{O}_r = \left\{ \left(S_r(t), E_r(t), I_r(t), R_r(t)\right) \in \mathbb{R}^4_+ \ :\ 0 \leqslant N_r(t) \leqslant \nu_r/\mu_r \right\}$$

with N_r satisfies (2.2). Note also that N satisfies (2.3). The next theorem shows that all feasible solutions of (MPV) associated to (PIC) are contained in the region $\mathcal{O}$.

Theorem 3.3. *All state variables of* (MPV) *system in* $\mathcal{O}$ *are uniformly bounded at any time.*

Proof. By combining all equations of (MPV) we get the following conservation law

$$\dot{N}(t) = \nu_h + \nu_r - \left[\mu_h N_h(t) + \mu_r N_r(t)\right] - \left(\delta_h\left[I_h(t) + H_h(t)\right] + \delta_r I_r(t)\right). \tag{3.1}$$

It follows, thanks to comparison theorem [17] (in particular, for $I_h = H_h = I_r = 0$), that

$$\dot{N}(t) + \left[\mu_h N_h(t) + \mu_r N_r(t)\right] \leqslant \nu_h + \nu_r\,. \tag{3.2}$$

The solution procedure of the differential inequality (3.2) yields

$$N(t) \leqslant \mathcal{N}_o \exp\left(-\left[\mu_h+\mu_r\right]t\right) + \left(\nu_h/\mu_h\right)\left[1-\exp\left(-\mu_h t\right)\right] + \left(\nu_r/\mu_r\right)\left[1-\exp\left(-\mu_r t\right)\right], \tag{3.3}$$

where $\mathcal{N}_o := N_h^0 \exp\left(-\mu_r\, t\right) + N_r^0 \exp\left(-\mu_h\, t\right)$ for some $N_h^0 \in \left[0, \nu_h/\mu_h\right]$ and $N_r^0 \in \left[0, \nu_r/\mu_r\right]$ denote the initial total populations of human and animal; respectively. However, passing the limit of (3.3) as t goes to infinity implies the solution $N(t)$ of (3.1) is bounded above by $\max\left\{\mathcal{N}_o, \left(\nu_h/\mu_h\right) + \left(\nu_r/\mu_r\right)\right\}$; which means that all state variables of (MPV) are uniformly bounded by the region $\mathcal{O}$ for every $t \in \left[0, \infty\right)$. □

Remark 3.4. From the above proof, one immediately sees that the region $\mathcal{O}$ is attracting all solutions in $\mathbb{R}^{12}_{\geqslant 0}$ when the solution set $\mathcal{O}_{\geqslant 0}$ is outside $\mathcal{O}$. Indeed, if $N \geqslant \left(\nu_h/\mu_h\right)+\left(\nu_r/\mu_r\right)$, then, it follows from (3.2) that $\dot{N} \leqslant 0$ which means that all nonnegative solutions enter $\mathcal{O}$ for all times. As a consequence, the set $\mathcal{O}$ is a feasible domain for the solutions of (MPV).

Corollary 3.5. *The* (MPV) *system is mathematically well-posed and biologically viable in* $\mathcal{O}$. *More precisely, the model of* (MPV) *admits a dynamical system on* $\mathcal{O}$.

4. Dynamical analysis of (MPV) system

This section presents the dynamical aspects of the (MPV) system on $\mathcal{O}$.

4.1. Mpox-free equilibrium for (MPV) *system*

The (MPV) system has a Mpox-free equilibrium in $\mathcal{O}$ at the steady state given by

$$\chi_o := \left(\chi_h^o, \chi_r^o\right) \in \mathcal{O}^o := \mathcal{O}_h^o \cup \mathcal{O}_r^o\,, \tag{MFE}$$

where $\mathcal{O}^o$ denotes the space of Mpox-free axis in $\mathcal{O}$ and

$$\chi_h^o := \left(\Lambda_1 \frac{\nu_h}{\mu_h}, \Lambda_2 \frac{\nu_h}{\mu_h}, 0, 0, 0, 0, 0, 0\right) \in \mathcal{O}_h^o\,, \quad \chi_r^o := \left(\frac{\nu_r}{\mu_r}, 0, 0, 0\right) \in \mathcal{O}_r^o$$

represent respectively the "*human*" and "*animal*" Mpox-free equilibria in $\mathcal{O}_h$ and $\mathcal{O}_r$ with

$$\Lambda_1 = \frac{\rho_2 + (1-q)\mu_h}{\rho_1 + \rho_2 + \mu_h}\,, \quad \Lambda_2 = \frac{\rho_1 + q\mu_h}{\rho_1 + \rho_2 + \mu_h} \quad \text{for some } \; 0 \leqslant q \leqslant 1\,. \tag{4.1}$$

Proposition 4.1. *The* (MPV) *system in* $\mathcal{O}$ *always possesses* (MFE) *for all parameters.*

4.2. Mpox-endemic equilibrium for (MPV) system

The Mpox-endemic equilibrium of (MPV) when there is a disease in $\mathcal{O}$ can be represented as

$$\chi^* := (\chi_h^*, \chi_r^*) \in \mathcal{O}^* := \mathcal{O}_h^* \cup \mathcal{O}_r^*, \tag{MEE}$$

where $\mathcal{O}^*$ denotes the space of Mpox-endemic axis in $\mathcal{O}$ and

$$\chi_h^* = (S_h^*, V_h^*, E_h^*, Q_h^*, A_h^*, I_h^*, H_h^*) \in \mathcal{O}_h^*, \quad \chi_r^* = (S_r^*, E_r^*, I_r^*, R_r^*) \in \mathcal{O}_r^* \tag{4.2}$$

represent respectively the "*human*" and "*animal*" Mpox-endemic equilibra in $\mathcal{O}_h$ and $\mathcal{O}_r$. All components of (MEE) can be expressed from (MPV) at the steady state as follows:

$$\begin{aligned}
S_h^* &= \frac{k_1 \nu_h}{k_2 + \beta_h^* k_3}, \quad V_h^* = \frac{\left[(\rho_1 + q\mu_h)(k_1 - q\mu_h) + \beta_h^*(k_0 k_1 + q k_3)\right]\nu_h}{(k_2 + \beta_h^* k_3)(k_1 - q\mu_h)}, \\
E_h^* &= \frac{\beta_h^* k_1 \nu_h}{(k_2 + \beta_h^* k_3)(\eta + \mu_h)}, \quad Q_h^* = \frac{\beta_h^* \eta k_1 \nu_h}{(k_2 + \beta_h^* k_3)(\eta + \mu_h)(\alpha + \theta + \mu_h)}, \\
A_h^* &= \frac{\alpha \beta_h^* \eta \tau k_1 \nu_h}{(k_2 + \beta_h^* k_3)(\eta + \mu_h)(\alpha + \theta + \mu_h)(\psi + \gamma_1 + \mu_h)}, \\
I_h^* &= \frac{\alpha \beta_h^* \eta k_1 \left[\psi + (1-\tau)(\gamma_1 + \mu_h)\right]\nu_h}{(k_2 + \beta_h^* k_3)(\eta + \mu_h)(\alpha + \theta + \mu_h)(\psi + \gamma_1 + \mu_h)(\omega + \delta_h + \mu_h)}, \\
H_h^* &= \frac{\alpha\beta_h^* \eta \omega k_1 \left[\psi + (1-\tau)(\gamma_1 + \mu_h)\right]\nu_h}{(k_2 + \beta_h^* k_3)(\eta+\mu_h)(\alpha+\theta+\mu_h)(\psi+\gamma_1+\mu_h)(\omega+\delta_h+\mu_h)(\gamma_2+\delta_h+\mu_h)}, \\
R_h^* &= \frac{\alpha \beta_h^* \eta k_1 \{\tau\gamma_1(\omega+\delta_h+\mu_h)(\gamma_2+\delta_h+\mu_h) + \omega\gamma_2\left[\psi+(1-\tau)(\gamma_1+\mu_h)\right]\}\nu_h}{(k_2+\beta_h^* k_3)(\eta+\mu_h)(\alpha+\theta+\mu_h)(\psi+\gamma_1+\mu_h)(\omega+\delta_h+\mu_h)(\gamma_2+\delta_h+\mu_h)(\varphi+\mu_h)}, \\
S_r^* &= \frac{\nu_r}{\beta_r^* + \mu_r}, \quad E_r^* = \frac{\beta_r^* \nu_r}{(\beta_r^* + \mu_r)(\sigma + \mu_r)}, \quad I_r^* = \frac{\sigma\beta_r^* \nu_r}{(\beta_r^* + \mu_r)(\sigma + \mu_r)(\gamma_3 + \delta_r + \mu_r)}, \\
R_r^* &= \frac{\gamma_3 \sigma \beta_r^* \nu_r}{(\beta_r^* + \mu_r)\mu_r(\sigma + \mu_r)(\gamma_3 + \delta_r + \mu_r)},
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
k_0 &= \frac{\alpha\eta\varphi\left\{(\omega + \delta_h + \mu_h)(\gamma_2 + \delta_h + \mu_h)\tau\gamma_1 + \omega\gamma_2\left[\psi + (1-\tau)(\gamma_1 + \mu_h)\right]\right\}}{(\eta + \mu_h)(\alpha + \theta + \mu_h)(\psi + \gamma_1 + \mu_h)(\omega + \delta_h + \mu_h)(\gamma_2 + \delta_h + \mu_h)(\varphi + \mu_h)}, \\
k_1 &= \rho_2 + (1-q)\mu_h, \qquad k_2 = \mu_h(\rho_1 + \rho_2 + \mu_h), \\
k_3 &= \frac{(\rho_2 + \mu_h)\left[(\eta + \mu_h)(\alpha + \theta + \mu_h) - \theta\eta\right] - \rho_2(\eta + \mu_h)(\alpha + \theta + \mu_h)k_0}{(\eta + \mu_h)(\alpha + \theta + \mu_h)}
\end{aligned} \tag{4.4}$$

for some $0 \leqslant q, \tau \leqslant 1$, and

$$\beta_h^* = \frac{\beta_1 A_h^* + \beta_2 I_h^*}{N_h^*} + \frac{\beta_3 I_r^*}{N_r^*}, \quad \beta_r^* = \frac{\beta_4 I_r^*}{N_r^*} \tag{4.5}$$

represent respectively the infection forces associated to χ_h^* and χ_r^* in (4.2) with

$$N_h^* = S_h^* + V_h^* + E_h^* + Q_h^* + A_h^* + I_h^* + H_h^* + R_h^*, \quad N_r^* = S_r^* + E_r^* + I_r^* + R_r^*. \tag{4.6}$$

Notice that there are possibly three cases of (MEE) induced by reasonably three cases of transmissions associated to human and/or animal infections as the following:

Case I: Animal-free transmission

Let us first consider the Mpox infection is transmitted only between humans when no infected animals exist in the (MPV) system (i.e., β_3 and β_4 are simultaneously nullable). In such case, the *animal-free* (MEE) point takes the form

$$\chi^*_{af} := \left(S^*_{h_{af}}, V^*_{h_{af}}, E^*_{h_{af}}, Q^*_{h_{af}}, A^*_{h_{af}}, I^*_{h_{af}}, H^*_{h_{af}}, R^*_{h_{af}}, \frac{\nu_r}{\mu_r}, 0, 0, 0\right) \in \mathcal{O}^*_{af}, \tag{4.7}$$

where $\mathcal{O}^*_{af}$ denotes the space of animal-free Mpox-endemic axis of $\mathcal{O}^*$, with

$$\beta^*_{h_{af}} := \frac{\beta_1 A^*_{h_{af}} + \beta_2 I^*_{h_{af}}}{N^*_{h_{af}}}. \tag{4.8}$$

It follows from (4.8) and (4.3) that all components of (4.7) can be given in terms of

$$A^*_{h_{af}} = \frac{\left(\beta_1 A^*_{h_{af}} + \beta_2 I^*_{h_{af}}\right) k_4}{N^*_{h_{af}} k_2 + \left(\beta_1 A^*_{h_{af}} + \beta_2 I^*_{h_{af}}\right) k_3}, \quad I^*_{h_{af}} = \frac{\left(\beta_1 A^*_{h_{af}} + \beta_2 I^*_{h_{af}}\right) k_5}{N^*_{h_{af}} k_2 + \left(\beta_1 A^*_{h_{af}} + \beta_2 I^*_{h_{af}}\right) k_3}, \tag{4.9}$$

where

$$k_4 = \frac{\alpha \eta \tau k_1 \nu_h}{(\eta + \mu_h)(\alpha + \theta + \mu_h)(\psi + \gamma_1 + \mu_h)}, \quad k_5 = \frac{k_4\left[\psi + (1-\tau)(\gamma_1 + \mu_h)\right]}{\tau(\omega + \delta_h + \mu_h)} \tag{4.10}$$

for positive constants k_1, k_2, k_3 given by (4.4). From (4.9), we obtain the following system

$$\begin{cases} \beta_1 k_3 \left(A^*_{h_{af}}\right)^2 + \beta_2 k_3 I^*_{h_{af}} A^*_{h_{af}} + \left(k_2 N^*_{h_{af}} - \beta_1 k_4\right) A^*_{h_{af}} - \beta_2 k_4 I^*_{h_{af}} = 0, \\ \beta_2 k_3 \left(I^*_{h_{af}}\right)^2 + \beta_1 k_3 A^*_{h_{af}} I^*_{h_{af}} + \left(k_2 N^*_{h_{af}} - \beta_2 k_5\right) I^*_{h_{af}} - \beta_1 k_5 A^*_{h_{af}} = 0. \end{cases} \tag{4.11}$$

By taking into account the two equations of (4.11) must be solved simultaneously, the solution procedure leads to the following two-variables quadratic formula

$$k_3\left[\left(\beta_1 A^*_{h_{af}}\right)^2 - \left(\beta_2 I^*_{h_{af}}\right)^2\right] + k_2 N^*_{h_{af}}\left(\beta_1 A^*_{h_{af}} - \beta_2 I^*_{h_{af}}\right) - \left(\beta_1 k_4 - \beta_2 k_5\right)\left(\beta_1 A^*_{h_{af}} + \beta_2 I^*_{h_{af}}\right) = 0.$$

Since the discriminant $\Delta := -4\left(\beta_1^2\right)\left(-\beta_2^2\right) k_3^2$ is strictly positive, we infer that (4.11) has two distinct nonnegative solutions. By setting $A^*_{h_{af}} = \left(k_4/k_5\right) I^*_{h_{af}}$, the last formulation can be reduced to

$$\Psi_1 \left(I^*_{h_{af}}\right)^2 + \Psi_2 I^*_{h_{af}} = 0, \tag{4.12}$$

where

$$\Psi_1 = k_3\left(\beta_1 k_4 + \beta_2 k_5\right) \quad \text{and} \quad \Psi_2 = \left[k_2 N^*_{h_{af}} - \left(\beta_1 k_4 + \beta_2 k_5\right)\right] k_5. \tag{4.13}$$

Obviously, one can see that Ψ_1 is always positive while Ψ_2 may have a negative sign. So, the real positive solution for (4.12) can be expressed explicitly as

$$I^*_{h_{af}} = \frac{\left(\beta_1 k_4 + \beta_2 k_5 - k_2 N^*_{h_{af}}\right) k_5}{k_3\left(\beta_1 k_4 + \beta_2 k_5\right)} \tag{4.14}$$

provided that $\beta_1 k_4 + \beta_2 k_5 - k_2 N^*_{h_{af}}$ has a positive sign. As the trivial solution $I^*_{h_{af}} = 0$ of (4.12) refers to the existence of a unique (MFE) (see Proposition 4.1), we infer that there exists a single positive "*animal-free*" (MEE) induced by (4.14) and solves (MPV) at the steady state.

Proposition 4.2. *The* (MPV) *system in* $\mathcal{O}$ *associated to animal-free transmission possesses at most one animal-free* (MEE) *given by* (4.7).

Case II: Human-free transmission

We next consider the Mpox infection is transmitted only from animals; especially when no infections from human to human (i.e., β_1 and β_2 are simultaneously nullable). As such, the *human-free* (MEE) for the (MPV) system in $\mathcal{O}$ will take the form

$$\chi^*_{hf} := \left(S^*_{h_{hf}}, V^*_{h_{hf}}, E^*_{h_{hf}}, Q^*_{h_{hf}}, A^*_{h_{hf}}, I^*_{h_{hf}}, H^*_{h_{hf}}, R^*_{h_{hf}}, S^*_r, E^*_r, I^*_r, R^*_r\right) \in \mathcal{O}^*_{hf}\,, \tag{4.15}$$

where $\mathcal{O}^*_{hf}$ denotes the space of human-free Mpox-endemic axis of $\mathcal{O}^*$, with

$$\beta^*_{h_{hf}} := \frac{\beta_3 I^*_r}{N^*_r} = \frac{\beta_3 \beta^*_r}{\beta_4}\,. \tag{4.16}$$

It follows from (4.16) and (4.3) that all components of (4.15) can be given in terms of

$$I^*_r = \frac{\sigma\beta_4 I^*_r \nu_r}{(\beta_4 I^*_r + \mu_r N^*_r)(\sigma + \mu_r)(\gamma_3 + \mu_r + \delta_r)}.$$

After some simplification, the last identity can be written as

$$I^*_r = \frac{\sigma\nu_r}{(\sigma + \mu_r)(\gamma_3 + \mu_r + \delta_r)} - \frac{\mu_r}{\beta_4} N^*_r \tag{4.17}$$

provided that $\beta_4\sigma\nu_r - (\sigma + \mu_r)(\gamma_3 + \mu_r + \delta_r)\mu_r N^*_r$ has a positive sign. Thus, we conclude there is a single positive "*human-free*" (MEE) induced by (4.17) and solves (MPV) at the steady state.

Proposition 4.3. *The* (MPV) *system in* $\mathcal{O}$ *associated to the human-free transmission possesses at most one human-free* (MEE) *given by* (4.15).

Case III: Animal-human transmission

We consider the Mpox infection is transmitted from human and animal together when there are infection forces associated to both animal and human (i.e., $\beta_1, \beta_2, \beta_3$ and β_4 are all nonnullable). This case gathers the previous cases of human and animal infections in the sense that (MEE) taking the same expressions as in (4.3) with β^*_h is given via (4.8) and (4.16) (i.e., $\beta^*_h = \beta^*_{h_{af}} + \beta^*_{h_{hf}}$). Indeed, it follows from (4.3) and (4.5) that all components of χ^*_h in (4.2) can be obtained in terms of

$$A^*_h = \frac{\left[\left(\beta_1 A^*_h + \beta_2 I^*_h\right) N^*_r + N^*_h \beta_3 I^*_r\right] k_4}{N^*_h k_2 N^*_r + \left[\left(\beta_1 A^*_h + \beta_2 I^*_h\right) N^*_r + N^*_h \beta_3 I^*_r\right] k_3}, \tag{4.18}$$

and

$$I^*_h = \frac{\left[\left(\beta_1 A^*_h + \beta_2 I^*_h\right) N^*_r + N^*_h \beta_3 I^*_r\right] k_5}{N^*_h k_2 N^*_r + \left[\left(\beta_1 A^*_h + \beta_2 I^*_h\right) N^*_r + N^*_h \beta_3 I^*_r\right] k_3}, \tag{4.19}$$

where k_2, k_3, N_h^*, N_r^*, k_4, k_5 and I_r^* are positive and defined respectively by (4.4), (4.6), (4.10) and (4.17). Utilizing (4.18) into (4.19) yields the following quadratic equation

$$\Phi_1 (I_h^*)^2 + \Phi_2 I_h^* + \Phi_3 = 0, \tag{4.20}$$

where $\Phi_1 = N_r^* \Psi_1, \Phi_2 = N_h^* k_3 k_5 \beta_3 I_r^* + \Psi_2 N_r^*$ and $\Phi_3 = -N_h^* k_5^2 \beta_3 I_r^*$. Here, Ψ_1 and Ψ_2 are given by (4.13) (with $\beta_3 \neq 0$). Clearly, Φ_1 is always positive while Φ_3 is negative (due to (4.17)) ensures that the discriminant $\Delta = \Phi_2^2 - 4\Phi_1\Phi_3 > 0$, and hence the equation (4.20) has two unequal real solutions which can be expressed explicity as follows

$$I_h^{*^\pm} = \frac{1}{2\Phi_1}\left[-\Phi_2 \pm (\Delta)^{1/2}\right],$$

where $I_h^{*^\pm}$ denote respectively the positive and negative values of real solutions. Notice that Φ_2 has a negative sign when $\left(\beta_1 k_4 + \beta_2 k_5 - N_h^* k_2\right) N_r^* - N_h^* k_3 \beta_3 I_r^*$ is a positive. As applying of Descartes' signs rule confirms that (4.20) has only one positive solution, we infer there exists a single positive "*human*" (MEE) point whose components are induced by $I_h^{*^+}$ and given as follows

$$\chi_h^{*^+} = \left(S_h^{*^+}, V_h^{*^+}, E_h^{*^+}, Q_h^{*^+}, A_h^{*^+}, I_h^{*^+}, H_h^{*^+}, R_h^{*^+}\right). \tag{4.21}$$

Combine (4.21) with the knowledge that all components of χ_r^* in (4.2) can be obtained explicity by (4.17) we conclude that the only positive (MEE) for the (MPV) system in $\mathcal{O}$ takes the form

$$\chi^{*^+} := \left(\chi_h^{*^+}, \chi_r^*\right) \in \mathcal{O}^{*^+} \tag{4.22}$$

which solves (MPV) at the steady state, where $\mathcal{O}^{*^+}$ denotes the space of *positive* Mpox-endemic axis of $\mathcal{O}^*$. This leads to establish the following statement:

Proposition 4.4. *The* (MPV) *system in* $\mathcal{O}$ *associated to animal-human transmission possesses at least one positive* (MEE) *given by* (4.22).

Remark 4.5. Observe that that the co-existence of (MPV) may occurrence only for Case III. Although we did not assume that infected humans transmitting the disease into animals or the only infected animals exist in our system, the (MPV) system identifies, beside to (MFE), three (MEE) states associated to human and/or animal infections in $\mathcal{O}$: the animal-free (MEE), the human-free (MEE), and the (MEE) state where the (MPV) co-exists.

5. Basic reproduction rate for (MPV) system

We utilize the next generation matrix method to calculate the *basic reproduction rate* [32] for the (MPV) system in $\mathcal{O}$. Here, our main focus will be only on transition terms associated to (MPV), which can be written in matrix-vector form:

$$\dot{z}(t) = \mathcal{F} \circ z(t) - \mathcal{V} \circ z(t), \quad z(t) = \left(E_h(t), Q_h(t), A_h(t), I_h(t), H_h(t), E_r(t), I_r(t)\right)^t,$$

where $\mathcal{F} = \left(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_7\right)^t$ with $\mathcal{F}_i$ denotes the rate of new infections occurring in compartment i and $\mathcal{V} = \left(\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_7\right)^t$ with $\mathcal{V}_i$ signifies the rates at which infections are transferred into and out of compartment i (for $i = 1, 2, \cdots, 7$). Accordingly, we have

$$\mathcal{F}_1 = \beta_h\, S_h(t)\,, \quad \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = 0\,, \quad \mathcal{F}_6 = \beta_r\, S_r(t)\,, \quad \mathcal{F}_7 = 0\,,$$

and

$$\begin{aligned}
&\mathcal{V}_1 = (\eta+\mu_h)\, E_h(t)\,,\ \mathcal{V}_2 = (\alpha+\theta+\mu_h)\, Q_h(t) - \eta\, E_h(t)\,,\ \mathcal{V}_3 = (\gamma_1+\psi+\mu_h)\, A_h(t) - \tau\,\alpha\, Q_h(t),\\
&\mathcal{V}_4 = (\omega+\mu_h+\delta_h)\, I_h(t) - (1-\tau)\,\alpha\, Q_h(t) - \psi A_h(t)\,,\ \mathcal{V}_5 = (\gamma_2+\mu_h+\delta_h)\, H_h(t) - \omega I_h(t)\,,\\
&\mathcal{V}_6 = (\sigma+\mu_r)\, E_r(t)\,,\ \mathcal{V}_7 = (\gamma_3+\mu_r+\delta_r)\, I_r(t) - \sigma E_r(t)\,,
\end{aligned}$$

where $0 < \tau < 1$. Here β_h and β_r are the forces of infection given by (2.4) and (2.5); respectively. Consequently, the Jacobian for $\mathcal{F}$ and $\mathcal{V}$ evaluated at (MFE) are given respectively by the nonnegative matrix of new infection terms:

$$\nabla_z \mathcal{F}\left[\chi_o\right] = \begin{bmatrix}
0 & 0 & f_{13} & f_{14} & 0 & 0 & f_{17}\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & f_{67}\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{5.1}$$

where $f_{13} = \beta_1 \Lambda_1$, $f_{14} = \beta_2 \Lambda_1$, $f_{17} = \beta_3 \Lambda\, \aleph$ and $f_{67} = \beta_4$ with

$$\aleph = \frac{\nu_h}{\mu_h}\frac{\mu_r}{\nu_r}, \tag{5.2}$$

and the nonsingular M-matrix:

$$\nabla_z \mathcal{V}\left[\chi_o\right] = \begin{bmatrix}
\nu_{11} & 0 & 0 & 0 & 0 & 0 & 0\\
-\eta & \nu_{22} & 0 & 0 & 0 & 0 & 0\\
0 & -\tau\alpha & \nu_{33} & 0 & 0 & 0 & 0\\
0 & (\tau-1)\alpha & -\psi & \nu_{44} & 0 & 0 & 0\\
0 & 0 & 0 & -\omega & \nu_{55} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \nu_{66} & 0\\
0 & 0 & 0 & 0 & 0 & -\sigma & \nu_{77}
\end{bmatrix}, \tag{5.3}$$

where

$$\begin{aligned}
&\nu_{11} = \eta + \mu_h,\ \ \nu_{22} = \alpha + \theta + \mu_h,\ \ \nu_{33} = \gamma_1 + \psi + \mu_h,\ \ \nu_{44} = \omega + \mu_h + \delta_h,\\
&\nu_{55} = \gamma_2 + \mu_h + \delta_h,\quad \nu_{66} = \sigma + \mu_r,\quad \nu_{77} = \gamma_3 + \mu_r + \delta_r
\end{aligned} \tag{5.4}$$

with the nonzero determinant $\left|\nabla \mathcal{V}\right| = \prod_{j=1}^{7} \nu_{jj}$. The matrices (5.1) and (5.3), from now on will be often denoted, indistinguishably, by $\mathcal{F}$ and $\mathcal{V}$; respectively. As customary $\mathcal{V}^{-1}$

always denotes to the inverse of $\mathcal{V}$ and it can be simply obtained as follows

$$\begin{bmatrix} \frac{1}{\nu_{11}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\eta}{\nu_{11}\nu_{22}} & \frac{1}{\nu_{22}} & 0 & 0 & 0 & 0 & 0 \\ \frac{\eta\,\tau\,\alpha}{\nu_{11}\nu_{22}\nu_{33}} & \frac{\tau\,\alpha}{\nu_{22}\nu_{33}} & \frac{1}{\nu_{33}} & 0 & 0 & 0 & 0 \\ \frac{\eta\,\alpha\,\xi}{\nu_{11}\nu_{22}\nu_{33}\nu_{44}} & \frac{\alpha\,\xi}{\nu_{22}\nu_{33}\nu_{44}} & \frac{\psi}{\nu_{33}\nu_{44}} & \frac{1}{\nu_{44}} & 0 & 0 & 0 \\ \frac{\eta\,\omega\,\alpha\,\xi}{\nu_{11}\nu_{22}\nu_{33}\nu_{44}\nu_{55}} & \frac{\omega\,\alpha\,\xi}{\nu_{22}\nu_{33}\nu_{44}\nu_{55}} & \frac{\omega\,\psi}{\nu_{33}\nu_{44}\nu_{55}} & \frac{\omega}{\nu_{44}\nu_{55}} & \frac{1}{\nu_{55}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\nu_{66}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sigma}{\nu_{66}\nu_{77}} & \frac{1}{\nu_{77}} \end{bmatrix},$$

where $\xi = \psi + (1-\tau)(\gamma_1 + \mu_h)$ and ν_{jj} for $j \in \{1, 2, \cdots, 7\}$ are given by (5.4). Therefore, the *basic reproduction rate* for the (MPV) system, which is usually denoted by $\mathcal{R}_o$, is given by the spectral radius $\rho(\mathcal{F}\mathcal{V}^{-1})$ of the next generation matrix $\mathcal{F}\mathcal{V}^{-1}$. Indeed, since

$$\mathcal{F}\mathcal{V}^{-1} = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & 0 & \Upsilon_{16} & \Upsilon_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Upsilon_{66} & \Upsilon_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\Upsilon_{11} = \frac{\eta\,\alpha\,\Lambda\,\zeta}{\nu_{11}\nu_{22}\nu_{33}\nu_{44}},\ \Upsilon_{12} = \frac{\alpha\,\Lambda\,\zeta}{\nu_{22}\nu_{33}\nu_{44}},\ \Upsilon_{13} = \frac{\Lambda[\beta_1(\varpi/\tau) + \beta_2\psi]}{\nu_{33}\nu_{44}},\ \Upsilon_{14} = \frac{\Lambda\,\beta_2}{\nu_{44}},$$
$$\Upsilon_{16} = \frac{\sigma\,\Lambda\,\beta_3\,\aleph}{\nu_{66}\nu_{77}},\ \Upsilon_{17} = \frac{\Lambda\,\beta_3\,\aleph}{\nu_{77}},\ \Upsilon_{66} = \frac{\sigma\,\beta_4}{\nu_{66}\nu_{77}},\ \Upsilon_{67} = \frac{\beta_4}{\nu_{77}}$$

with $\zeta = \varpi\beta_1 + \xi\beta_2$ and $\varpi = \tau\nu_{44}$, we infer that $\mathcal{R}_o$ can be determined by the overall

$$\mathcal{R}_o = \max\left\{\mathcal{R}_h^o, \mathcal{R}_r^o\right\}, \tag{5.5}$$

where

$$\mathcal{R}_h^o = \frac{\eta\alpha\left[\rho_2 + (1-q)\mu_h\right]\left(\beta_1\tau(\omega + \mu_h + \delta_h) + \beta_2\left[\psi + (1-\tau)(\gamma_1 + \mu_h)\right]\right)}{(\rho_1 + \rho_2 + \mu_h)(\eta + \mu_h)(\alpha + \theta + \mu_h)(\gamma_1 + \psi + \mu_h)(\omega + \mu_h + \delta_h)} \tag{5.6}$$

and

$$\mathcal{R}_r^o = \frac{\beta_4\sigma}{(\sigma + \mu_r)(\gamma_3 + \mu_r + \delta_r)} \tag{5.7}$$

are called respectively the *"human"* and *"animal"* threshold rates. Consequently, the next fact which is straightforward to verify its claims will be used often in the sequel of this work.

Lemma 5.1. *Suppose that* (5.5), (5.6) *and* (5.7) *are holding for the* (MPV) *system in* $\mathcal{O}$. *Then,* $\mathcal{R}_h^o > 1$ *if and only if* $\eta\,\alpha\,\Lambda\,\zeta - \nu_{11}\nu_{22}\nu_{33}\nu_{44}$ *has a positive sign, and* $\mathcal{R}_h^o \leqslant 1$ *otherwise. Moreover,* $\mathcal{R}_r^o > 1$ *if and only if* $\beta_4\,\sigma - \nu_{66}\nu_{77}$ *has a positive sign, and* $\mathcal{R}_r^o \leqslant 1$ *otherwise. Furthermore,* $\mathcal{R}_o > 1$ *if and only if either of* $\mathcal{R}_h^o$ *or* $\mathcal{R}_r^o$ *exceeds the unity, and* $\mathcal{R}_o \leqslant 1$ *otherwise.*

In the next theorem, we shall view the results of Propositions 4.2, 4.3 and 4.4 as a one statement by reformulating them in the light of (5.5); however, in this form, it displays a much stronger conclusion than those forms because it is related to them but the proof is different from there where it relies on the fact that one knows a priori that $\mathcal{R}_o \in (1,\infty)$.

Theorem 5.2. *Consider the cases of animal-free, human-free, or animal-human transmissions. Then, the* (MPV) *system in* $\mathcal{O}$ *admits a unique* (MEE) *whenever* $\mathcal{R}_o$ *surpasses 1.*

Proof. Let $\mathcal{R}_o$, $\mathcal{R}_h^o$ and $\mathcal{R}_r^o$ be given by (5.5), (5.6) and (5.7); respectively. Assume that $\mathcal{R}_o > 1$. First, if $1 = \mathcal{R}_r^o < \mathcal{R}_h^o$, then $\mathcal{R}_o > 1$ means that $\mathcal{R}_h^o > 1$ (thanks to Lemma 5.1). Since from (4.14) we deduce

$$I_{h_{af}}^* = \Pi\left[\mathcal{R}_h^o - \frac{\mu_h}{\nu_h}N_{h_{af}}^*\right], \tag{5.8}$$

where $\mathcal{R}_h^o := \mathcal{R}_{h_{af}}^o$ (due to nonexistent of β_3 in (5.6)) and

$$\Pi = \frac{(\rho_1+\rho_2+\mu_h)\left[\psi+(1-\tau)(\gamma_1+\mu_h)\right]\nu_h}{k_3\left\{\beta_1\tau(\omega+\mu_h+\delta_h)+\beta_2\left[\psi+(1-\tau)(\gamma_1+\mu_h)\right]\right\}} > 0$$

whenever $0 < \tau < 1$ and $k_3 > 0$ is defined by (4.4), one can easily show that $\mathcal{R}_o > 1$ yields $I_{h_{af}}^* > 0$ follows from $N_{h_{af}}^*(t) \leqslant (\nu_h/\mu_h)$ in $\mathcal{O}_{h_{af}}^*$ for every $t > 0$.
Next, if $1 = \mathcal{R}_h^o < \mathcal{R}_r^o$, then $\mathcal{R}_o > 1$ means that $\mathcal{R}_r^o > 1$ (according to Lemma 5.1 above). Since $N_r^*(t) \leqslant (\nu_r/\mu_r)$ in $\mathcal{O}_r^*$ for every $t > 0$ one can see immediately from (4.17) that

$$I_r^* = \frac{\nu_r}{\beta_4}\left[\mathcal{R}_o - 1\right] > 0 \quad \text{whenrver} \quad \mathcal{R}_o > 1. \tag{5.9}$$

Further, if $1 < \mathcal{R}_h^o < \mathcal{R}_r^o$ or $\mathcal{R}_h^o < 1 < \mathcal{R}_r^o$, then using Lemma 5.1 gives $\mathcal{R}_o = \mathcal{R}_r^o > 1$. Applying (5.6) and (5.9) to the leading and constant coefficients of (4.20) yield respectively

$$\Phi_1 = k_3\,\nu_h(\nu_r/\mu_r)\mathcal{R}_h^o > 0, \tag{5.10}$$

where $k_3 > 0$ is defined by (4.4), and

$$\Phi_3 = \Theta_1(k_5/k_3)\left[1-\mathcal{R}_o\right] < 0, \tag{5.11}$$

where $k_5 > 0$ is defined by (4.10) and $\Theta_1 = (\nu_h/\mu_h)(\nu_r/\beta_4)\beta_3 k_3 k_5$ is also positive. Moreover, using (5.10) and (5.11) together with the fact that the linear coefficient of (4.20) satisfies

$$\Phi_2 = \Theta_1\left[\mathcal{R}_o - 1\right] + \Theta_2\left[1-\mathcal{R}_h^o\right] < 0 \quad \text{if} \quad \mathcal{R}_o < (\Theta_2/\Theta_1)\left[\mathcal{R}_h^o - 1\right] + 1, \tag{5.12}$$

where $\Theta_2 = \nu_h(\rho_2 + \mu_h)k_5$ is a positive, this gives necessary conditions for the existence of distinct real solutions (4.2). Indeed, employing (5.10), (5.11) with (5.12) into the discriminant of (4.20) leads to

$$\Delta = \Theta_1^2[\mathcal{R}_o - 1]^2 + \Theta_2^2[\mathcal{R}_h^o - 1]^2 + 2\Theta_1[\Theta_2 + (2\Theta_3 - \Theta_2)\mathcal{R}_h^o][\mathcal{R}_o - 1]$$

has a positive sign whenever $\Theta_3 = (\nu_r/\mu_r)\nu_h k_5$ is a greater than $\Theta_2/2$. It follows that the single positive value of (4.20) can be given explicitly by the expression

$$\begin{aligned} I_h^{*^+} = \frac{\mu_r k_5}{2\mathcal{R}_h^o}\Bigg(&\frac{\beta_3}{\beta_4\mu_h}[\mathcal{R}_o - 1] + \frac{\rho_2 + \mu_h}{k_3\nu_r}[1 - \mathcal{R}_h^o] \\ &+ \left[\left(\frac{\beta_3}{\beta_4\mu_h}[\mathcal{R}_o - 1] - \frac{\rho_2 + \mu_h}{k_3\nu_r}[\mathcal{R}_h^o - 1]\right)^2 + \frac{\beta_3}{\beta_4\mu_h}\frac{4\mathcal{R}_h^o}{k_3\mu_r}[\mathcal{R}_o - 1]\right]^{1/2}\Bigg) \end{aligned} \tag{5.13}$$

provided that $k_3\beta_3\nu_r < \beta_4\mu_h(\rho_2 + \mu_h)$. Thus, each component of (4.7), (4.15) or (4.22) exhibits only one positive value induced respectively either by (5.8), (5.9) or (5.13) for $\mathcal{R}_o > 1$. □

Notice that, in all cases of the proof above, the connection between $\mathcal{R}_h^o$, $\mathcal{R}_r^o$ and $\mathcal{R}_o$ is given by (5.5). More precisely, the theorem we proved above basically establishes the connection between $\mathcal{R}_o$ in $(1, \infty)$ and the existence of a single (MEE) for the (MPV) system.

Remark 5.3. In all reasonably three cases of transmissions associated to human and/or animal infections (see Theorem 5.2), the uniqueness results for (MEE) and (MFE) can be obtained concomitantly from our method of proof (since (MFE) always exists!). However, our (MPV) system always admits (MFE) even when $\mathcal{R}_o \in (0, 1]$. This is another case, which is not covered by the statement of Theorem 5.2, but follows easily from it.

6. Steady state analysis of (MPV) system

This section discusses concepts related to stability states of the (MPV) system at each of (MFE) and (MEE).

6.1. Local stability for (MFE)

The following theorem shows that the Mpox will disappear over time when the initial subpopulations sizes of (MPV) system lie in the attraction basin of (MFE).

Theorem 6.1. *The* (MFE) *of* (MPV) *system is a LAS if* $\mathcal{R}_o \leqslant 1$*, and unstable otherwise.*

Proof. Consider the case $\mathcal{R}_o < 1$. So, we must have $\mathcal{R}_h^o < 1$ and $\mathcal{R}_r^o < 1$ (due to Lemma 5.1). In order to assess a *LAS* of (MFE), we need to analyze the Jacobian representation of (MPV) system at the fixed point; χ_o, with respect to the forces of infection β_h and β_r which can be given by

$$\mathcal{J}_{12}^{hr}[\chi_o](\beta_h; \beta_r) = \begin{bmatrix} M_{8\times 8}^h[\chi_o](\beta_1, \beta_2) & M_{4\times 8}^h[\chi_o](\beta_3) \\ O_{8\times 4} & M_{4\times 4}^r[\chi_o](\beta_4) \end{bmatrix}, \tag{6.1}$$

where $O_{8\times4}$ denotes to the 8×4-null matrix, and

$$M^{h}_{8\times8}[\chi_o](\beta_1,\beta_2) := \begin{bmatrix} -h_1 & \rho_2 & 0 & \theta & -\beta_1\Lambda_1 & -\beta_2\Lambda_1 & 0 & 0 \\ \rho_1 & -h_2 & 0 & 0 & 0 & 0 & 0 & \varphi \\ 0 & 0 & -h_3 & 0 & \beta_1\Lambda_1 & \beta_2\Lambda_1 & 0 & 0 \\ 0 & 0 & \eta & -h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau\alpha & -h_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-\tau)\alpha & \psi & -h_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & -h_7 & 0 \\ 0 & 0 & 0 & 0 & \gamma_1 & 0 & \gamma_2 & -h_8 \end{bmatrix},$$

$$M^{h}_{4\times8}[\chi_o](\beta_3) := \begin{bmatrix} 0 & 0 & -\beta_3\Lambda_1\aleph & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_3\Lambda_1\aleph & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M^{r}_{4\times4}[\chi_o](\beta_4) := \begin{bmatrix} -\mu_r & 0 & -\beta_4 & 0 \\ 0 & -r_1 & \beta_4 & \nu 0 \\ 0 & \sigma & -r_2 & 0 \\ 0 & 0 & \gamma_3 & -\mu_r \end{bmatrix},$$

with

$$\begin{aligned} &h_i = \rho_i + \mu_h \text{ (for } i = 1,2),\ h_3 = \eta + \mu_h,\ h_4 = \alpha + \theta + \mu_h,\ h_5 = \gamma_1 + \psi + \mu_h, \\ &h_6 = \omega + \mu_h + \delta_h,\ h_7 = \gamma_2 + \mu_h + \delta_h,\ h_8 = \varphi + \mu_h,\ r_1 = \sigma + \mu_r,\ r_2 = \gamma_3 + \mu_r + \delta_r \end{aligned} \tag{6.2}$$

and Λ_1 and $\aleph$ are given by (4.1) and (5.2); respectively. Observe that the characteristic polynomial of (6.1) can be expressed as the determinant $\left|\mathcal{J}^{hr}_{12}[\chi_o] - \lambda\,\mathrm{Id}_{12}\right|$, where λ and Id_{12} stand for eigenvalues and 12×12-identity matrix; respectively. Therefore, we have

$$\mathcal{P}^{hr}_{12}(\lambda) = \mathcal{P}^{h}_{8}(\lambda)\,\mathcal{P}^{r}_{4}(\lambda), \tag{6.3}$$

where

$$\mathcal{P}^{h}_{8}(\lambda) = \mathcal{P}^{h}_{2}(\lambda)\mathcal{P}^{h}_{4}(\lambda)(-h_7-\lambda)(-h_8-\lambda) \quad \text{and} \quad \mathcal{P}^{r}_{4}(\lambda) = (-\mu_r-\lambda)^2\,\mathcal{P}^{r}_{2}(\lambda)$$

represent the characteristic polynomials for the "*human*" and "*animal*" blocks $M^{h}_{8\times8}[\chi_o]$ and $M^{r}_{4\times4}[\chi_o]$; respectively. It is straightforward to provide four roots of (6.3); say, $\lambda_7 = -(\gamma_2+\mu_h+\delta_h)$, $\lambda_8 = -(\varphi+\mu_h)$ for $\mathcal{P}^{h}_{8}$ and $\lambda_9 = -\mu_r = \lambda_{12}$ for $\mathcal{P}^{r}_{4}$. Moreover, we can see that the first two roots λ_1 and λ_2 of (6.3) have both negative real parts as well whenever $h_1h_2 - \rho_1\rho_2$ has a positive sign. Indeed, it is easy to show that $\lambda_1 = -\mu_h$ and $\lambda_2 = -(\rho_1+\rho_2+\mu_h)$ from the quadratic "*human*"-polynomial

$$\mathcal{P}^{h}_{2}(\lambda) = \lambda^2 + (h_1+h_2)\lambda + h_1h_2 - \rho_1\rho_2.$$

Consequently, the *LAS* of (MFE) is ascertained by the remaining six roots of (6.3) which can be found from the equations for E_h, Q_h, A_h, I_h, E_r, I_r by the following submatrices:

$$M^{h}_{4\times4}[\chi_o](\beta_1,\beta_2) = \begin{bmatrix} -h_3 & 0 & -\beta_1\Lambda_1 & -\beta_2\Lambda_1 \\ \eta & -h_4 & 0 & 0 \\ 0 & \tau\alpha & -h_5 & 0 \\ 0 & (1-\tau)\alpha & \psi & -h_6 \end{bmatrix} \text{ and } M^{r}_{2\times2}[\chi_o](\beta_4) = \begin{bmatrix} -r_1 & \beta_4 \\ \sigma & -r_2 \end{bmatrix},$$

where the h's and r's entries are given (as above!) by (6.2). As the characteristic polynomials for these subblocks can be respectively described as follows

$$\mathcal{P}_4^h(\lambda)=\lambda^4+\mathcal{H}_3\lambda^3+\mathcal{H}_2\lambda^2+\mathcal{H}_1\lambda+\mathcal{H}_0\left(1-\mathcal{R}_h^o\right) \quad \text{and} \quad \mathcal{P}_2^r(\lambda)=\lambda^2+\mathcal{R}_1\lambda+\mathcal{R}_0\left(1-\mathcal{R}_r^o\right), \tag{6.4}$$

where $\mathcal{H}_0 = h_3h_4h_5h_6$, $\mathcal{H}_1 = h_3h_4\left(h_5+h_6\right)+\left(h_3+h_4\right)h_5h_6-\eta\alpha\Lambda_1\left[\tau\beta_1+(1-\tau)\beta_2\right]$, $\mathcal{H}_2 = h_3\left(h_4+h_5\right)+h_4\left(h_5+h_6\right)+h_5h_6$, $\mathcal{H}_3 = h_3+h_4+h_5+h_6$, $\mathcal{R}_0 = r_1r_2$ and $\mathcal{R}_1 = r_1+r_2$. Clearly, the two roots; say, λ_{10} and λ_{11} of (6.3), which can be simply found from the quadratic "*animal*"-polynomial $\mathcal{P}_2^r$, have both negative real parts since $\mathcal{R}_1 > 0$, $\mathcal{R}_0\left(1-\mathcal{R}_r^o\right) > 0$ if and only if $\mathcal{R}_r^o < 1$, and $\mathcal{R}_1^2 > 4\mathcal{R}_0\left(1-\mathcal{R}_r^o\right)$ where $\mathcal{R}_0 = r_1r_2$ for $r_1 > 0$, $r_2 > 0$. Indeed, such these roots can be expressed explicitly as follows

$$\lambda_{10,11} = -\left[\left(\mu_r+\frac{\sigma+\delta_r+\gamma_3}{2}\right) \mp \sqrt{\left(\frac{\sigma-\delta_r-\gamma_3}{2}\right)^2+\sigma\beta_4}\right] < 0$$

provided that $r_1r_2-\sigma\beta_4$ has a positive sign. Furthermore, according to the Routh-Hurwitz criterion, the last four roots λ_i (for $i = 3,4,5,6$) of (6.3), which can be obtained from $\mathcal{P}_4^h$, all possess negative real parts too provided that the following hypotheses are satisfied:

$$\mathcal{H}_3 > 0\,,\ \mathcal{H}_2 > 0 \quad \text{and} \quad \mathcal{H}_3\mathcal{H}_2 > \mathcal{H}_1\,,$$

$$\mathcal{H}_0\left(1-\mathcal{R}_h^o\right) > 0 \quad \text{if and only if} \quad \mathcal{R}_h^o < 1 \quad \text{and} \quad \mathcal{H}_0 > 0\,,$$

$$\mathcal{H}_3\mathcal{H}_2\mathcal{H}_1 > 0\,, \text{and}$$

$$\mathcal{H}_3\mathcal{H}_2\mathcal{H}_1-\mathcal{H}_1^2 > \mathcal{H}_3^2\mathcal{H}_0\left(1-\mathcal{R}_h^o\right), \text{where } \mathcal{H}_0 = h_3h_4h_5h_6 \text{ for } h_3>0,\ h_4>0,\ h_5>0,\ h_6>0.$$

Thus, all roots of (6.3) have negative real parts whenever $\mathcal{R}_o < 1$, thereby indicating a *LAS* of (MFE) and this finishes the proof of stability. For the instability of (MFE), it is direct to show that each polynomial in (6.4) has a positive root when $\mathcal{R}_h^o$ or $\mathcal{R}_r^o$ is grather than 1. □

6.2. Global stability for (MFE)

The next theorem shows that Mpox eradication is independent of the initial population sizes.

Theorem 6.2. *The* (MFE) *of* (MPV) *system with* $q = \rho_1 = \rho_2 = \varphi = \beta_3 = 0$ *is a GAS in* $\mathcal{O}$ *if* $\mathcal{R}_o \leqslant 1$*, and unstable otherwise.*

Proof. For any $t > 0$ we consider the Lyapunov functional on $\mathcal{O}$

$$L(t) = l_1E_{h_{af}}(t)+l_2Q_{h_{af}}(t)+l_3A_{h_{af}}(t)+l_4I_{h_{af}}(t)+l_5E_r(t)+l_6I_r(t), \tag{6.5}$$

where the coefficients l_i for $i \in \{1,2,\cdots,6\}$ are nonnegative chosen constants in such away that L is continuous, well-defined, and positive for all $\left(E_{h_{af}}, Q_{h_{af}}, A_{h_{af}}, I_{h_{af}}, E_r, I_r\right) > 0$

in $\mathcal{O}$ with $L = 0$ only at the null $(0,0,0,0,0,0)$. Differentiating (6.5) with respect to t and using the fact $N_{h_{af}}(t) \leqslant (\nu_h/\mu_h)$ and $N_r(t) \leqslant (\nu_r/\mu_r)$ in $\mathcal{O}$ for any $t > 0$ we obtain

$$\begin{aligned}\dot{L}(t) \leqslant\ & l_1\left[\beta_1 A_{h_{af}} + \beta_2 I_{h_{af}} - h_3 E_{h_{af}}\right] + l_2\left[\eta E_{h_{af}} - h_4 Q_{h_{af}}\right] + l_3\left[\tau\alpha Q_{h_{af}}\right. \\ & \left. - h_5 A_{h_{af}}\right] + l_4\left[\alpha(1-\tau) Q_{h_{af}} - h_6 I_{h_{af}}\right] + l_5\left[\beta_4 I_r - r_1 E_r\right] + l_6\left[\sigma E_r - r_2 I_r\right],\end{aligned} \tag{6.6}$$

where h_3, h_4, h_5, h_6, r_1, and r_2 are given by (6.2). Setting each sum of relevant coefficients to variables $E_{h_{af}}, A_{h_{af}}, I_{h_{af}}, I_r$ and solving the new system for l_i, $i \in \{1,2,\cdots,6\}$ to get

$$\begin{aligned} l_1 &= \frac{\nu_h}{\mu_h}, \quad l_2 = \frac{\nu_h(\eta+\mu_h)}{\mu_h\eta}, \quad l_3 = \frac{\nu_h\left[\beta_1(\omega+\mu_h+\delta_h)+\beta_2\psi\right]}{\mu_h(\psi+\gamma_1+\mu_h)(\omega+\mu_h+\delta_h)}, \\ l_4 &= \frac{\nu_h\,\beta_2}{\mu_h(\omega+\mu_h+\delta_h)}, \quad l_5 = \frac{\nu_r}{\mu_r}, \quad l_6 = \frac{\nu_r\,\beta_4}{\mu_r(\gamma_3+\mu_r+\delta_r)}. \end{aligned}$$

Inserting the l's values into (6.5) allows to express (6.6) as follows

$$\dot{L}(t) \leqslant \nu_h\left(\frac{1}{\eta}+\frac{1}{\mu_h}\right)\left[\alpha\tau+(1-\tau)\alpha+\theta+\mu_h\right]\left[\mathcal{R}_h^o-1\right]Q_{h_{af}} + \nu_r\left(1+\frac{\sigma}{\mu_r}\right)\left[\mathcal{R}_r^o-1\right]E_r,$$

where $\mathcal{R}_h^o$ and $\mathcal{R}_r^o$ are given by (5.6) and (5.7); respectively. Now, to show that $\dot{L}(t) \leqslant 0$, first note that it is trivially true if $\mathcal{R}_o \leqslant 1$ (i.e., $\mathcal{R}_h^o \leqslant 1$ and $\mathcal{R}_r^o \leqslant 1$ — see Lemma 5.1 above). Furthermore,

$$\dot{L}(t) = 0 \ \text{ if and only if } \ E_{h_{af}} = Q_{h_{af}} = A_{h_{af}} = I_{h_{af}} = E_r = I_r = 0.$$

It follows from LaSalle invariance principle [18] that

$$\left(S_{h_{af}}, V_{h_{af}}, E_{h_{af}}, Q_{h_{af}}, A_{h_{af}}, I_{h_{af}}, H_{h_{af}}, R_{h_{af}}, S_r, E_r, I_r, R_r\right) \xrightarrow[t\to\infty]{} (\text{MFE}).$$

Thus, the biggest invariant subspace included in a space of points in $\mathcal{O}$ where $\dot{L}$ vanishes is reduced to only (MFE); i.e., $\mathcal{O}^o := \{\chi_o\}$. □

Notice that Theorem 6.2 shows that in the absence of immigrants, effective vaccination of susceptible and recovered individuals, and cross infection from animal to human, the Mpox could still be effectively eliminated from the population (when $\mathcal{R}_o \leqslant 1$) only by quarantne and hospitalization strategies.

6.3. Global stability for (MEE)

We will show the (MPV) system will eventually settle into a steady state of persistent infection within the population irrespective of the initial population sizes. To do that, we first recall that the disease-free axis Ω^o consisting of all $(S_h, V_h, 0,0,0,0,0,0, S_r, 0,0,0) \in \Omega$ is a stable manifold of (MFE), and then we claim:

Theorem 6.3. *The unique* (MEE) *of* (MPV) *system with* $\theta = \varphi = \psi = 0$ *is a GAS in* $\mathcal{O}\backslash\mathcal{O}^o$ *for* $\mathcal{R}_o > 1$*, and unstable otherwise.*

Proof. First note that $\mathcal{R}_o > 1$ implies (MEE) with $\theta = \varphi = \psi = 0$ exists and unique (thanks to Theorem 5.2). For any time $t > 0$ we consider the Lyapunov functional in $\mathcal{O}\backslash\mathcal{O}^o$ as follows:

$$\begin{aligned}
L(t) = {} & \left[\frac{S_h}{S_h^*} - \ln\frac{S_h}{S_h^*} - 1\right]S_h^* + \left[\frac{V_h}{V_h^*} - \ln\frac{V_h}{V_h^*} - 1\right]V_h^* + \left[\frac{E_h}{E_h^*} - \ln\frac{E_h}{E_h^*} - 1\right]E_h^* + \frac{\beta_3 I_r^* S_h^*}{\eta E_h^*}\left[\frac{Q_h}{Q_h^*}\right. \\
& \left. - \ln\frac{Q_h}{Q_h^*} - 1\right]Q_h^* + \frac{\beta_1 A_h^* S_h^*}{\alpha\tau Q_h^*}\left[\frac{A_h}{A_h^*} - \ln\frac{A_h}{A_h^*} - 1\right]A_h^* + \frac{\beta_2 I_h^* S_h^*}{\alpha(1-\tau)Q_h^*}\left[\frac{I_h}{I_h^*} - \ln\frac{I_h}{I_h^*} - 1\right]I_h^* \\
& + \left[\frac{S_r}{S_r^*} - \ln\frac{S_r}{S_r^*} - 1\right]S_r^* + \left[\frac{E_r}{E_r^*} - \ln\frac{E_r}{E_r^*} - 1\right]E_r^* + \frac{\beta_4 I_r^* S_r^*}{\sigma E_r^*}\left[\frac{I_r}{I_r^*} - \ln\frac{I_r}{I_r^*} - 1\right]I_r^*,
\end{aligned}$$

where the fixed point $(S_h^*, V_h^*, E_h^*, Q_h^*, A_h^*, I_h^*, S_r^*, E_r^*, I_r^*)$ given by (4.3) is the only extremum and global minimum of L in $\mathcal{O}\backslash\mathcal{O}^o$. Consequently, the time-derivative of L at the endemic steady state can be written as

$$\begin{aligned}
\dot{L}(t) = {} & \left[1 - \frac{S_h^*}{S_h}\right]\left[\beta_h^* S_h^* - \beta_h S_h - (\rho_1 + \mu_h)(S_h - S_h^*) - \rho_2(V_h - V_h^*)\right] + \rho_1\left[1 - \frac{V_h^*}{V_h}\right]\left[\rho_1(S_h - S_h^*)\right. \\
& \left. - (\rho_2 + \mu_h)(V_h - V_h^*)\right] + \left[1 - \frac{E_h^*}{E_h}\right]\left[\beta_h S_h - \beta_h^* S_h^*\frac{E_h}{E_h^*}\right] + \beta_3 I_r^* S_h^*\left[1 - \frac{Q_h^*}{Q_h}\right]\left[\frac{E_h}{E_h^*} - \frac{Q_h}{Q_h^*}\right] \\
& + \beta_1 A_h^* S_h^*\left[1 - \frac{A_h^*}{A_h}\right]\left[\frac{Q_h}{Q_h^*} - \frac{A_h}{A_h^*}\right] + \beta_2 I_h^* S_h^*\left[1 - \frac{I_h^*}{I_h}\right]\left[\frac{Q_h}{Q_h^*} - \frac{I_h}{I_h^*}\right] + \left[1 - \frac{S_r^*}{S_r}\right]\left[\beta_r^* S_r^*\right. \\
& \left. - \beta_r S_r - \mu_r(S_r - S_r^*)\right] + \left[1 - \frac{E_r^*}{E_r}\right]\left[\beta_r S_r - \beta_r^* S_r^*\frac{E_r}{E_r^*}\right] + \beta_4 I_r^* S_r^*\left[1 - \frac{I_r^*}{I_r}\right]\left[\frac{E_r}{E_r^*} - E_r^*\frac{I_r}{I_r^*}\right].
\end{aligned}$$

After further simplifications we can reduce the last expression to just

$$\begin{aligned}
\dot{L}(t) \leqslant & -(\rho_1 + \mu_h)\frac{(S_h - S_h^*)^2}{S_h} - (\rho_2 + \mu_h)\frac{(V_h - V_h^*)^2}{V_h} \\
& - \mu_r\frac{(S_r - S_r^*)^2}{S_r} - \Phi(S_h, V_h, E_h, Q_h, A_h, I_h, S_r, E_r, I_r),
\end{aligned}$$

where the function

$$\begin{aligned}
\Phi = {} & \rho_1 S_h^*\left[\frac{S_h}{S_h^*} + \frac{V_h^*}{V_h} - \frac{S_h V_h^*}{S_h^* V_h} - 1\right] + \rho_2 V_h^*\left[\frac{V_h}{V_h^*} + \frac{S_h^*}{S_h} - \frac{S_h^* V_h}{S_h V_h^*} - 1\right] + \beta_1 S_h^* A_h^*\left[\frac{S_h^*}{S_h} + \frac{E_h}{E_h^*}\right. \\
& \left. - \frac{Q_h}{Q_h^*} + \frac{Q_h A_h^*}{Q_h^* A_h} + \frac{S_h E_h^* A_h}{S_h^* E_h A_h^*} - 3\right] + \beta_2 S_h^* I_h^*\left[\frac{S_h^*}{S_h} + \frac{E_h}{E_h^*} - \frac{Q_h}{Q_h^*} + \frac{Q_h I_h^*}{Q_h^* I_h} + \frac{S_h E_h^* I_h}{S_h^* E_h I_h^*} - 3\right] \\
& + \beta_3 S_h^* I_r^*\left[\frac{S_h^*}{S_h} + \frac{E_h Q_h^*}{E_h^* Q_h} + \frac{Q_h I_r^*}{Q_h^* I_r} + \frac{S_h E_h^* I_r}{S_h^* E_h I_r^*} - 4\right] + \beta_4 S_r^* I_r^*\left[\frac{S_r^*}{S_r} + \frac{E_r I_r^*}{E_r^* I_r} + \frac{S_r E_r^* I_r}{S_r^* E_r I_r^*} - 3\right]
\end{aligned}$$

is nonnegative (using the approach in [22]). It follows that $\dot{L} \leqslant 0$ for $\mathcal{R}_o > 1$ with $\dot{L} = 0$ only at $(S_h^*, V_h^*, E_h^*, Q_h^*, A_h^*, I_h^*, S_r^*, E_r^*, I_r^*)$. Applying the LaSalle's invariance principle [18] yields all trajectories of (MPV) system (with $\theta = \varphi = \psi = 0$) which intersect $\mathcal{O}\backslash\mathcal{O}^o$ will go to the invariant space $\mathcal{O}^*$ consisting only of (MEE). □

7. Bifurcation analysis of (MPV) system

This section investigates the local dynamics of (MEE) for the (MPV) system close to a bifurcation point and verify that (MFE) will be a nonhyperbolic equilibrium. More precisely, we adapt Theorem 4.1 provided in [8] to analyze the existence of a bifurcation for our system whenever $\mathcal{R}_o$ is equal to unity around (MFE). To clarify our notation, we will define a new parameter for β as $\mathcal{R}_o - 1$ after transform (MPV) system into the general form of ODE's:

$$\dot{\chi}(t) = \mathfrak{X}(\chi(t), \beta) \quad \text{under prescribed } \chi(0) = \chi_o,$$

where the vector field $\mathfrak{X} = (\mathfrak{X}_1, \cdots, \mathfrak{X}_{12})^t$ is given after the change of variables as follows:

$$\begin{aligned}
\mathfrak{X}_1 &= (1-q)\nu_h - (\rho_1+\mu_h)\chi_1 + \rho_2\chi_2 + \theta\chi_3 - (\chi_1/N_h)(\beta_1\chi_5+\beta_2\chi_6) - (\chi_1/N_r)\beta_3\chi_{11}\,,\\
\mathfrak{X}_2 &= q\nu_h + \rho_1\chi_1 - (\rho_2+\mu_h)\chi_2 + \varphi\chi_8\,,\\
\mathfrak{X}_3 &= (\chi_1/N_h)(\beta_1\chi_5+\beta_2\chi_6) + (\chi_1/N_r)\beta_3\chi_{11} - (\eta+\mu_h)\chi_3\,,\\
\mathfrak{X}_4 &= \eta\chi_3 - (\tau\alpha + (1-\tau)\alpha + \theta + \mu_h)\chi_4\,,\\
\mathfrak{X}_5 &= \tau\alpha\chi_4 - (\psi+\gamma_1+\mu_h)\chi_5\,,\\
\mathfrak{X}_6 &= (1-\tau)\alpha\chi_4 + \psi\chi_5 - (\omega+\delta_h+\mu_h)\chi_6\,,\\
\mathfrak{X}_7 &= \omega\chi_6 - (\gamma_2+\delta_h+\mu_h)\chi_7\,,\\
\mathfrak{X}_8 &= \gamma_1\chi_5 + \gamma_2\chi_7 - (\varphi+\mu_h)\chi_8\,,\\
\mathfrak{X}_9 &= \nu_r - \mu_r\chi_9 - (\chi_9/N_r)\beta_4\chi_{11}\,,\\
\mathfrak{X}_{10} &= (\chi_9/N_r)\beta_4\chi_{11} - (\sigma+\mu_r)\chi_{10}\,,\\
\mathfrak{X}_{11} &= \sigma\chi_{10} - (\gamma_3+\delta_r+\mu_r)\chi_{11}\,,\\
\mathfrak{X}_{12} &= \gamma_3\chi_{11} - \mu_r\chi_{12}\,.
\end{aligned} \tag{7.1}$$

7.1. Existence of bifurcation

We consider the case of $\beta_h = \beta_1 = \beta_2$ and $\beta_r = \beta_4$ are selected as the bifurcation parameters. Setting $\mathcal{R}_o = 1$ in (5.5) and simplify for $\beta_h^* = \beta_h$ and $\beta_r^* = \beta_r$ in (5.6) and (5.7); respectively, implies

$$\beta_h^* = \frac{(\rho_1+\rho_2+\mu_h)(\eta+\mu_h)(\alpha+\theta+\mu_h)(\gamma_1+\psi+\mu_h)(\omega+\mu_h+\delta_h)}{\alpha\eta[\rho_2+(1-q)\mu_h]\,(\tau[\psi+\omega+\mu_h+\delta_h]+(1-\tau)(\gamma_1+\mu_h))} \tag{7.2}$$

and

$$\beta_r^* = \frac{1}{\sigma}(\sigma+\mu_r)(\gamma_3+\mu_r+\delta_r). \tag{7.3}$$

Then, the Jacobian matrix of transformed system (7.1) around (MFE), denoted by $\mathcal{J}[\chi_o](\beta_h^*;\beta_r^*)$, is similarly given via (6.1) (with interchange β_h and β_r by the new corresponding parameters β_h^* and β_r^*, in light of (7.2) and (7.3)). Thus, from (6.4) one can see easily that $\mathcal{J}[\chi_o](\beta_h^*;\beta_r^*)$ has a zero eigenvalue and hence a bifurcation occurs.

7.2. Determining the direction of bifurcation

The right eigenvector of $\mathcal{J}[\chi_o](\beta_h^*;\beta_r^*)$ corresponding to the zero eigenvalue is given by $z=(z_1,z_2,\cdots,z_{12})^t$, where

$$
\begin{aligned}
z_1 &= \frac{(\rho_2+\mu_h)\left[\theta z_4-\beta_h^*\Lambda_1(z_5+z_6)\right]+\rho_2\,\varphi\,z_8}{\mu_h(\rho_1+\rho_2+\mu_h)},\\
z_2 &= \frac{\rho_1\left[\theta z_4-\beta_h^*\Lambda_1(z_5+z_6)\right]+(\rho_1+\mu_h)\,\varphi\,z_8}{\mu_h(\rho_1+\rho_2+\mu_h)},\quad z_3=\frac{\alpha+\theta+\mu_h}{\eta}z_4>0,\\
z_4 &= z_4>0,\quad z_5=\frac{\tau\alpha}{\psi+\gamma_1+\mu_h}z_4>0,\quad z_6=\frac{\alpha\left[\psi+(1-\tau)(\gamma_1+\mu_h)\right]z_4}{(\psi+\gamma_1+\mu_h)(\omega+\delta_h+\mu_h)}>0,\\
z_7 &= \frac{\alpha\,\omega\left[\psi+(1-\tau)(\gamma_1+\mu_h)\right]z_4}{(\psi+\gamma_1+\mu_h)(\omega+\delta_h+\mu_h)(\gamma_2+\delta_h+\mu_h)}>0,\\
z_8 &= \frac{\alpha\left(\tau\gamma_1(\omega+\delta_h+\mu_h)(\gamma_2+\delta_h+\mu_h)+\omega\gamma_2\left[\psi+(1-\tau)(\gamma_1+\mu_h)\right]\right)z_4}{(\psi+\gamma_1+\mu_h)(\omega+\delta_h+\mu_h)(\gamma_2+\delta_h+\mu_h)(\varphi+\mu_h)}>0,
\end{aligned}
$$

and $z_9=z_{10}=z_{11}=z_{12}=0$. Moreover, the left eigenvector of $\mathcal{J}[\chi_o](\beta_h^*;\beta_r^*)$ corresponding to the zero eigenvalue is given by $y=(y_1,y_2,\cdots,y_{12})^t$ such that $y\cdot z=1$, where

$$
\begin{aligned}
y_1 &= y_2=0,\quad y_3=\frac{\eta\,y_4}{\eta+\mu_h}>0,\quad y_4=y_4>0,\\
y_5 &= \frac{\beta_h^*\,\eta\,\Lambda_1(\psi+\omega+\delta_h+\mu_h)\,y_4}{(\eta+\mu_h)(\psi+\gamma_1+\mu_h)(\omega+\delta_h+\mu_h)}>0,\\
y_6 &= \frac{\beta_h^*\,\eta\,\Lambda_1\,y_4}{(\eta+\mu_h)(\omega+\delta_h+\mu_h)},\quad y_7=y_8=y_9=0,\\
y_{10} &= \frac{\beta_3^*\nu_h\,\mu_r\,\Lambda_1\,\eta\,\sigma\,y_4}{\mu_h\nu_r(\eta+\mu_h)\left[(\sigma+\mu_r)(\gamma_3+\delta_r+\mu_r)-\beta_r^*\sigma\right]}>0,\\
y_{11} &= \left(1+\frac{\mu_r}{\sigma}\right)y_{10}>0,\quad y_{12}=0.
\end{aligned}
$$

Next, we evaluate all the nonzero 2$^{\text{nd}}$-order partial derivatives of $\mathcal{X}$ at (MFE) to identify

$$
a=\sum_{k,i,j=1}^{12} y_k\,z_i\,z_j\,\partial_{\chi_i\chi_j}\mathcal{X}_k(\chi_o;\beta^*)=\frac{2\,\beta_h^*\,\eta\,y_4}{\mu_h\,N_h\,(\eta+\mu_h)\,(\rho_1+\rho_2+\mu_h)}\left[\mathcal{M}-\mathcal{L}\right],
$$

where

$$
\mathcal{M}=\frac{(\rho_1+\rho_2+\mu_h)\left[\rho_2(\theta z_4+\varphi z_8)+\mu_h\theta z_4\right](z_5+z_6)+q\mu_h\beta_h^*(\rho_2+\mu_h)(z_5+z_6)^2}{\rho_1+\rho_2+\mu_h}>0
$$

and

$$
\mathcal{L}=\frac{\beta_h^*\,(\rho_2+\mu_h)^2(z_5+z_6)^2}{\rho_1+\rho_2+\mu_h}>0.
$$

As $y_4 > 0$ is free we infer the value of $\mathfrak{a}$ has to be a positive if and only if $\mathcal{L}$ is lower than $\mathcal{M}$. So, one may define the *basic reinfection rate* for the (MPV) system as follows

$$\mathcal{R}_o^* = \frac{(\rho_1+\rho_2+\mu_h)\left[(\rho_2+\mu_h)\theta z_4+\rho_2\varphi z_8\right]+q\mu_h\beta_h^*(\rho_2+\mu_h)(z_5+z_6)}{\beta_h^*(\rho_2+\mu_h)^2(z_5+z_6)}. \tag{7.4}$$

Clearly, $\mathfrak{a}$ has a positive sign whenever $\mathcal{R}_o^* > 1$. Furthermore, the value of $\mathfrak{b}$ can be shown as follows

$$\mathfrak{b} = \sum_{k,i=1}^{12} y_k z_i \partial_{\chi_i\beta^*}\mathcal{X}_k(\chi_o;\beta^*) = \frac{\eta\nu_h\left[\rho_2+(1-q)\mu_h\right]y_4}{\mu_h N_h(\eta+\mu_h)(\rho_1+\rho_2+\mu_h)}(z_5+z_6)$$

which is always positive. Hence, we have the following main result: let's first recall (5.5).

Theorem 7.1. *Let $\mathcal{R}_o^*$ be the human-basic reinfection for the (MPV) system given by (7.4). Then, the (MPV) system exhibits a transcritical bifurcation at $\mathcal{R}_o = 1$ and its direction is a backward whenever $\mathcal{R}_o^*$ exceeds the unity in $\mathcal{O}$.*

Three important consequences arise in view of Theorem 7.1: we recall (5.6) and (5.7).

Corollary 7.2. *Consider $\mathcal{R}_o^*$ is less than the unity. Then, the (MPV) system in $\mathcal{O}$ admits a forward bifurcation at $\mathcal{R}_h^o = 1$ for $\mathcal{R}_r^o < 1$ and at $\mathcal{R}_r^o = 1$ for any value of $\mathcal{R}_h^o > 0$.*

Remark 7.3. The fact in Corollary 7.2 can be achieved particularly in the absence of immigrants, reinfection of quarantine, and effective vaccination of recovered individuals.

The second is just we use Corollary 7.2 and Item (iv) of Theorem 4.1 in [8] to conclude:

Corollary 7.4. *The special case of (MPV) system in $\mathcal{O}$ with $q = \theta = \varphi = 0$ posses a unique (MEE) for some $\mathcal{R}_h^o > \mathcal{R}_r^o \geqslant 1$, and no (MEE) otherwise. In this case, the (MEE) is a LAS close to unstable (MFE) for $\mathcal{R}_h^o \gg 1$, and unstable otherwise.*

Finally, since the reinfection rate of prompt quarantine at (MFE) can be evaluated as

$$\theta^* = \frac{\beta_h^*\left[\rho_2+(1-q)\mu_h\right](\rho_2+\mu_h)(z_5+z_6)-(\rho_1+\rho_2+\mu_h)\rho_2\,\varphi\,z_8}{(\rho_1+\rho_2+\mu_h)(\rho_2+\mu_h)z_4}, \tag{7.5}$$

we recast the substatements above, in light of (7.5), regarding the bifurcation phenomena in the next mainful conclusion.

Corollary 7.5. *Let θ^* be given by (7.5). Then, the (MPV) system in $\mathcal{O}$ admits a backward (or; respectively, forward) bifurcation phenomena if and only if $\theta^* < \theta$ (or; respectively, $\theta^* > \theta$).*

Notice that for θ is higher than the calculated value θ^* the backward bifurcation property of (MPV) system arises while it dispeares when θ does not exceed the value of θ^*. This means that any slight change in the reinfection rate of prompt quarantine on population can impact the success of control interventions.

8. Conclusion

This study developed and analyzed a comprehensive nonlinear mathematical model consisting of twelve compartments to investigate the transmission dynamics of Mpox, accounting for complex human-animal interactions. By integrating key intervention of vaccination, quarantine and hospitalization, the model especially highlighted the role of prompt quarantine strategies and the impact of migration and temporary immunity on disease persistence. Qualitative analysis established fundamental properties of the system, including the existence and uniqueness of equilibria. Using the next-generation matrix method, the basic reproduction number $\mathcal{R}_o$ was obtained as a crucial threshold parameter. Stability analysis using the Routh-Hurwitz criterion and Lyapunov functions showed that the disease-free equilibrium is stable when $\mathcal{R}_o$ is less than one. However, application of center manifold theory revealed the occurrence of backward bifurcation at $\mathcal{R}_o = 1$. This phenomenon, induced by the loss of immunity and the influx of infected immigrants, suggests that reduction of $\mathcal{R}_o$ below unity is necessary but may not be sufficient for complete eradication to disease unless specific thresholds for reinfection $\mathcal{R}_o^*$ are maintained. Our results emphasize that the success of public health interventions is highly sensitive to the frequency of immediate quarantine and management of healthy individuals with temporary immunity. As a result, this work will provide a theoretical basis for healthcare professionals to prioritize early isolation and robust vaccination campaigns. For future work, we recommend extending this framework to the setting of fractional-order derivatives to better capture the memory effects of infections. In addition, incorporating stochastic fluctuations might evolve deeper insight into the environmental uncertainties affecting Mpox outbreaks.

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Conflicts of Interest

No conflict of interest to disclose.

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