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Note for Line and Total SuperHyperGraphs: Connecting Vertices, Edges, Edges of Edges, Edges of Edges of Edges in Hierarchical Systems

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Abstract

Hypergraphs extend classical graphs by allowing *hyperedges* to connect any nonempty subset of vertices, thereby capturing complex group-level relationships. Superhypergraphs advance this framework by introducing recursively nested powerset layers, enabling the representation of hierarchical and self-referential links among hyperedges. A *line graph* encodes the adjacencies between edges of an original graph by transforming each edge into a vertex and connecting two vertices if their corresponding edges share a common endpoint. A *total graph* incorporates both the vertices and edges of the original graph as its own vertices, with edges representing adjacency or incidence between these entities. An *iterated line graph* arises from the repeated application of the line graph construction, where each iteration takes the previous line graph as its input. Similarly, an *iterated total graph* is generated by iteratively applying the total graph transformation a specified number of times. This paper investigates the hypergraph and superhypergraph analogues of these constructions, providing a foundation for further theoretical development.

Keywords: SuperHyperGraph, HyperGraph, Line Graph, Total Graph, Iterated line graph, Iterated total graph.
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1. Introduction

1.1. Hypergraphs and Superhypergraphs

Graph theory investigates mathematical structures built from vertices and edges, providing tools to model relationships, networks, connectivity, optimization, and many real-world systems [1, 2]. However, classical (simple) graphs struggle to encode certain phenomena—most notably higher-order interactions and hierarchical organization. To overcome these limitations, *HyperGraphs* and *SuperHyperGraphs* have been introduced and developed. A *hypergraph* generalizes a simple graph by allowing each edge (a *hyperedge*) to connect an arbitrary nonempty subset of the vertex set, thereby representing relations

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among more than two entities [3, 4, 5]. A *superhypergraph* further enlarges this framework by iterating the powerset operation, which yields layered (recursively defined) vertex sets and hierarchically nested links among hyperedges; see, e.g., [6, 7, 8, 9]. These topics continue to be actively investigated [10, 11, 12, 13]. Table 1 contrasts graphs, hypergraphs, and superhypergraphs. Unless stated otherwise, n denotes a natural number.

Concept	**Notation**	**Edge Type**	**Extension Mechanism**
Graph	$G = (V, E)$	$E \subseteq \binom{V}{2} = \{\{u, v\} : u, v \in V,\ u \neq v\}$	Standard edges always join exactly two vertices.
Hypergraph	$H = (V, E)$	$E \subseteq \mathrm{POWS}(V) \setminus \{\emptyset\}$	Hyperedges may join any nonempty subset of V (arity $\geqslant 1$).
Superhypergraph	$\mathrm{SuHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathrm{POWS}^n(V_0), \quad E \subseteq \mathrm{POWS}(V)$	Applies an n-fold powerset to produce layered vertex sets and nested connections.

Table 1: Comparison of Graph, Hypergraph, and Superhypergraph.

1.2. *Total Graph and Line Graph*

A wide variety of graph extensions and graph classes have been explored in the literature. In this paper we focus on the *Line Graph* and the *Total Graph*. The line graph of a graph represents its edges as vertices and joins two such vertices whenever the corresponding original edges share an endpoint (cf. [14, 15, 16]). The total graph has as vertices both the original vertices and edges, linking them according to adjacency or incidence in the source graph (cf. [17, 18, 19, 20]). An *iterated line graph* is obtained by repeatedly applying the line-graph construction, each time to the output of the previous step[21, 22]; analogously, an *iterated total graph* arises from iterating the total-graph transformation a prescribed number of times. These concepts have also been the subject of numerous recent studies [23, 24, 25]. For reference, an overview of Graph, Line Graph, Total Graph, Iterated Line Graph, and Iterated Total Graph is provided in Table 2.

1.3. *Our Contribution*

While hypergraphs/superhypergraphs and line/total graphs are each well-motivated, their fused counterparts—especially superhypergraph analogues of the line and total graph constructions—remain insufficiently characterized. To bridge this gap, we extend the line-graph and total-graph operations to the settings of HyperGraphs and SuperHyperGraphs and examine their basic properties. This paper thus lays a foundation for further theoretical development of line- and total-type constructions within hierarchical, higher-order

Construction	Vertices	Edges / Adjacency rule	Notes
Graph $G = (V, E)$	V	$\{u, v\} \in E$ joins $u, v \in V$.	Simple and loopless unless stated otherwise.
Line Graph $L(G)$	$E(G)$	$\{e, f\}$ is an edge iff $e \cap f \neq \emptyset$ in G.	Edges of G become vertices; captures edge–edge contact.
Total Graph $T(G)$	$V(G) \dot\cup E(G)$	Vertex–vertex if adjacent in G; edge–edge if share an endpoint in G; vertex–edge if incident in G.	Unifies adjacency and incidence in one graph.
Iterated Line $L^k(G)$	$V(L^k(G))$	$L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$ for $k \geqslant 1$.	Encodes "transfers of transfers"; structure evolves with k.
Iterated Total $T^k(G)$	$V(T^{k-1}(G)) \dot\cup E(T^{k-1}(G))$	$T^0(G) = G$, $T^k(G) = T(T^{k-1}(G))$; same three rules at each level.	Adds higher-layer incidence; contains $L^k(G)$ as an induced subgraph.

Table 2: Graph, Line Graph, Total Graph, Iterated Line Graph, and Iterated Total Graph.

network models. Tables 3 and 4 present an overview of HyperGraph, Line/Total HyperGraphs and their iterated variants, as well as SuperHyperGraph, Line/Total SuperHyperGraphs and their iterated variants.

Construction	Vertices	Edges / Adjacency rule	Notes
HyperGraph $H = (V, \mathcal{E})$	V	$\mathcal{E} \subseteq \mathrm{POWS}(V) \setminus \{\emptyset\}$.	Hyperedge arity $\geqslant 1$.
Line HyperGraph $L(H)$	$\mathcal{E}(H)$	Edges are stars $\mathrm{Star}_H(v) = \{E \in \mathcal{E} : v \in E\}$.	Incidence-based; 2-uniform $\Rightarrow$ $[L(H)]_2 = L(G)$.
Total HyperGraph $T(H)$	$V \dot\cup \mathcal{E}$	$\mathcal{A} = \mathcal{E}$, $\mathcal{B} = \{\mathcal{E}_H(v)\}$, $\mathcal{C} = \{\{v\} \cup \mathcal{E}_H(v)\}$.	Unifies adjacency+incidence; $H = H_G \Rightarrow [T(H)]_2 = T(G)$.
Iterated Line $L^k(H)$	$V(L^k(H))$	$L^0(H) = H$, $L^k(H) = L(L^{k-1}(H))$.	$[L^{2m}(H)]_2 \cong [H]_2$, $[L^{2m+1}(H)]_2 \cong [L(H)]_2$.
Iterated Total $T^k(H)$	$V(T^{k-1}(H)) \dot\cup \mathcal{E}(T^{k-1}(H))$	$T^0(H) = H$, $T^k(H) = T(T^{k-1}(H))$.	$L^k(H) \subseteq_{\mathrm{ind}} T^k(H)$; $H = H_G \Rightarrow [T^k(H)]_2 = T^k(G)$.

Table 3: Summary of HyperGraph, Line/Total HyperGraphs, and iterated variants.

2. Preliminaries

In this section, we review the key concepts and notation used throughout this paper. All graphs and HyperGraphs considered here are finite.

2.1. *Hypergraphs and Superhypergraphs*

The following describes the definitions of Hypergraphs and Superhypergraphs.

Definition 2.1 (Base Set)**.** [26] Let S be a nonempty set, called the *base set*. All higher-order objects, such as powersets and supervertices, are constructed from S:

$$S = \{\, x \mid x \text{ is an element of the domain} \,\}.$$

Construction	Vertices	Edges / Adjacency rule	Notes
Level-n SuperHyperGraph $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$	$V_n \subseteq \mathrm{POWS}^n(V_0)$	$\emptyset \neq \mathcal{E} \subseteq \mathrm{POWS}(V_n) \setminus \{\emptyset\}$.	$n = 0 \Rightarrow$ hypergraph.
Line SuperHyperGraph $\mathbf{L}(\mathcal{H}^{(n)})$	$\mathcal{E}$	Edges are stars $\mathrm{Star}_{\mathcal{H}}(v)$ for $v \in V_n$.	Level shift $n \to n+1$; $n=0$ 2-uniform $\Rightarrow$ 2-sec $= L(G)$.
Total SuperHyperGraph $\mathbf{T}(\mathcal{H}^{(n)})$	$U_{n+1} = \iota(V_n) \dot{\cup} \mathcal{E}$	$\mathcal{A} = \{\iota(E)\}$, $\mathcal{B} = \{\mathrm{Star}_{\mathcal{H}}(v)\}$, $\mathcal{C} = \{\{\iota(v)\} \cup \mathrm{Star}_{\mathcal{H}}(v)\}$.	Level shift $n \to n+1$; $\mathbf{L} = \mathbf{T}[\mathcal{E}]$; $n=0 \Rightarrow T(H)$.
Iterated Line $\mathbf{L}^t(\mathcal{H}^{(n)})$	$V(\mathbf{L}^t)$	$\mathbf{L}^0 = \mathcal{H}^{(n)}$, $\mathbf{L}^{t+1} = \mathbf{L}(\mathbf{L}^t)$.	Level $n+t$; parity: $[\mathbf{L}^{2m}(K)]_2 \cong [K]_2$.
Iterated Total $\mathbf{T}^t(\mathcal{H}^{(n)})$	$V(\mathbf{T}^{t-1}) \dot{\cup} \mathcal{E}(\mathbf{T}^{t-1})$	$\mathbf{T}^0 = \mathcal{H}^{(n)}$, $\mathbf{T}^{t+1} = \mathbf{T}(\mathbf{T}^t)$.	Level $n+t$; contains $\mathbf{L}^t$ (induced); for H_G: $[\mathbf{T}^t(H_G)]_2 = T^t(G)$.

Table 4: Summary of SuperHyperGraph, Line/Total SuperHyperGraphs, and iterated variants.

Definition 2.2 (Powerset). [27] For any set S, its *powerset* $\mathrm{POWS}(S)$ is the collection of all subsets of S, including $\emptyset$ and S itself:

$$\mathrm{POWS}(S) = \{ A \mid A \subseteq S \}.$$

Definition 2.3 (n-th Powerset). [28, 29, 30]. Let H be a set. The n*-th powerset* $\mathrm{POWS}^n(H)$ is defined recursively by

$$\mathrm{POWS}^0(H) = H, \qquad \mathrm{POWS}^{k+1}(H) = \mathrm{POWS}\big(\mathrm{POWS}^k(H)\big), \quad k \geqslant 0.$$

The *nonempty* n*-th powerset* $\mathrm{POWS}^{*n}(H)$ is defined similarly but removes the empty set at each stage:

$$\mathrm{POWS}^{*0}(H) = H, \qquad \mathrm{POWS}^{*(k+1)}(H) = \mathrm{POWS}^*\big(\mathrm{POWS}^{*k}(H)\big),$$

where $\mathrm{POWS}^*(X) = \mathrm{POWS}(X) \setminus \{\emptyset\}$

Example 2.4 (n-th Powerset Example). Let $H = \{a, b\}$. Then:

$$\begin{aligned}
\mathrm{POWS}^0(H) &= \{a, b\},\\
\mathrm{POWS}^1(H) &= \{\emptyset, \{a\}, \{b\}, \{a, b\}\},\\
\mathrm{POWS}^2(H) &= \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a, b\}\},\\
&\quad \{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}, \{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{a\}, \{a, b\}\},\\
&\quad \{\emptyset, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}, \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\}.
\end{aligned}$$

This illustrates the recursive growth of $\mathrm{POWS}^n(H)$ with n.

Example 2.5 (Real-life view of POWS^n: meal planning). Let the base set of *ingredient-types* be

$$H = \{\text{protein } (P),\ \text{vegetable } (V),\ \text{grain } (G)\}.$$

Then:

$$\mathrm{POWS}^0(H) = H = \{P, V, G\} \quad \text{(single ingredients)}.$$

One powerset step (dish options).

$$\mathrm{POWS}^1(H) = \{\emptyset, \{P\}, \{V\}, \{G\}, \{P, V\}, \{P, G\}, \{V, G\}, \{P, V, G\}\}.$$

Each element is a *dish composition* (which ingredient-types are used together). If empty dishes are disallowed, use the nonempty variant $\mathrm{POWS}^{*1}(H) = \mathrm{POWS}^1(H) \setminus \{\emptyset\}$.

Two powerset steps (menus as sets of dishes). $\mathrm{POWS}^2(H) = \mathrm{POWS}(\mathrm{POWS}^1(H))$ consists of *menus*, i.e., collections of dish options. For instance,

$$\{\{P\}, \{V, G\}\}, \quad \{\{P, V\}, \{P, G\}, \{V, G\}\}, \quad \{\{P, V, G\}\} \in \mathrm{POWS}^2(H).$$

If we exclude empty dishes at every stage, then

$$\mathrm{POWS}^{*2}(H) = \mathrm{POWS}\big(\mathrm{POWS}^{*1}(H)\big) \setminus \{\emptyset\}$$

collects *nonempty menus made of nonempty dishes*.

Counts (with and without empties). With $|H| = 3$,

$$|\mathrm{POWS}^1(H)| = 2^3 = 8, \qquad |\mathrm{POWS}^2(H)| = 2^8 = 256.$$

For the nonempty variant,

$$|\mathrm{POWS}^{*1}(H)| = 2^3 - 1 = 7, \qquad |\mathrm{POWS}^{*2}(H)| = 2^7 - 1 = 127.$$

$\mathrm{POWS}^1(H)$ enumerates all possible **dishes** (ingredient-type combinations). $\mathrm{POWS}^2(H)$ enumerates all possible **menus** (sets of dishes). Continuing, $\mathrm{POWS}^3(H)$ would encode **collections of menus** (e.g., weekly plans), and so on.

Definition 2.6 (HyperGraph)**.** [31, 32] A (finite) *HyperGraph* is an ordered pair $H = (V, E)$ where $V \neq \emptyset$ is the vertex set and

$$E \subseteq \mathrm{POWS}(V) \setminus \{\emptyset\}$$

is a finite family of nonempty subsets of V called *hyperedges*. In contrast to ordinary graphs, a hyperedge may contain an arbitrary number of vertices, allowing one to encode higher–arity relations. Throughout this note we assume V and E are finite.

Example 2.7 (Real-life HyperGraph: University Study Groups)**.** Let the vertex set be the students

$$V = \{\text{Alice, Bob, Cara, Dan, Eve}\}.$$

Define the family of hyperedges $E = \{e_1, e_2, e_3, e_4\} \subseteq \mathrm{POWS}(V) \setminus \{\emptyset\}$ by

$$e_1 = \{\text{Alice}, \text{Bob}, \text{Cara}\}, \quad e_2 = \{\text{Bob}, \text{Dan}\}, \quad e_3 = \{\text{Alice}, \text{Cara}, \text{Dan}, \text{Eve}\}, \quad e_4 = \{\text{Eve}\}.$$

Interpretation:

- e_1: a three–person Machine Learning project team,

- e_2: a two–person laboratory pair,
- e_3: a four–person capstone group,
- e_4: a solo independent study.

Then $H = (V, E)$ is a finite hypergraph with $|V| = 5$ and $|E| = 4$. The varying sizes $|e_1| = 3$, $|e_2| = 2$, $|e_3| = 4$, $|e_4| = 1$ model collaborations of different arities in a single structure.

Definition 2.8 (Level-n SuperHyperGraph (incidence model))**.** (cf. [33, 6, 34, 35]) Let V_0 be a finite base set and let $n \in \mathbb{N} \cup \{0\}$. Define the iterated powerset by

$$\mathrm{POWS}^0(V_0) = V_0, \qquad \mathrm{POWS}^{k+1}(V_0) = \mathrm{POWS}\big(\mathrm{POWS}^k(V_0)\big) \ \ (k \geqslant 0).$$

Choose a finite set $V_n \subseteq \mathrm{POWS}^n(V_0)$; its elements are the n-*supervertices*. A *level-*n *SuperHyperGraph* is a pair

$$\mathcal{H}^{(n)} = \big(V_n, \mathcal{E}\big), \qquad \varnothing \neq \mathcal{E} \subseteq \mathrm{POWS}(V_n) \setminus \{\varnothing\},$$

whose members $E \in \mathcal{E}$ are n-*superedges*. For $n = 0$ this recovers the usual finite hypergraph; if, in addition, every $E \in \mathcal{E}$ has $|E| = 2$, one obtains an ordinary (simple) graph.

Notation 2.9 (Stars)**.** For $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$ and $v \in V_n$, the *star* of v is

$$\mathsf{Star}_{\mathcal{H}}(v) \ := \ \{\, E \in \mathcal{E} \ : \ v \in E \,\} \ \subseteq \ \mathcal{E}.$$

We also write $\mathcal{E}^{\neq \emptyset}(v) := \mathsf{Star}_{\mathcal{H}}(v)$ and $\mathcal{E}^{(\geqslant 2)}(v) := \{\, E \in \mathcal{E} : v \in E \text{ and } |E| \geqslant 2 \,\}$ when we wish to exclude size-1 edges in the star.

Example 2.10 (A Level-2 SuperHyperGraph and its Stars)**.** Let the base set be $V_0 = \{a, b, c\}$. The level-1 supervertices are

$$V_1 = \big\{\{a, b\}, \{b, c\}, \{a\}\big\}.$$

The level-2 supervertices are taken as

$$V_2 = \big\{\{\{a, b\}, \{b, c\}\},\ \{\{a\}, \{b, c\}\},\ \{\{a, b\}\}\big\}.$$

Define the superedges by

$$\mathcal{E} = \big\{\{\{\{a, b\}, \{b, c\}\},\ \{\{a\}, \{b, c\}\}\},\ \{\{\{a\}, \{b, c\}\},\ \{\{a, b\}\},\ \{\{a, b\}, \{b, c\}\}\}\big\}.$$

Then $\mathcal{H}^{(2)} = (V_2, \mathcal{E})$ is a level-2 SuperHyperGraph.

For example, the star of the vertex $v = \{\{a, b\}, \{b, c\}\}$ is

$$\mathsf{Star}_{\mathcal{H}}(v) = \big\{\{\{\{a, b\}, \{b, c\}\},\ \{\{a\}, \{b, c\}\}\},\ \{\{\{a\}, \{b, c\}\},\ \{\{a, b\}\},\ \{\{a, b\}, \{b, c\}\}\}\big\}.$$

Example 2.11 (Real-life n-SuperHyperGraph ($n = 2$): Corporate Program of Committees)**.** Let the base set of employees be

$$V_0 = \{\text{Aiko},\ \text{Bruno},\ \text{Chloe},\ \text{Diego}\}.$$

Level 1 supervertices (committees) are chosen as a finite subset $V_1 \subseteq \mathrm{POWS}(V_0)$:

$$C_1 = \{\text{Aiko}, \text{Bruno}\}, \quad C_2 = \{\text{Bruno}, \text{Chloe}\}, \quad C_3 = \{\text{Chloe}, \text{Diego}\}, \quad V_1 = \{C_1, C_2, C_3\}.$$

Level 2 supervertices (committee clusters) are finite subsets of V_1; pick

$$S_1 = \{C_1, C_2\}, \qquad S_2 = \{C_2, C_3\}, \qquad V_2 = \{S_1, S_2\} \subseteq \mathrm{POWS}(V_1).$$

Define level–2 superedges $\mathcal{E} \subseteq \mathrm{POWS}(V_2) \setminus \{\varnothing\}$ by

$$E_1 = \{S_1, S_2\} \quad \text{(a cross–department “Innovation Program”)},$$

$$E_2 = \{S_1\} \quad \text{(a “Compliance Sprint”)}.$$

Then $\mathcal{H}^{(2)} = (V_2, \mathcal{E})$ is a level–2 superhypergraph with $|V_0| = 4$, $|V_1| = 3$, $|V_2| = 2$, and $|\mathcal{E}| = 2$. Semantics:

- Elements of V_1 (e.g., C_2) are committees (subsets of employees).
- Elements of V_2 (e.g., S_1) are clusters of committees (subsets of V_1).
- A level–2 superedge (e.g., E_1) joins multiple clusters when those clusters are jointly assigned to a single corporate initiative.

This hierarchy captures “teams of teams” and their joint engagements in a way that a usual hypergraph on V_0 cannot.

3. Review and Result: Line Graph

In this section, we address the concepts of Line Graph, Line HyperGraph, and Line SuperHyperGraph.

3.1. Line Graph

A line graph represents edges of a graph as vertices, linking them if the original edges share a common endpoint (cf.[14, 15, 16, 36, 37, 38]).

Definition 3.1 (Line graph)**.** Let $G = (V(G), E(G))$ be a (loopless) undirected simple graph. The *line graph* $L(G)$ is the (simple) graph defined by

$$V\big(L(G)\big) = E(G), \qquad \{e, f\} \in E\big(L(G)\big) \iff e \cap f \neq \emptyset,$$

i.e., two vertices of $L(G)$ are adjacent exactly when the corresponding two edges of G share an endpoint.

Example 3.2 (Line Graph — flight connections)**.** Consider a small airline network with airports $V = \{A, B, C, D\}$ and direct flights

$$E(G) = \{\, e_1 = AB,\ e_2 = AC,\ e_3 = BC,\ e_4 = BD \,\}.$$

The *line graph* $L(G)$ has one vertex for each flight e_i, and two vertices are adjacent iff the flights share an airport. Explicitly,

$$E\big(L(G)\big) = \{\ e_1e_2,\ e_1e_3,\ e_1e_4,\ e_2e_3,\ e_3e_4\ \},$$

so $L(G)$ is a K_4 missing the edge e_2e_4. Degrees are

$$\deg_{L(G)}(e_1) = 3, \quad \deg_{L(G)}(e_2) = 2, \quad \deg_{L(G)}(e_3) = 3, \quad \deg_{L(G)}(e_4) = 2.$$

(Verification by the standard formula: $\deg_G(A) = 2$, $\deg_G(B) = 3$, $\deg_G(C) = 2$, $\deg_G(D) = 1$ gives $|E(L(G))| = \sum_{v\in V} \binom{\deg_G(v)}{2} = 1+3+1+0 = 5$ as listed.) This models *connection possibilities*: a path $e_i - e_j$ in $L(G)$ corresponds to a feasible one-stop transfer sharing an airport.

3.2. Line HyperGraph

A line hypergraph transforms each hyperedge into a vertex, connecting vertices via hyperedges that share at least one original vertex (cf.[39, 40, 41, 42, 43]).

Definition 3.3 (Clique of rank r)**.** Given a finite set X and an integer $r \geqslant 0$, the *clique of rank* r on X is the (simple) uniform hypergraph whose vertex set is X and whose edge set is $\binom{X}{r}$ (with the conventions that rank 2 is an ordinary complete graph on X; rank 1 has edges the singletons of X; rank 0 is a single isolated vertex).

Definition 3.4 (Line hypergraph (Tyshkevich–Zverovich))**.** Let $H = (V, E)$ be a hypergraph without isolated vertices, and list $V = \{v_1, \dots, v_n\}$. Define two integer vectors

$$\mathbf{1}_H = (\deg(v_1), \dots, \deg(v_n)), \qquad \mathbf{0}_H = (\,0_{v_i}\,)_{i=1}^{n} \ \text{with}\ \ 0_{v_i} = \begin{cases} 0, & \deg(v_i) = 1, \\ 2, & \deg(v_i) \geqslant 2. \end{cases}$$

Let $\mathcal{D}_H := \{D = (d_{v_i})_{i=1}^{n} :\ \mathbf{0}_H \leqslant D \leqslant \mathbf{1}_H \text{ componentwise}\}$. For any $D \in \mathcal{D}_H$, define for each $v \in V$ the clique F_v of rank d_v on the vertex set $E(v)$, and set

$$L_D(H) \ :=\ \bigcup_{v\in V} F_v.$$

The (multi-valued) *line hypergraph* of H is the set

$$L(H) \ :=\ \{\, L_D(H) : D \in \mathcal{D}_H \,\}.$$

Example 3.5 (Line Hypergraph — meetings sharing attendees)**.** Let $H = (V, \mathcal{E})$ encode meetings with *people* as vertices

$$V = \{\text{Alice},\ \text{Bob},\ \text{Chloe},\ \text{Dan}\},$$

and *meetings* (hyperedges)

$$\mathcal{E} = \{\, E_1 = \{\text{Alice}, \text{Bob}, \text{Chloe}\},\ E_2 = \{\text{Bob}, \text{Dan}\},\ E_3 = \{\text{Chloe}, \text{Dan}\}\,\}.$$

The *line hypergraph* $L(H)$ has vertex set $\mathcal{E}$, and for each person $v \in V$ we add a hyperedge collecting all meetings that v attends:

$$\mathrm{Star}_H(\text{Alice}) = \{E_1\}, \quad \mathrm{Star}_H(\text{Bob}) = \{E_1, E_2\},$$

$$\mathrm{Star}_H(\text{Chloe}) = \{E_1, E_3\}, \quad \mathrm{Star}_H(\text{Dan}) = \{E_2, E_3\}.$$

Hence

$$L(H) = \Big(\{E_1, E_2, E_3\}, \ \big\{\{E_1\}, \{E_1, E_2\}, \{E_1, E_3\}, \{E_2, E_3\}\big\} \Big).$$

Interpretation: vertices are meetings; a hyperedge $\{E_i, E_j\}$ says "the two meetings share at least one common attendee" (here, Bob or Dan or Chloe).

3.3. Line SuperHyperGraph

A line superhypergraph maps each superedge to a vertex, linking them through superedges containing a common supervertex in the original structure.

Definition 3.6 (Line SuperHyperGraph). Let $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$ be a level-n SuperHyperGraph. Its *line SuperHyperGraph* is the pair

$$L^{(n)}(\mathcal{H}) \;=\; (V'_{n+1}, \, \mathcal{E}'_{n+1})$$

defined by

$$V'_{n+1} \;:=\; \mathcal{E} \qquad \text{and} \qquad \mathcal{E}'_{n+1} \;:=\; \big\{ \mathrm{Star}_{\mathcal{H}}(v) \subseteq \mathcal{E} \;:\; v \in V_n, \ \mathrm{Star}_{\mathcal{H}}(v) \neq \emptyset \big\}.$$

Intuitively, $L^{(n)}(\mathcal{H})$ has one vertex for each superedge of $\mathcal{H}$; for every supervertex $v \in V_n$, we add a (hyper)edge collecting *all* superedges that contain v.

Remark 3.7 (Level bookkeeping and singletons). Since $V_n \subseteq \mathrm{POWS}^n(V_0)$, we have $V'_{n+1} = \mathcal{E} \subseteq \mathrm{POWS}(V_n) \subseteq \mathrm{POWS}\big(\mathrm{POWS}^n(V_0)\big) = \mathrm{POWS}^{n+1}(V_0)$. Thus vertices of $L^{(n)}(\mathcal{H})$ naturally live one level up. Optionally replacing $\mathrm{Star}_{\mathcal{H}}(v)$ by $\mathcal{E}^{(\geqslant 2)}(v)$ removes singleton edges without changing the 2-section adjacency. We keep the nonempty stars for generality.

Example 3.8 (Line SuperHyperGraph — programs sharing teams). Start with a level-1 SuperHyperGraph that models *teams of people* and *programs grouping teams*. Let the base set of individuals be $V_0 = \{a, b, c, d\}$. Take the level-1 supervertex set (teams)

$$V_1 = \{ T_1 = \{a, b\}, \ T_2 = \{b, c\}, \ T_3 = \{c, d\} \} \subseteq \mathrm{POW}(V_0),$$

and define superedges (programs) as sets of teams:

$$\mathcal{E} = \{ P_1 = \{T_1, T_2\}, \ P_2 = \{T_2, T_3\} \} \subseteq \mathrm{POW}(V_1).$$

The *line SuperHyperGraph* $L^{(1)}(\mathcal{H})$ has vertex set $V'_2 = \mathcal{E} = \{P_1, P_2\}$. For each team $T \in V_1$, add one (super)hyperedge that collects all programs containing T:

$$\mathrm{Star}_{\mathcal{H}}(T_1) = \{P_1\}, \quad \mathrm{Star}_{\mathcal{H}}(T_2) = \{P_1, P_2\}, \quad \mathrm{Star}_{\mathcal{H}}(T_3) = \{P_2\}.$$

Therefore

$$L^{(1)}(\mathcal{H}) = \Big(\{P_1, P_2\}, \ \big\{\{P_1\}, \{P_1, P_2\}, \{P_2\}\big\} \Big).$$

Real-world reading: vertices are *programs*; a (super)hyperedge joins all programs that *share the same team*. Thus $\{P_1, P_2\}$ indicates that the two programs overlap operationally through team T_2.

Example 3.9 (Line SuperHyperGraph — research grants sharing laboratories). **Step 0: Build a level-1 superhypergraph.** Let the base set of *researchers* be

$$V_0 = \{a, b, c, d\}.$$

Form level-1 supervertices as *laboratories* (each a subset of researchers)

$$V_1 = \{ L_1 = \{a, b\},\ L_2 = \{b, c\},\ L_3 = \{c, d\} \} \subseteq \mathrm{POWS}(V_0),$$

and define *grants* (superedges) as sets of laboratories

$$\mathcal{E} = \{ G_A = \{L_1, L_2\},\ \ G_B = \{L_2\},\ \ G_C = \{L_2, L_3\} \} \subseteq \mathrm{POWS}(V_1).$$

Thus $\mathcal{H}^{(1)} = (V_1, \mathcal{E})$ is a level-1 SuperHyperGraph whose supervertices are labs and whose superedges are multi-lab grants.

Step 1: Form the line SuperHyperGraph. By Definition (Line SuperHyperGraph), the vertex set of $\mathsf{L}^{(1)}(\mathcal{H})$ is the grant set:

$$V_2' = \mathcal{E} = \{G_A, G_B, G_C\}.$$

For each lab $L \in V_1$, add one (super)hyperedge equal to the *star* $\mathrm{Star}_{\mathcal{H}}(L) = \{ G \in \mathcal{E} : L \in G \}$:

$$\mathrm{Star}_{\mathcal{H}}(L_1) = \{G_A\}, \qquad \mathrm{Star}_{\mathcal{H}}(L_2) = \{G_A, G_B, G_C\}, \qquad \mathrm{Star}_{\mathcal{H}}(L_3) = \{G_C\}.$$

Hence the line SuperHyperGraph is

$$\mathsf{L}^{(1)}(\mathcal{H}) = \Big(\{G_A, G_B, G_C\},\ \big\{ \{G_A\},\ \{G_A, G_B, G_C\},\ \{G_C\} \big\} \Big).$$

Vertices represent *grants*. A (super)hyperedge collects all grants that *share the same laboratory*. In particular, the hyperedge $\{G_A, G_B, G_C\}$ indicates that all three grants involve lab L_2, revealing operational overlap through a common lab; singletons $\{G_A\}$ and $\{G_C\}$ reflect labs L_1 and L_3 used by only one grant.

Theorem 3.10 (Well-definedness and SuperHyperGraph property). *For any level-n SuperHyperGraph $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$, the line construction $\mathsf{L}^{(n)}(\mathcal{H}) = (V'_{n+1}, \mathcal{E}'_{n+1})$ of Definition 3.6 is a level-$(n+1)$ SuperHyperGraph. In particular,*

$$V'_{n+1} \subseteq \mathrm{POWS}^{n+1}(V_0) \quad \textit{and} \quad \emptyset \neq \mathcal{E}'_{n+1} \subseteq \mathrm{POWS}(V'_{n+1}) \setminus \{\emptyset\}.$$

Proof. First, $V'_{n+1} = \mathcal{E} \subseteq \mathrm{POWS}(V_n) \subseteq \mathrm{POWS}^{n+1}(V_0)$ as noted in Remark 3.7; hence V'_{n+1} is a finite set of $(n{+}1)$-level objects. Second, by construction every $\mathsf{Star}_{\mathcal{H}}(v)$ is a subset of $\mathcal{E} = V'_{n+1}$. Nonemptiness of $\mathsf{Star}_{\mathcal{H}}(v)$ is part of the definition of $\mathcal{E}'_{n+1}$, so $\mathcal{E}'_{n+1} \subseteq \mathrm{POWS}(V'_{n+1}) \setminus \{\emptyset\}$. Finally, because $\mathcal{H}^{(n)}$ is finite, only finitely many $v \in V_n$ yield nonempty stars, so $\mathcal{E}'_{n+1}$ is finite and nonempty whenever $\mathcal{E} \neq \emptyset$. Therefore $\mathsf{L}^{(n)}(\mathcal{H})$ satisfies Definition 2.8 with level $n{+}1$. □

Definition 3.11 (2–section of a (Super)HyperGraph). For a pair $(U, \mathcal{F})$ with U finite and $\emptyset \neq \mathcal{F} \subseteq \mathrm{POWS}(U) \setminus \{\emptyset\}$, its *2–section* $[(U, \mathcal{F})]_2$ is the (simple) graph on vertex set U in which distinct $x, y \in U$ are adjacent iff there exists $F \in \mathcal{F}$ with $\{x, y\} \subseteq F$.

Example 3.12 (2-section of a hypergraph: overlapping groups)**.** Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and let

$$\mathcal{F} = \{F_1 = \{u_1, u_2, u_3\},\ \ F_2 = \{u_2, u_3, u_4\},\ \ F_3 = \{u_4, u_5\}\} \subseteq \mathrm{POWS}(U) \setminus \{\emptyset\}.$$

By Definition 3.11, the 2–section $[(U, \mathcal{F})]_2$ is the (simple) graph on vertex set U whose edges are precisely the 2–subsets contained in some F_i. Enumerating all such pairs:

$$\begin{aligned}
&\text{from } F_1: \ \{u_1, u_2\},\ \{u_1, u_3\},\ \{u_2, u_3\},\\
&\text{from } F_2: \ \{u_2, u_3\} \text{ (already counted)},\ \{u_2, u_4\},\ \{u_3, u_4\},\\
&\text{from } F_3: \ \{u_4, u_5\}.
\end{aligned}$$

Hence

$$E([(U, \mathcal{F})]_2) = \{\{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_2, u_4\}, \{u_3, u_4\}, \{u_4, u_5\}\}.$$

Interpretation: the 2–section contains a triangle on $\{u_1, u_2, u_3\}$, vertex u_4 is adjacent to u_2 and u_3, and u_5 attaches to u_4. For completeness, the degrees in the 2–section are

$$\deg(u_1) = 2, \quad \deg(u_2) = 3, \quad \deg(u_3) = 3, \quad \deg(u_4) = 3, \quad \deg(u_5) = 1,$$

all arising from membership overlaps inside F_1, F_2, F_3.

Theorem 3.13 (Graph case)**.** *Let $G = (V, E)$ be a finite simple loopless graph (so $E \subseteq \mathrm{POWS}(V)$, $|e| = 2$ for all $e \in E$). Regard G as a level-0 SuperHyperGraph $\mathcal{H}^{(0)} = (V, \mathcal{E})$ with $\mathcal{E} = E$. Then the 2–section of its line SuperHyperGraph coincides with the classical line graph:*

$$[L^{(0)}(\mathcal{H})]_2 \ = \ L(G).$$

Proof. By Definition 3.6, the vertex set of $L^{(0)}(\mathcal{H})$ is $V'_1 = \mathcal{E} = E$; we therefore identify vertices with edges of G. For $v \in V$, the star $\mathrm{Star}_{\mathcal{H}}(v) = \{e \in E : v \in e\}$ is precisely the set of all edges of G incident with v. In the 2–section $[L^{(0)}(\mathcal{H})]_2$, two distinct vertices $e, f \in E$ are adjacent iff there exists $v \in V$ with $\{e, f\} \subseteq \mathrm{Star}_{\mathcal{H}}(v)$, i.e. iff e and f are both incident with v, equivalently $e \cap f \neq \emptyset$. This is exactly the adjacency rule of the line graph $L(G)$. □

Theorem 3.14 (Hypergraph case: $L^{(0)}(H)$ is the incidence line hypergraph)**.** *Let $H = (V, \mathcal{E})$ be a finite hypergraph (i.e. a level-0 SuperHyperGraph). Then $L^{(0)}(H) = (\mathcal{E}, \mathcal{E}')$ with*

$$\mathcal{E}' \ = \ \{\{E \in \mathcal{E} : v \in E\} \ : \ v \in V,\ \{E \in \mathcal{E} : v \in E\} \neq \emptyset\}$$

is exactly the standard incidence *line hypergraph of H: its vertices are the edges of H, and for each $v \in V$ we add one hyperedge consisting of all edges incident with v.*

Proof. This is immediate from Definition 3.6 with $n = 0$. By construction, the vertex set is $\mathcal{E}$ (the original hyperedges), and for each $v \in V$ the star $\mathrm{Star}_{\mathcal{H}}(v)$ collects exactly those edges of H that contain v. □

Corollary 3.15 (Compatibility with the classical line graph via 2–section)**.** *For a graph G viewed as a rank-2 hypergraph $H_G = (V, \mathcal{E})$ with $\mathcal{E} = \{\{u, v\} : uv \in E(G)\}$, the 2–section $[L^{(0)}(H_G)]_2$ equals the classical line graph $L(G)$.*

Proof. Apply Theorem 3.13 with $\mathcal{H}^{(0)} = H_G$. □

4. Review and Result: Total Graph

In this section, we address the concepts of Total Graph, Total HyperGraph, and Total SuperHyperGraph.

4.1. Total Graph

A total graph has vertices for both vertices and edges of a graph, joining them by adjacency or incidence relationships (cf.[17, 18, 19, 20]).

Definition 4.1 (Total graph)**.** Let $G = (V(G), E(G))$ be a loopless simple undirected graph. The *total graph* $T(G)$ is the (simple) graph with

$$V\big(T(G)\big) = V(G) \,\dot\cup\, E(G),$$

where two distinct vertices $x, y \in V(G) \cup E(G)$ are adjacent in $T(G)$ iff one of the following holds:

1. (vertex–vertex) $x, y \in V(G)$ and $xy \in E(G)$;
2. (edge–edge) $x, y \in E(G)$ and $x \cap y \neq \emptyset$ in G;
3. (vertex–edge) $\{x, y\} = \{v, e\}$ with $v \in V(G)$, $e \in E(G)$, and $v \in e$.

Example 4.2 (Total Graph — a tiny road network)**.** Consider three intersections $V(G) = \{A, B, C\}$ with two roads $E(G) = \{AB,\ BC\}$. The *total graph* $T(G)$ has vertex set

$$V\big(T(G)\big) = \{A, B, C,\ AB,\ BC\}.$$

Edges in $T(G)$ encode: (i) road adjacency between intersections, (ii) two roads sharing an endpoint, and (iii) incidence between an intersection and a road passing through it:

$$\begin{array}{ll} \text{(vertex–vertex)} & \{A, B\},\ \{B, C\}; \\ \text{(edge–edge)} & \{AB, BC\} \quad \text{(they share B);} \\ \text{(vertex–edge)} & \{A, AB\},\ \{B, AB\},\ \{B, BC\},\ \{C, BC\}. \end{array}$$

Interpretation: $T(G)$ simultaneously represents the places (intersections) and the links (roads), and how they relate (including transfer from a place onto a road, and between adjacent roads).

4.2. Total HyperGraph

A total hypergraph contains original vertices and hyperedges, connecting them through vertex–vertex, edge–edge, and vertex–edge incidence hyperedges.

Definition 4.3 (Total hypergraph)**.** Let $H = (V, E)$ be a hypergraph. Form the disjoint union of the ground objects

$$U \;:=\; V \,\dot\cup\, E,$$

where we regard vertices $v \in V$ and hyperedges $e \in E$ as distinct types of elements of U. Define three families of subsets of U:

$$\mathcal{A} \;:=\; \{\, e \subseteq V \;:\; e \in E \,\},$$

$$\mathcal{B} := \{ E_H(v) \subseteq E \,:\, v \in V,\ |E_H(v)| \geqslant 2 \},$$
$$\mathcal{C} := \{ \{v\} \cup E_H(v) \subseteq U \,:\, v \in V \}.$$

The *total hypergraph* of H is the hypergraph

$$T(H) := \big(U,\ \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}\big).$$

Intuitively: $\mathcal{A}$ records vertex–vertex co-membership (original hyperedges), $\mathcal{B}$ records edge–edge intersection at a common vertex, and $\mathcal{C}$ records vertex–edge incidence.

Example 4.4 (Total HyperGraph — meetings and attendees). Let $H = (V, \mathcal{E})$ encode people and meetings:

$$V = \{\text{Alice}, \text{Bob}, \text{Chloe}\}, \qquad \mathcal{E} = \{E_1 = \{\text{Alice}, \text{Bob}\},\ E_2 = \{\text{Bob}, \text{Chloe}\}\}.$$

Form $U := V \dot\cup \mathcal{E} = \{\text{Alice}, \text{Bob}, \text{Chloe}, E_1, E_2\}$. The *total hypergraph* $T(H) = (U, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$ has:

$$\mathcal{A} = \{E_1, E_2\}, \quad \mathcal{B} = \big\{\ \mathcal{E}_H(\text{Bob}) = \{E_1, E_2\}\ \big\}, \quad \mathcal{C} = \big\{\ \{\text{Alice}, E_1\}, \{\text{Bob}, E_1, E_2\}, \{\text{Chloe}, E_2\}\ \big\}.$$

Here $\mathcal{A}$ records each original meeting (as a subset of people), $\mathcal{B}$ groups meetings sharing the same attendee (Bob attends both), and $\mathcal{C}$ couples each person with all meetings they attend. Thus $T(H)$ captures people, meetings, and their incidences within one hypergraph.

Theorem 4.5 (Well-definedness). *For every hypergraph $H = (V, E)$, the pair $T(H) = \big(U, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}\big)$ is a hypergraph.*

Proof. By construction $U = V \dot\cup E$ is a finite set. Each member of $\mathcal{A}$ is a nonempty subset of $V \subseteq U$ because $E \subseteq \mathrm{POWS}(V) \setminus \{\emptyset\}$. Each member of $\mathcal{B}$ is a subset of $E \subseteq U$ and is required to have size at least 2, hence nonempty. Each member of $\mathcal{C}$ has the form $\{v\} \cup E_H(v)$ and is therefore nonempty (it contains v). Thus $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \subseteq \mathrm{POWS}(U) \setminus \{\emptyset\}$, so $T(H)$ meets the definition of a hypergraph. □

Definition 4.6 (2–section of a hypergraph). For a hypergraph $K = (U, \mathcal{F})$, its *2–section* $[K]_2$ is the (simple) graph with vertex set U in which distinct $x, y \in U$ are adjacent iff there exists $F \in \mathcal{F}$ with $\{x, y\} \subseteq F$.

Theorem 4.7 (Generalization). *Let $G = (V(G), E(G))$ be a loopless simple graph and let H_G be the rank-2 hypergraph with*

$$V(H_G) = V(G), \qquad E(H_G) = \{\{u, v\} : uv \in E(G)\}.$$

Then the 2–section of the total hypergraph of H_G coincides with the total graph of G:

$$\big[T(H_G)\big]_2 = T(G).$$

Proof. Write $U = V(G) \dot\cup E(G)$. We must show that two distinct elements of U are adjacent in $[T(H_G)]_2$ iff they are adjacent in $T(G)$.

($\Rightarrow$) Suppose x, y are adjacent in $[T(H_G)]_2$. Then $\{x, y\} \subseteq F$ for some $F \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ as defined for H_G. We consider cases.

1) If $F \in \mathcal{A}$, then $F = e = \{u, v\}$ for some edge $e = uv \in E(G)$. Hence $x, y \in V(G)$ and $\{x, y\} = \{u, v\}$, so x and y are adjacent in $T(G)$ by rule (vertex–vertex).

2) If $F \in \mathcal{B}$, then $F = E_{H_G}(v) = \{ e \in E(G) : v \in e \}$ for some $v \in V(G)$ with $|E_{H_G}(v)| \geqslant 2$. Thus $x, y \in E(G)$ and both are incident with v; hence x and y share an endpoint in G and are adjacent in $T(G)$ by rule (edge–edge).

3) If $F \in \mathcal{C}$, then $F = \{v\} \cup E_{H_G}(v)$ for some $v \in V(G)$. If $x, y \in V(G)$, impossible since F contains exactly one vertex element v. If $x, y \in E(G)$, then both lie in $E_{H_G}(v)$ and we are back to case 2). Otherwise $\{x, y\} = \{v, e\}$ with $e \in E(G)$ incident to v; hence x and y are adjacent in $T(G)$ by rule (vertex–edge).

Thus in all cases, adjacency in $[T(H_G)]_2$ implies adjacency in $T(G)$.

($\Leftarrow$) Conversely, suppose x, y are adjacent in $T(G)$. Again split by rule.

1) If $x, y \in V(G)$ with $xy \in E(G)$, then $\{x, y\} \subseteq e$ for $e = \{x, y\} \in \mathcal{A}$, so x, y are adjacent in $[T(H_G)]_2$.

2) If $x, y \in E(G)$ share an endpoint v, then $\{x, y\} \subseteq E_{H_G}(v) \in \mathcal{B}$, hence adjacent in $[T(H_G)]_2$.

3) If $\{x, y\} = \{v, e\}$ with $v \in V(G)$, $e \in E(G)$, $v \in e$, then $\{x, y\} \subseteq \{v\} \cup E_{H_G}(v) \in \mathcal{C}$, hence adjacent in $[T(H_G)]_2$.

Therefore adjacency in $T(G)$ implies adjacency in $[T(H_G)]_2$.

Combining both directions yields $[\,T(H_G)\,]_2 = T(G)$. □

4.3. *Total SuperHyperGraph*

A total superhypergraph extends total hypergraphs to hierarchical supervertex structures, linking supervertices and superedges by membership, intersection, or incidence.

Definition 4.8 (Total superhypergraph). Let $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$. Define three edge-families in U_{n+1}:

$$\mathcal{A} := \{\, \iota(E) \subseteq \iota(V_n) \;:\; E \in \mathcal{E} \,\}, \qquad \mathcal{B} := \{\, \mathrm{Star}_{\mathcal{H}}(v) \subseteq \mathcal{E} \;:\; v \in V_n,\ |\mathrm{Star}_{\mathcal{H}}(v)| \geqslant 2 \,\},$$

$$\mathcal{C} := \{\, \{\iota(v)\} \cup \mathrm{Star}_{\mathcal{H}}(v) \;\subseteq U_{n+1} \;:\; v \in V_n \,\}.$$

The *total superhypergraph* of $\mathcal{H}^{(n)}$ is

$$\mathbf{T}\big(\mathcal{H}^{(n)}\big) \;:=\; \big(U_{n+1},\ \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}\big).$$

(Respectively: vertex–vertex co-membership, edge–edge "meet at a vertex," and vertex–edge incidence.)

Example 4.9 (Total SuperHyperGraph — programs, teams, and members). Start from a level-1 superhypergraph whose supervertices are *teams of people* and whose superedges are *programs made of teams*. Let the base set of individuals be $V_0 = \{a, b, c\}$. Take the team set

$$V_1 = \{T_1 = \{a, b\},\ T_2 = \{b, c\}\} \subseteq \mathrm{POWS}(V_0),$$

and the programs

$$\mathcal{E} = \{P_1 = \{T_1, T_2\},\ P_2 = \{T_2\}\} \subseteq \mathrm{POWS}(V_1).$$

Promote teams via $\iota(T) := \{T\}$ and form the level-2 universe

$$U_2 = \{\iota(T_1), \iota(T_2)\} \,\dot{\cup}\, \{P_1, P_2\}.$$

The *total superhypergraph* $\mathbf{T}(\mathcal{H}^{(1)}) = (U_2, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$ has

$$\mathcal{A} = \{ \iota(P_1) = \{\iota(T_1), \iota(T_2)\}, \ \ \iota(P_2) = \{\iota(T_2)\} \},$$

$$\mathcal{B} = \Big\{ \mathrm{Star}_{\mathcal{H}}(T_2) = \{P_1, P_2\} \Big\} \quad (\mathrm{Star}_{\mathcal{H}}(T_1) = \{P_1\} \text{ is size } 1),$$

$$\mathcal{C} = \Big\{ \{\iota(T_1), P_1\}, \{\iota(T_2), P_1, P_2\} \Big\}.$$

Reading: vertices are "singleton teams" $\iota(T_i)$ and programs P_j. The family $\mathcal{A}$ encodes which teams each program contains; $\mathcal{B}$ links programs that share a team (here T_2 participates in two programs); and $\mathcal{C}$ records, for each team, the joint presence of that team with all programs using it.

Example 4.10 (Total SuperHyperGraph — degree programs, courses, and students)**. Level and objects.** Let the base set of *students* be

$$V_0 = \{s_1, s_2, s_3, s_4\}.$$

Form level-1 supervertices as *courses* (each a subset of students)

$$V_1 = \{ C_1 = \{s_1, s_2\}, \ C_2 = \{s_2, s_3\}, \ C_3 = \{s_3, s_4\}\} \subseteq \mathrm{POWS}(V_0),$$

and define *degree programs* (superedges) as sets of courses

$$\mathcal{E} = \{ \mathrm{Pr}_1 = \{C_1, C_2\}, \ \mathrm{Pr}_2 = \{C_1, C_3\}, \ \mathrm{Pr}_3 = \{C_2\}\} \subseteq \mathrm{POWS}(V_1).$$

Thus $\mathcal{H}^{(1)} = (V_1, \mathcal{E})$ is a level-1 superhypergraph.

Stars.

$$\mathrm{Star}_{\mathcal{H}}(C_1) = \{\mathrm{Pr}_1, \mathrm{Pr}_2\}, \quad \mathrm{Star}_{\mathcal{H}}(C_2) = \{\mathrm{Pr}_1, \mathrm{Pr}_3\}, \quad \mathrm{Star}_{\mathcal{H}}(C_3) = \{\mathrm{Pr}_2\}.$$

Total superhypergraph. Promote courses by $\iota(C) = \{C\}$ and set the level-2 universe

$$U_2 = \{\iota(C_1), \iota(C_2), \iota(C_3)\} \,\dot\cup\, \{\mathrm{Pr}_1, \mathrm{Pr}_2, \mathrm{Pr}_3\}.$$

By Definition 4.8,

$$\mathcal{A} = \Big\{ \iota(\mathrm{Pr}_1) = \{\iota(C_1), \iota(C_2)\}, \ \ \iota(\mathrm{Pr}_2) = \{\iota(C_1), \iota(C_3)\}, \ \ \iota(\mathrm{Pr}_3) = \{\iota(C_2)\} \Big\},$$

$$\mathcal{B} = \Big\{ \{\mathrm{Pr}_1, \mathrm{Pr}_2\}, \{\mathrm{Pr}_1, \mathrm{Pr}_3\} \Big\} \quad (\text{from } C_1, C_2),$$

$$\mathcal{C} = \Big\{ \{\iota(C_1), \mathrm{Pr}_1, \mathrm{Pr}_2\}, \{\iota(C_2), \mathrm{Pr}_1, \mathrm{Pr}_3\}, \{\iota(C_3), \mathrm{Pr}_2\} \Big\}.$$

Hence $\mathbf{T}(\mathcal{H}^{(1)}) = (U_2, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$.

Reading. Vertices are "singleton courses" $\iota(C_i)$ and programs Pr_j. $\mathcal{A}$ encodes which courses each program contains; $\mathcal{B}$ links programs that share a course; $\mathcal{C}$ records, for each course, its incidence with all programs using it.

Example 4.11 (Total SuperHyperGraph — warehouses, shipping lanes, and goods)**. Level and objects.** Let the base set of *goods* be

$$V_0 = \{g_1, g_2, g_3\}.$$

Form level-1 supervertices as *warehouses* (each stores a subset of goods)

$$V_1 = \{\, W_1 = \{g_1, g_2\},\ W_2 = \{g_2, g_3\},\ W_3 = \{g_1\}\} \subseteq \mathrm{POWS}(V_0),$$

and define *shipping lanes* (superedges) as sets of warehouses

$$\mathcal{E} = \{\, L_1 = \{W_1, W_2\},\ L_2 = \{W_2, W_3\}\} \subseteq \mathrm{POWS}(V_1).$$

Thus $\mathcal{H}^{(1)} = (V_1, \mathcal{E})$.

Stars.

$$\mathrm{Star}_{\mathcal{H}}(W_1) = \{L_1\}, \qquad \mathrm{Star}_{\mathcal{H}}(W_2) = \{L_1, L_2\}, \qquad \mathrm{Star}_{\mathcal{H}}(W_3) = \{L_2\}.$$

Total superhypergraph. Promote warehouses by $\iota(W) = \{W\}$ and let

$$U_2 = \{\iota(W_1), \iota(W_2), \iota(W_3)\} \,\dot\cup\, \{L_1, L_2\}.$$

Then, by Definition 4.8,

$$\mathcal{A} = \Big\{\ \iota(L_1) = \{\iota(W_1), \iota(W_2)\},\ \ \iota(L_2) = \{\iota(W_2), \iota(W_3)\}\ \Big\},$$

$$\mathcal{B} = \Big\{\ \{L_1, L_2\}\ \Big\} \quad (\text{from } W_2),$$

$$\mathcal{C} = \Big\{\ \{\iota(W_1), L_1\},\ \{\iota(W_2), L_1, L_2\},\ \{\iota(W_3), L_2\}\ \Big\}.$$

Therefore $\mathbf{T}(\mathcal{H}^{(1)}) = (U_2, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$.

Vertices comprise singleton warehouses $\iota(W_i)$ and shipping lanes L_j. $\mathcal{A}$ tells which warehouses each lane connects; $\mathcal{B}$ links lanes that share a warehouse (transshipment point); and $\mathcal{C}$ captures, for each warehouse, its joint incidence with all lanes serving it.

Proposition 4.12 (Well-definedness and level shift)**.** *If $\mathcal{H}^{(n)}$ is a level-n SuperHyperGraph, then $\mathbf{T}(\mathcal{H}^{(n)})$ is a level-$(n{+}1)$ SuperHyperGraph.*

Proof. By the Definition, $U_{n+1} \subseteq \mathrm{POWS}(V_n) \subseteq \mathrm{POWS}^{n+1}(V_0)$. By the Definition, every member of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a nonempty subset of U_{n+1}. Finiteness is inherited from $\mathcal{H}^{(n)}$. Hence $\mathbf{T}(\mathcal{H}^{(n)})$ satisfies the Definition at level $n{+}1$. □

Theorem 4.13 (Reduction to the Total HyperGraph when $n = 0$)**.** *Let $H = (V, \mathcal{E})$ be a hypergraph (so level $n = 0$). Identify $\iota(V) \cong V$ via $\iota(v) \leftrightarrow v$. Then there is a canonical isomorphism*

$$\mathbf{T}(H) \;\cong\; T(H).$$

Proof. For $n = 0$, the universe $U_1 = \iota(V)\dot\cup\mathcal{E}$ of $\mathbf{T}(H)$ corresponds bijectively to $V\dot\cup\mathcal{E}$ of $T(H)$ by $\iota(v) \leftrightarrow v$ and $E \leftrightarrow E$. Under this identification:

$$\mathcal{A} \text{ of } \mathbf{T}(H) \leftrightarrow \{E : E \in \mathcal{E}\} \text{ of } T(H), \qquad \mathcal{B} \leftrightarrow \{\mathcal{E}_H(v) : v \in V,\ |\mathcal{E}_H(v)| \geqslant 2\},$$

$$\mathcal{C} \leftrightarrow \{\{v\} \cup \mathcal{E}_H(v) : v \in V\}.$$

Thus vertex and edge sets correspond termwise, giving an isomorphism $\mathbf{T}(H) \cong T(H)$. □

Theorem 4.14 (Line SuperHyperGraph as an induced part of the Total SuperHyperGraph)**.** *For every level-n SuperHyperGraph $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$, the Line SuperHyperGraph appears as the induced sub(super)hypergraph on the vertex set $\mathcal{E}$:*

$$\mathbf{L}(\mathcal{H}^{(n)}) = \mathbf{T}(\mathcal{H}^{(n)})[\,\mathcal{E}\,].$$

Proof. In $\mathbf{T}(\mathcal{H}^{(n)})$ the vertices are $\iota(V_n)\dot\cup\mathcal{E}$. Restricting to the vertex set $\mathcal{E}$ deletes all $\iota(v)$'s and keeps only hyperedges contained in $\mathcal{E}$. Among $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, the only members contained in $\mathcal{E}$ are precisely the $\mathcal{B}$-family $\{\mathrm{Star}_{\mathcal{H}}(v) : v \in V_n\}$, which is exactly the edge set of $\mathbf{L}(\mathcal{H}^{(n)})$ by the Definition. Hence $\mathbf{T}(\mathcal{H}^{(n)})[\,\mathcal{E}\,] = \mathbf{L}(\mathcal{H}^{(n)})$. □

5. Review and Result: iterated line graphs

In this section, we address the concepts of iterated line Graph, iterated line HyperGraph, and iterated line SuperHyperGraph.

5.1. iterated line graphs

An iterated line graph is formed by repeatedly applying the line graph transformation to a graph, using each result as the next input [44, 45, 46, 47].

Definition 5.1 (Iterated line graphs)**.** Define $L^0(G) := G$. For each integer $n \geqslant 1$, the *n-th iterated line graph* of G is defined recursively by

$$L^n(G) := L\big(L^{n-1}(G)\big).$$

Equivalently, many authors write $L_1(G) = L(G)$ and $L_n(G) = L\big(L_{n-1}(G)\big)$ for $n \geqslant 2$.

Example 5.2 (Iterated line graphs — subway transfers and transfer-of-transfers)**.** Consider a tiny subway map with stations $V(G) = \{A, B, C, D\}$ and tracks

$$E(G) = \{\, e_1 = AB,\ e_2 = BC,\ e_3 = CD \,\}.$$

The line graph $L(G)$ has one vertex per track and joins two tracks when they meet at a station:

$$V\big(L(G)\big) = \{e_1, e_2, e_3\}, \qquad E\big(L(G)\big) = \{\, e_1e_2,\ e_2e_3 \,\} \cong P_2.$$

Here $L(G)$ encodes *one-stop transfers*: e_1 connects to e_2 at B, and e_2 connects to e_3 at C. The iterated line graph $L^2(G) = L(L(G))$ has

$$V\big(L^2(G)\big) = E\big(L(G)\big) = \{\, e_1e_2,\ e_2e_3 \,\}, \qquad E\big(L^2(G)\big) = \{\, (e_1e_2)(e_2e_3) \,\} \cong K_2.$$

Thus $L^2(G)$ represents *transfer-of-transfer* possibilities (two consecutive transfers through the middle track e_2).

5.2. iterated line hypergraphs

An iterated line hypergraph applies the line hypergraph construction repeatedly to a hypergraph, capturing higher-order adjacency of hyperedges over iterations.

Definition 5.3 (Iterated line hypergraphs)**.** Define $L^0(H) := H$ and, for each $n \geqslant 1$, set

$$L^n(H) \;:=\; L\big(L^{n-1}(H)\big).$$

Example 5.4 (Iterated line hypergraphs — meetings sharing attendees, then "overlaps of overlaps")**.** Let a schedule be a hypergraph $H = (V, \mathcal{E})$ with people

$$V = \{\text{Alice}, \text{Bob}, \text{Chloe}, \text{Dan}\}$$

and meetings (hyperedges)

$$\mathcal{E} = \{\, E_1 = \{\text{Alice}, \text{Bob}\},\; E_2 = \{\text{Bob}, \text{Chloe}\},\; E_3 = \{\text{Chloe}, \text{Dan}\}\,\}.$$

The line hypergraph $L(H)$ has vertices $\{E_1, E_2, E_3\}$ and, for each person v, one hyperedge collecting all meetings that v attends:

$$\text{Star}_H(\text{Alice}) = \{E_1\}, \quad \text{Star}_H(\text{Bob}) = \{E_1, E_2\}, \quad \text{Star}_H(\text{Chloe}) = \{E_2, E_3\}, \quad \text{Star}_H(\text{Dan}) = \{E_3\}.$$

Hence

$$L(H) = \Big(\, \{E_1, E_2, E_3\},\; \big\{\{E_1\}, \{E_1, E_2\}, \{E_2, E_3\}, \{E_3\}\big\} \,\Big).$$

Iterating once more, $L^2(H) = L(L(H))$ has vertex set

$$V\big(L^2(H)\big) = \big\{\, \{E_1\},\, \{E_1, E_2\},\, \{E_2, E_3\},\, \{E_3\} \,\big\},$$

and its hyperedges correspond to each original meeting E_i: gather all vertices of $L^2(H)$ that contain E_i. Concretely,

$$\text{for } E_1 : \{\{E_1\}, \{E_1, E_2\}\}, \qquad \text{for } E_2 : \{\{E_1, E_2\}, \{E_2, E_3\}\}, \qquad \text{for } E_3 : \{\{E_2, E_3\}, \{E_3\}\}.$$

Thus $L^2(H)$ captures *overlaps of overlaps*: two "meeting-groups" are linked when they share a common meeting.

Theorem 5.5 (Trivial recovery of the line hypergraph)**.** *For every hypergraph H, the first iterate coincides with the line hypergraph:*

$$L^1(H) = L(H).$$

Proof. By the recursive definition $L^1(H) := L(L^0(H)) = L(H)$. □

Lemma 5.6 (2–section after two line-steps)**.** *For any hypergraph $K = (U, \mathcal{F})$,*

$$\big[L^2(K)\big]_2 \;\cong\; [K]_2.$$

Proof. By definition,

$$L(K) = \big(\,\mathcal{F},\ \{\mathrm{Star}_K(u) : u \in U,\ \mathrm{Star}_K(u) \neq \emptyset\}\,\big).$$

Hence

$$L^2(K) \;=\; L\big(L(K)\big)$$

has vertex set $\{\mathrm{Star}_K(u) : u \in U,\ \mathrm{Star}_K(u) \neq \emptyset\}$. Two distinct vertices $\mathrm{Star}_K(u)$ and $\mathrm{Star}_K(v)$ of $L^2(K)$ are adjacent in the 2–section $[L^2(K)]_2$ iff there exists a hyperedge of $L(K)$ containing both of them, i.e. iff there exists $F \in \mathcal{F}$ such that

$$\mathrm{Star}_K(u) \ni F \quad \text{and} \quad \mathrm{Star}_K(v) \ni F,$$

which is equivalent to $u \in F$ and $v \in F$. This, in turn, is exactly the adjacency condition for u and v in $[K]_2$. The map $\mathrm{Star}_K(u) \mapsto u$ is therefore a graph isomorphism $[L^2(K)]_2 \to [K]_2$. □

Corollary 5.7 (All iterates, seen through 2–sections)**.** *For any hypergraph K and any integer $m \geqslant 0$,*

$$\big[L^{2m}(K)\big]_2 \;\cong\; [K]_2, \qquad \big[L^{2m+1}(K)\big]_2 \;\cong\; \big[L(K)\big]_2.$$

Proof. Apply Lemma 5.6 repeatedly:

$$[L^{2(m+1)}(K)]_2 \cong [L^{2m}(K)]_2 \cong \cdots \cong [K]_2$$

, and similarly

$$[L^{2m+3}(K)]_2 \cong [L^{2m+1}(K)]_2 \cong [L(K)]_2$$

. □

Definition 5.8 (Graphs as 2–uniform hypergraphs)**.** For a (finite, loopless, simple) graph $G = (V, E)$, write $H_G := (V, \mathcal{E})$ with $\mathcal{E} = \{\{u, v\} : uv \in E\}$. Then $[H_G]_2 = G$.

Lemma 5.9 (One step: line graph via 2–section)**.** *For every graph G,*

$$\big[L(H_G)\big]_2 \;=\; L(G).$$

Proof. Vertices of $L(H_G)$ are edges of G. Two such vertices (i.e. two edges of G) are adjacent in the 2–section iff they belong to a common star $\mathrm{Star}_{H_G}(v)$, that is, iff they are both incident with v in G. This is exactly the definition of adjacency in $L(G)$. □

Theorem 5.10 (Iterated line hypergraphs generalize line graphs and their iteration (up to the natural 2–section))**.** *Let G be a finite simple graph and H_G its associated 2–uniform hypergraph. For every integer $m \geqslant 0$,*

$$\big[L^{2m}(H_G)\big]_2 \;=\; G, \qquad \big[L^{2m+1}(H_G)\big]_2 \;=\; L(G).$$

Consequently:

- *taking $m = 0$ in the second equality recovers the classical line graph $L(G)$ from the (first) line hypergraph of H_G;*

- *taking $n = 1$ in Theorem 5.5 recovers the usual line hypergraph of an arbitrary hypergraph;*
- *the iterated line hypergraph sequence $\{L^n(H_G)\}_{n \geqslant 0}$ forms a* hypergraph lift *of the classical iterated line graph process, whose 2–sections alternate between* G *and* $L(G)$.

Proof. By Corollary 5.7 applied to $K = H_G$, $[L^{2m}(H_G)]_2 \cong [H_G]_2 = G$. Similarly, $[L^{2m+1}(H_G)]_2 \cong [L(H_G)]_2$, and the Lemma gives $[L(H_G)]_2 = L(G)$. □

5.3. iterated line superhypergraphs

An iterated line superhypergraph repeatedly applies the line superhypergraph transformation to a superhypergraph, modeling evolving hierarchical incidence patterns across multiple levels.

Definition 5.11 (Iterated line superhypergraphs)**.** For an initial level-n superhypergraph $\mathcal{H}^{(n)}$, define

$$\mathbf{L}^0(\mathcal{H}^{(n)}) := \mathcal{H}^{(n)}, \qquad \mathbf{L}^{t+1}(\mathcal{H}^{(n)}) := \mathbf{L}\big(\mathbf{L}^t(\mathcal{H}^{(n)})\big) \quad (t \geqslant 0).$$

By Proposition, $\mathbf{L}^t(\mathcal{H}^{(n)})$ is level $n+t$.

Example 5.12 (Iterated line superhypergraphs — programs (by teams) and shared-team structure across two iterations)**.** Start from a level-1 superhypergraph where supervertices are teams of people and superedges are programs (collections of teams). Let the individual set be $V_0 = \{a, b, c, d\}$. Take teams

$$V_1 = \{\, T_1 = \{a, b\},\ T_2 = \{b, c\},\ T_3 = \{c, d\}\,\} \subseteq \mathrm{POWS}(V_0),$$

and programs (superedges)

$$\mathcal{E} = \{\, P_1 = \{T_1, T_2\},\ P_2 = \{T_2, T_3\}\,\} \subseteq \mathrm{POWS}(V_1).$$

The line superhypergraph $\mathbf{L}^1(\mathcal{H})$ has vertex set $\{P_1, P_2\}$ and, for each team T, a (super)hyperedge collecting the programs that include T:

$$\mathrm{Star}_{\mathcal{H}}(T_1) = \{P_1\}, \quad \mathrm{Star}_{\mathcal{H}}(T_2) = \{P_1, P_2\}, \quad \mathrm{Star}_{\mathcal{H}}(T_3) = \{P_2\}.$$

Hence

$$\mathbf{L}^1(\mathcal{H}) = \Big(\ \{P_1, P_2\},\ \{\{P_1\}, \{P_1, P_2\}, \{P_2\}\}\ \Big).$$

Iterating again, $\mathbf{L}^2(\mathcal{H}) = \mathbf{L}\big(\mathbf{L}^1(\mathcal{H})\big)$ has vertices

$$V\big(\mathbf{L}^2(\mathcal{H})\big) = \big\{\ S_1 = \{P_1\},\ S_{12} = \{P_1, P_2\},\ S_2 = \{P_2\}\ \big\},$$

and for each program P_j we add a (super)hyperedge consisting of all S-vertices that contain P_j:

$$\text{for } P_1 : \{S_1, S_{12}\}, \qquad \text{for } P_2 : \{S_{12}, S_2\}.$$

Interpretation: $\mathbf{L}^1$ tells which programs share a team; $\mathbf{L}^2$ then records how those *program-groups* themselves are linked by sharing a common program.

Example 5.13 (An iterated line 2–superhypergraph (two iterations)). **Step 0: Build a level-2 superhypergraph.** Let the base set of individuals be $V_0 = \{a, b, c, d\}$. Form level-1 supervertices (teams)

$$V_1 = \{ T_1 = \{a, b\},\ T_2 = \{b, c\},\ T_3 = \{c, d\} \} \subseteq \mathrm{POWS}(V_0),$$

and level-2 supervertices (collections of teams)

$$V_2 = \{ X_1 = \{T_1, T_2\},\ \ X_2 = \{T_2, T_3\},\ \ X_3 = \{T_1\} \} \subseteq \mathrm{POWS}(V_1).$$

Define the level-2 superedges

$$\mathcal{E} = \{ P_1 = \{X_1, X_2\},\ \ P_2 = \{X_2, X_3\} \} \subseteq \mathrm{POWS}(V_2),$$

and set $\mathcal{H}^{(2)} = (V_2, \mathcal{E})$.

Step 1: First line step $\mathbf{L}^1(\mathcal{H}^{(2)}) = \mathbf{L}(\mathcal{H}^{(2)})$. By definition, the vertex set is the superedge set:

$$V\big(\mathbf{L}^1\big) = \mathcal{E} = \{P_1, P_2\}.$$

For each $v \in V_2$, form the star $\mathrm{Star}_{\mathcal{H}}(v) = \{E \in \mathcal{E} : v \in E\}$:

$$\mathrm{Star}_{\mathcal{H}}(X_1) = \{P_1\}, \qquad \mathrm{Star}_{\mathcal{H}}(X_2) = \{P_1, P_2\}, \qquad \mathrm{Star}_{\mathcal{H}}(X_3) = \{P_2\}.$$

Thus the (super)edge set of $\mathbf{L}^1$ is

$$\mathcal{E}^{(1)} = \{ \{P_1\},\ \{P_1, P_2\},\ \{P_2\} \},$$

and

$$\mathbf{L}^1(\mathcal{H}^{(2)}) = \Big(\{P_1, P_2\},\ \{\{P_1\}, \{P_1, P_2\}, \{P_2\}\} \Big).$$

Step 2: Second line step $\mathbf{L}^2(\mathcal{H}^{(2)}) = \mathbf{L}\big(\mathbf{L}^1(\mathcal{H}^{(2)})\big)$. Now the vertices are the superedges of $\mathbf{L}^1$:

$$V\big(\mathbf{L}^2\big) = \{ S_1 = \{P_1\},\ S_{12} = \{P_1, P_2\},\ S_2 = \{P_2\} \}.$$

For each old vertex $u \in V(\mathbf{L}^1) = \{P_1, P_2\}$, compute

$$\mathrm{Star}_{\mathbf{L}^1}(P_1) = \{ S_1, S_{12} \}, \qquad \mathrm{Star}_{\mathbf{L}^1}(P_2) = \{ S_{12}, S_2 \}.$$

Hence the (super)edge set of $\mathbf{L}^2$ is

$$\mathcal{E}^{(2)} = \{ \{S_1, S_{12}\},\ \{S_{12}, S_2\} \},$$

and

$$\mathbf{L}^2(\mathcal{H}^{(2)}) = \Big(\{S_1, S_{12}, S_2\},\ \{\{S_1, S_{12}\}, \{S_{12}, S_2\}\} \Big).$$

Interpretation. $\mathbf{L}^1$ records which programs P_1, P_2 share a common level-2 supervertex (here X_2). $\mathbf{L}^2$ then links the "program-groups" S_1, S_{12}, S_2 when they share a common program, yielding a simple chain $\{S_1, S_{12}\}, \{S_{12}, S_2\}$ at the next level.

Theorem 5.14 (First iterate recovers the line superhypergraph). *For every level-n superhypergraph $\mathcal{H}^{(n)}$, $\mathbf{L}^1(\mathcal{H}^{(n)}) = \mathbf{L}(\mathcal{H}^{(n)})$ as in the Definition.*

Proof. Directly from Definition 5.11 with $t = 0$. □

Theorem 5.15 (Hypergraph case: one step equals the incidence line hypergraph). *Let* $H = (V, \mathcal{E})$ *be an ordinary hypergraph* ($n = 0$)*. Then*

$$\mathbf{L}(H) \;=\; \big(\,\mathcal{E},\, \{\, \{E \in \mathcal{E} : v \in E\,\} : v \in V,\ \mathrm{Star}_H(v) \neq \emptyset\,\}\,\big),$$

i.e., the standard incidence-based line hypergraph. Consequently, $\mathbf{L}^t(H)$ *is the* iterated line hypergraph *sequence.*

Proof. This is exactly the Definition with $n = 0$. □

Lemma 5.16 (2–section parity identity). *For any (super)hypergraph* $K = (U, \mathcal{F})$,

$$\big[\mathbf{L}^2(K)\big]_2 \;\cong\; [K]_2.$$

Proof. Write $\mathbf{L}(K) = (\mathcal{F}, \{\mathrm{Star}_K(u) : u \in U,\ \mathrm{Star}_K(u) \neq \emptyset\})$. Then $\mathbf{L}^2(K) = \mathbf{L}(\mathbf{L}(K))$ has vertex set $\{\mathrm{Star}_K(u) : u \in U,\ \mathrm{Star}_K(u) \neq \emptyset\}$. Two distinct such vertices $\mathrm{Star}_K(u), \mathrm{Star}_K(v)$ are adjacent in the 2–section iff they lie together in some hyperedge of $\mathbf{L}(K)$, i.e., iff there exists $F \in \mathcal{F}$ with $F \in \mathrm{Star}_K(u) \cap \mathrm{Star}_K(v)$, equivalently $u, v \in F$. This is exactly the adjacency condition in $[K]_2$. The map $\mathrm{Star}_K(u) \mapsto u$ is the desired graph isomorphism. □

Corollary 5.17 (Alternation for all iterates). *For any* K *and* $m \geqslant 0$,

$$\big[\mathbf{L}^{2m}(K)\big]_2 \;\cong\; [K]_2, \qquad \big[\mathbf{L}^{2m+1}(K)\big]_2 \;\cong\; \big[\mathbf{L}(K)\big]_2.$$

Theorem 5.18 (Graphs as 2–uniform hypergraphs: lift of iterated line graphs). *Let* $G = (V, E)$ *be a finite simple loopless graph and encode it as a 2–uniform hypergraph* $H_G = (V, \mathcal{E})$ *with* $\mathcal{E} = \{\{u, v\} : uv \in E\}$*. Then, for all* $m \geqslant 0$,

$$\big[\mathbf{L}^{2m}(H_G)\big]_2 \;=\; G, \qquad \big[\mathbf{L}^{2m+1}(H_G)\big]_2 \;=\; L(G).$$

In particular, $\mathbf{L}^1(H_G)$ *projects (via 2–section) to the classical line graph, and the superhypergraph chain* $\{\mathbf{L}^t(H_G)\}_{t \geqslant 0}$ *is a* superhypergraph lift *of the iterated line–graph process, alternating between* G *and* $L(G)$ *under* $[\,\cdot\,]_2$.

Proof. By Corollary 5.17 with $K = H_G$, $[\mathbf{L}^{2m}(H_G)]_2 \cong [H_G]_2 = G$. Also $[\mathbf{L}^{2m+1}(H_G)]_2 \cong [\mathbf{L}(H_G)]_2$. But in $\mathbf{L}(H_G)$, vertices are edges of G, and two such vertices are adjacent in the 2–section iff they belong to a common star $\mathrm{Star}_{H_G}(v)$, i.e., iff the corresponding edges of G share an endpoint v. Thus $[\mathbf{L}(H_G)]_2 = L(G)$. □

6. Review and Result: iterated total graphs

In this section, we address the concepts of iterated total Graph, iterated total HyperGraph, and iterated total SuperHyperGraph.

6.1. iterated total graphs

An iterated total graph repeatedly applies the total graph operation to a graph, incorporating vertices, edges, and all incidence relationships at each stage [48, 49, 50].

Definition 6.1 (Iterated total graphs)**.** Define $T^0(G) := G$, and for each integer $k \geqslant 1$ set

$$T^k(G) := T\big(T^{k-1}(G)\big).$$

(Thus $T^1(G) = T(G)$, $T^2(G) = T(T(G))$, etc.) This notation is also used in the literature.

Example 6.2 (Iterated total graphs — roads, then "roads-of-roads")**.** Let G be a tiny road map with intersections $V(G) = \{A, B, C\}$ and roads

$$E(G) = \{AB,\ BC\}.$$

The total graph $T(G)$ has

$$V\big(T(G)\big) = \{A, B, C,\ AB,\ BC\},$$

and edges encoding: (i) intersection–intersection $\{A, B\}, \{B, C\}$, (ii) road–road $\{AB, BC\}$ (they share B), and (iii) intersection–road $\{A, AB\}, \{B, AB\}, \{B, BC\}, \{C, BC\}$. Thus

$$E\big(T(G)\big) = \{\{A, B\}, \{B, C\}, \{AB, BC\}, \{A, AB\}, \{B, AB\}, \{B, BC\}, \{C, BC\}\}.$$

Iterating once more, $T^2(G) = T(T(G))$ has vertex set

$$V\big(T^2(G)\big) = V\big(T(G)\big) \,\dot{\cup}\, E\big(T(G)\big),$$

i.e., the five previous vertices together with the seven "edge-vertices"

$$\{A, B\}, \{B, C\}, \{AB, BC\}, \{A, AB\}, \{B, AB\}, \{B, BC\}, \{C, BC\}$$

. Now $T^2(G)$ records, in one graph, *places and roads* (from $T(G)$) *and* how any two of those items co-occur inside a single rule of $T(G)$—a compact way to encode two layers of incidence.

Lemma 6.3 (Line sits inside total)**.** *For every graph X, the line graph $L(X)$ is an induced subgraph of the total graph $T(X)$.*

Proof. Map each vertex $e \in V(L(X)) = E(X)$ to the vertex $e \in V(T(X))$ (the same edge, now viewed as a vertex of $T(X)$). Two vertices e, f are adjacent in $L(X)$ iff e and f share an endpoint in X, which is precisely the edge–edge adjacency rule in $T(X)$. No additional edges among $\{e, f\}$ appear in $T(X)$ beyond this rule, so the embedding is induced. □

Lemma 6.4 (Total preserves induced subgraphs)**.** *If H is an induced subgraph of G, then $T(H)$ is an induced subgraph of $T(G)$.*

Proof. Vertices of $T(H)$ are $V(H) \cup E(H) \subseteq V(G) \cup E(G) = V(T(G))$. Adjacency in $T(H)$ is determined by adjacency/incidence inside H, and because H is induced in G, the same adjacencies hold when viewed in G. Thus no extra edges among vertices from $V(H) \cup E(H)$ arise in $T(G)$; the inclusion is induced. □

Theorem 6.5 (Iterated total graphs generalize iterated line graphs). *For every finite simple graph* G *and every* $k \geqslant 1$*, the iterated line graph* $L^k(G)$ *is an induced subgraph of the iterated total graph* $T^k(G)$*. Consequently, the sequence* $\{T^k(G)\}_{k\geqslant 0}$ generalizes $\{L^k(G)\}_{k\geqslant 0}$ *in the sense that each* $L^k(G)$ *occurs canonically inside* $T^k(G)$*.*

Proof. We proceed by induction on k.

Base case $k = 1$. By Lemma 6.3, $L(G)$ is an induced subgraph of $T(G)$.

Inductive step. Assume $L^{k-1}(G)$ is an induced subgraph of $T^{k-1}(G)$. Apply Lemma 6.4 with $H = L^{k-1}(G)$ and $G' = T^{k-1}(G)$: then $T\big(L^{k-1}(G)\big)$ is an induced subgraph of $T\big(T^{k-1}(G)\big) = T^k(G)$. Finally, apply Lemma 6.3 to $X = L^{k-1}(G)$ to see that

$$L^k(G) = L\big(L^{k-1}(G)\big) \text{ is an induced subgraph of } T\big(L^{k-1}(G)\big).$$

By transitivity of "is an induced subgraph of," we conclude $L^k(G) \subseteq_{\text{ind}} T^k(G)$. □

6.2. iterated total hypergraphs

An iterated total hypergraph is obtained by successively applying the total hypergraph transformation, encoding vertices, hyperedges, and their incidences through multiple iteration layers.

Definition 6.6 (Iterated total hypergraphs). Let $T^0(H) := H$ and, for $t \geqslant 1$, define recursively

$$T^t(H) := T\big(T^{t-1}(H)\big).$$

Example 6.7 (Iterated total HyperGraph — meetings, people, and two layers of incidence). Let $H = (V, \mathcal{E})$ encode meetings:

$$V = \{\text{Alice}, \text{Bob}, \text{Chloe}\}, \qquad \mathcal{E} = \{\, E_1 = \{\text{Alice}, \text{Bob}\},\ E_2 = \{\text{Bob}, \text{Chloe}\}\,\}.$$

The total hypergraph $T(H) = (U, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$ has

$$U = \{\text{Alice}, \text{Bob}, \text{Chloe}, E_1, E_2\},$$

$$\mathcal{A} = \{E_1, E_2\}, \quad \mathcal{B} = \big\{\, \mathcal{E}_H(\text{Bob}) = \{E_1, E_2\} \,\big\}, \quad \mathcal{C} = \big\{\, \{\text{Alice}, E_1\}, \{\text{Bob}, E_1, E_2\}, \{\text{Chloe}, E_2\} \,\big\}.$$

Iterating, $T^2(H) = T(T(H))$ has vertex set $U \,\dot{\cup}\, (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$; e.g., the six previous hyperedges now also appear as vertices:

$$E_1,\ E_2,\ \{E_1, E_2\},\ \{\text{Alice}, E_1\},\ \{\text{Bob}, E_1, E_2\},\ \{\text{Chloe}, E_2\}.$$

Some illustrative hyperedges of $T^2(H)$ are:

$$\underbrace{\{E_1\}}_{\text{from } \mathcal{A}}, \qquad \underbrace{\big\{\{E_1, E_2\}\big\}}_{\text{from } \mathcal{B} \text{ at Bob}}, \qquad \underbrace{\big\{\{\text{Bob}\}, \{\text{Bob}, E_1, E_2\}\big\}}_{\text{from } \mathcal{C} \text{ at Bob}}.$$

Interpretation: $T(H)$ mixes *people*, *meetings*, and *who attends what*; $T^2(H)$ adds a second layer that also relates these "attendance patterns" to one another inside a single object.

Theorem 6.8 (Well-definedness for all iterates). *For every finite hypergraph* H *and every* $t \geqslant 1$, $T^t(H)$ *is a finite hypergraph.*

Proof. From the Definition, U is finite and each member of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a nonempty subset of U. Thus $T(H)$ is a finite hypergraph. Inductively $T^t(H) = T\big(T^{t-1}(H)\big)$ is finite for all t. □

Theorem 6.9. *For every hypergraph* H*,* $T^1(H) = T(H)$.

Proof. Immediate from Definition 6.6. □

Lemma 6.10 (Line sits inside total, and T preserves induced subhypergraphs)**.** *For every hypergraph* H*,* $L(H)$ *is the induced subhypergraph* $T(H)[\,\mathcal{E}\,]$ *on the vertex set* $\mathcal{E}$*. Moreover, if* K *is an induced subhypergraph of* H*, then* $T(K)$ *is an induced subhypergraph of* $T(H)$*.*

Proof. In $T(H)$, the vertices in $\mathcal{E}$ support exactly the hyperedge family $\mathcal{B} = \{\mathcal{E}_H(v)\}$, which equals the edge set of $L(H)$; hence $L(H) = T(H)[\mathcal{E}]$. For the second claim, let $K = H[W]$ with $W \subseteq V(H)$. Then $V\big(T(K)\big) = W \,\dot\cup\, \mathcal{E}_K$ is a subset of $V\big(T(H)\big) = V(H) \,\dot\cup\, \mathcal{E}_H$. Each hyperedge of $T(K)$ is obtained from the corresponding one in $T(H)$ by intersecting with $W \,\dot\cup\, \mathcal{E}_K$:

$$\mathcal{A}_K = \mathcal{A}_H \cap \mathrm{POWS}(W),$$

$$\mathcal{B}_K = \{\, \mathcal{E}_H(v) \cap \mathcal{E}_K : v \in W \,\},$$

$$\mathcal{C}_K = \{\, (\{v\} \cup \mathcal{E}_H(v)) \cap (W \cup \mathcal{E}_K\,) : v \in W \,\}.$$

Thus $T(K)$ is induced in $T(H)$. □

Theorem 6.11 (Iterated line hypergraphs inside iterated total hypergraphs)**.** *For every hypergraph* H *and every* $t \geqslant 1$*,* $L^t(H)$ *is an induced subhypergraph of* $T^t(H)$*.*

Proof. $t = 1$ follows from Lemma 6.10. Assume $L^{t-1}(H) \subseteq_{\mathrm{ind}} T^{t-1}(H)$. By Lemma 6.10, $L\big(L^{t-1}(H)\big) \subseteq_{\mathrm{ind}} T\big(L^{t-1}(H)\big) \subseteq_{\mathrm{ind}} T\big(T^{t-1}(H)\big) = T^t(H)$. Since $L^t(H) = L\big(L^{t-1}(H)\big)$, the claim follows. □

Lemma 6.12 (One step: total graph as 2–section of total hypergraph)**.** $[\,T(H_G)\,]_2 = T(G)$.

Proof. Vertices of $[T(H_G)]_2$ are $V \,\dot\cup\, E$. Two original vertices are adjacent in $[T(H_G)]_2$ iff they lie together in some $e \in \mathcal{A}$, i.e. iff they are adjacent in G (vertex–vertex rule). Two edge-vertices $e, f \in E$ are adjacent in $[T(H_G)]_2$ iff they lie together in some $\mathcal{E}_{H_G}(v) \in \mathcal{B}$, i.e. iff e, f share an endpoint v in G (edge–edge rule). Finally $v \in V$ and $e \in E$ are adjacent in $[T(H_G)]_2$ iff $\{v, e\} \subseteq \{v\} \cup \mathcal{E}_{H_G}(v) \in \mathcal{C}$, i.e. iff $v \in e$ (vertex–edge rule). These are exactly the adjacency rules of $T(G)$. □

Lemma 6.13 (Edge–hyperedge correspondence inside the total tower)**.** *For each* $s \geqslant 0$ *there exists a natural bijection*

$$\Phi_s : \; E\big(T^s(G)\big) \;\longleftrightarrow\; \mathcal{E}\big(T^s(H_G)\big)$$

such that, for any distinct $x, y \in V\big(T^s(G)\big)$*,*

$$\{x, y\} \in E\big(T^s(G)\big) \quad \Longleftrightarrow \quad \Phi_s(\{x, y\}) \text{ is a hyperedge of } T^s(H_G) \text{ containing } x \text{ and } y.$$

Proof. By induction on s. For $s = 0$, $E(T^0(G)) = E(G)$ and $\mathcal{E}(T^0(H_G)) = \mathcal{E}(H_G)$, with the identification $\{u, v\} \leftrightarrow \{u, v\}$. Assume Φ_s exists. In $T^{s+1}(G) = T(T^s(G))$, each edge arises uniquely from one of the three total-graph rules on $T^s(G)$: (vertex–vertex via $\mathcal{A}$, edge–edge via $\mathcal{B}$, vertex–edge via $\mathcal{C}$). In $T^{s+1}(H_G) = T(T^s(H_G))$, the corresponding witnessing hyperedge is exactly the member of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ that contains the same pair. Uniqueness holds because a pair of vertices of $T^s(H_G)$ belongs to at most one of the three disjoint types: two "vertex-type", two "edge-type", or one of each. Define Φ_{s+1} by mapping each edge of $T^{s+1}(G)$ to this unique witnessing hyperedge. This extends the induction and preserves containment of endpoints. □

Theorem 6.14 (Iterated total hypergraphs lift iterated total graphs)**.** *For every finite simple graph* G *and every* $t \geqslant 1$,

$$\left[T^t(H_G)\right]_2 = T^t(G).$$

Proof. By induction on t. For $t = 1$, the Lemma holds. Assume $[T^{t-1}(H_G)]_2 = T^{t-1}(G)$. Vertices of $[T^t(H_G)]_2$ are

$$V\big(T^{t-1}(G)\big) \,\dot\cup\, \mathcal{E}\big(T^{t-1}(H_G)\big)$$

, while vertices of $T^t(G)$ are

$$V\big(T^{t-1}(G)\big) \,\dot\cup\, E\big(T^{t-1}(G)\big)$$

. By Lemma 6.13 (with $s = t - 1$), the bijection Φ_{t-1} identifies

$$\mathcal{E}\big(T^{t-1}(H_G)\big)$$

with $E\big(T^{t-1}(G)\big)$ so that a pair of vertices is adjacent in $[T^t(H_G)]_2$ iff it is adjacent in $T^t(G)$ (the three total rules are matched one-to-one by Φ_{t-1}). Hence the graphs are equal. □

6.3. *iterated total superhypergraphs*

An iterated total superhypergraph results from repeatedly performing the total superhypergraph construction, capturing hierarchical vertices, superedges, and complex incidence structures across increasing levels.

Definition 6.15 (Iterated total superhypergraphs)**.** For $t \geqslant 0$ define

$$\mathbf{T}^0\big(\mathcal{H}^{(n)}\big) := \mathcal{H}^{(n)}, \qquad \mathbf{T}^{t+1}\big(\mathcal{H}^{(n)}\big) := \mathbf{T}\Big(\mathbf{T}^t\big(\mathcal{H}^{(n)}\big)\Big).$$

So $\mathbf{T}^t(\mathcal{H}^{(n)})$ is level $n+t$.

Example 6.16 (Iterated total SuperHyperGraph — programs (of teams) with two incidence layers)**.** Start with a level-1 superhypergraph: supervertices are teams of people, superedges are programs (sets of teams). Let $V_0 = \{a, b, c\}$ (people), teams

$$V_1 = \{\, T_1 = \{a, b\},\ T_2 = \{b, c\}\} \subseteq \mathrm{POWS}(V_0),$$

and programs

$$\mathcal{E} = \{\, P_1 = \{T_1, T_2\},\ P_2 = \{T_2\}\} \subseteq \mathrm{POWS}(V_1).$$

The total superhypergraph $\mathbf{T}(\mathcal{H}^{(1)}) = (U_2, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$ has

$$U_2 = \{\{T_1\}, \{T_2\},\ P_1,\ P_2\},$$

$$\mathcal{A} = \{\ \{T_1\}, \{T_2\} \text{ (as the image of } P_1),\ \ \{T_2\} \text{ (as the image of } P_2)\ \},$$

$$\mathcal{B} = \{\ \{P_1, P_2\} \text{ (the programs meeting at team } T_2)\ \},$$

$$\mathcal{C} = \{\ \{\{T_1\}, P_1\},\ \{\{T_2\}, P_1, P_2\}\ \}.$$

Iterating once more, $\mathbf{T}^2(\mathcal{H}^{(1)}) = \mathbf{T}(\mathbf{T}(\mathcal{H}^{(1)}))$ has vertex set $U_2 \dot\cup (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$; for instance, $\{P_1, P_2\}$ and $\{\{T_2\}, P_1, P_2\}$ become vertices. A sample hyperedge of $\mathbf{T}^2$ arising from the "incidence at $\{T_2\}$" rule is

$$\{\ \{\{T_2\}\},\ \{\{T_2\}, P_1, P_2\}\ \},$$

which ties the singleton vertex $\{T_2\}$ to the pattern "$\{T_2\}$ participates in P_1 and P_2". Thus $\mathbf{T}$ captures programs/teams incidence in one level, while $\mathbf{T}^2$ additionally relates those incidence *patterns* to each other.

Example 6.17 (An iterated total 3–superhypergraph (two iterations))**. Step 0: A level-3 superhypergraph.** Let the base set be $V_0 = \{a, b\}$. Choose level-1 supervertices (subsets of V_0)

$$V_1 = \{\ A = \{a\},\ B = \{b\}\ \},$$

level-2 supervertices (subsets of V_1)

$$V_2 = \{\ X_1 = \{A\},\ X_2 = \{A, B\}\ \},$$

and level-3 supervertices (subsets of V_2)

$$V_3 = \{\ Y_1 = \{X_1, X_2\},\ \ Y_2 = \{X_2\}\ \} \subseteq \mathrm{POWS}^3(V_0).$$

Define level-3 superedges as subsets of V_3:

$$\mathcal{E} = \{\ E_1 = \{Y_1, Y_2\},\ \ E_2 = \{Y_2\}\ \} \subseteq \mathrm{POWS}(V_3) \setminus \{\emptyset\}.$$

Then $\mathcal{H}^{(3)} = (V_3, \mathcal{E})$ is a level-3 SuperHyperGraph. For stars (incidence neighborhoods) in $\mathcal{H}^{(3)}$:

$$\mathrm{Star}_{\mathcal{H}}(Y_1) = \{E_1\}, \qquad \mathrm{Star}_{\mathcal{H}}(Y_2) = \{E_1, E_2\}.$$

Step 1: First total step $\mathbf{T}^1(\mathcal{H}^{(3)}) = \mathbf{T}(\mathcal{H}^{(3)})$. Embed V_3 into $\mathrm{POWS}(V_3)$ via $\iota(v) := \{v\}$ and set

$$U_4 \ := \ \iota(V_3) \dot\cup \mathcal{E} \ = \ \{\iota(Y_1), \iota(Y_2)\} \dot\cup \{E_1, E_2\}.$$

The three total families (all nonempty subsets of U_4) are:

$$\mathcal{A} = \{\ \iota(E_1) = \{\iota(Y_1), \iota(Y_2)\},\ \ \iota(E_2) = \{\iota(Y_2)\}\ \},$$

$$\mathcal{B} = \{\ \mathrm{Star}_{\mathcal{H}}(Y_2) = \{E_1, E_2\}\ \} \quad (\mathrm{Star}_{\mathcal{H}}(Y_1) = \{E_1\} \text{ has size } 1),$$

$$\mathcal{C} = \{\ \{\iota(Y_1), E_1\},\ \{\iota(Y_2), E_1, E_2\}\ \}.$$

Thus

$$\mathbf{T}(\mathcal{H}^{(3)}) = \Big(U_4, \underbrace{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}}_{=: \mathcal{F}_1} \Big),$$

which is a level-4 SuperHyperGraph.

Step 2: Second total step $\mathbf{T}^2(\mathcal{H}^{(3)}) = \mathbf{T}\big(\mathbf{T}(\mathcal{H}^{(3)})\big)$. Label the hyperedges of $\mathcal{F}_1$ for clarity:

$$A_1 := \{\iota(Y_1), \iota(Y_2)\}, \quad A_2 := \{\iota(Y_2)\}, \quad B_1 := \{E_1, E_2\}, \quad C_1 := \{\iota(Y_1), E_1\}, \quad C_2 := \{\iota(Y_2), E_1, E_2\}.$$

The level-5 vertex set is the disjoint union

$$U_5 := \iota(U_4) \,\dot\cup\, \mathcal{F}_1 = \{\iota(\iota(Y_1)), \iota(\iota(Y_2)), \iota(E_1), \iota(E_2)\} \,\dot\cup\, \{A_1, A_2, B_1, C_1, C_2\}.$$

The three total families at this stage are obtained exactly as before:

$$\mathcal{A}^{(2)} = \big\{ \iota(A_1), \ \iota(A_2), \ \iota(B_1), \ \iota(C_1), \ \iota(C_2) \big\},$$

$$\mathcal{B}^{(2)} = \big\{ \mathrm{Star}_{\mathbf{T}}(\iota(Y_1)) = \{A_1, C_1\},$$
$$\mathrm{Star}_{\mathbf{T}}(\iota(Y_2)) = \{A_1, A_2, C_2\},$$
$$\mathrm{Star}_{\mathbf{T}}(E_1) = \{B_1, C_1, C_2\}, \ \ \mathrm{Star}_{\mathbf{T}}(E_2) = \{B_1, C_2\} \big\},$$

$$\mathcal{C}^{(2)} = \big\{ \{\iota(\iota(Y_1)), A_1, C_1\}, \ \{\iota(\iota(Y_2)), A_1, A_2, C_2\}, \ \{\iota(E_1), B_1, C_1, C_2\}, \ \{\iota(E_2), B_1, C_2\} \big\}.$$

Hence

$$\mathbf{T}^2(\mathcal{H}^{(3)}) = \Big(U_5, \ \mathcal{A}^{(2)} \cup \mathcal{B}^{(2)} \cup \mathcal{C}^{(2)} \Big),$$

which is a level-5 SuperHyperGraph.

Interpretation. The first total step $\mathbf{T}$ combines the level-3 supervertices $\iota(Y_i)$ with superedges E_j and records: (i) original co-membership ($\mathcal{A}$), (ii) edge–edge intersection at a common supervertex ($\mathcal{B}$), and (iii) vertex–edge incidence ($\mathcal{C}$). The second step $\mathbf{T}^2$ then treats these incidence patterns themselves as vertices and re-applies the same three rules, yielding a concrete instance of an *iterated total 3–superhypergraph*.

Theorem 6.18. *For every* $\mathcal{H}^{(n)}$, $\mathbf{T}^1(\mathcal{H}^{(n)}) = \mathbf{T}(\mathcal{H}^{(n)})$.

Proof. Immediate from Definition 6.15. □

Theorem 6.19 ($\mathbf{T}^t$ reduces to T^t at $n = 0$ (canonical isomorphism))**.** *Let* $H = (V, \mathcal{E})$ *be a hypergraph. Identify* $\iota(V) \cong V$ *via* $\iota(v) \leftrightarrow v$*. Then for all* $t \geqslant 1$,

$$\mathbf{T}^t(H) \cong T^t(H).$$

Proof. For $t = 1$, Definition 4.8 with $n = 0$ and the identification $\iota(v) \leftrightarrow v$ yields exactly $T(H)$. Inductively, the promotion at each step replaces V by $\iota(V)$, which we re-identify with V; the three edge-families correspond termwise. Hence $\mathbf{T}^t(H) \cong T^t(H)$ for all t. □

Lemma 6.20 (One step: total graph from total superhypergraph)**.** $[\mathbf{T}(H_G)]_2 = T(G)$.

Proof. Vertices: $V([\mathbf{T}(H_G)]_2) = \iota(V)\dot{\cup}E \cong V\dot{\cup}E$. Adjacency in the 2–section occurs exactly when a pair lies together in some member of $\mathcal{A}$ (vertex–vertex), $\mathcal{B}$ (edge–edge through a shared endpoint), or $\mathcal{C}$ (vertex–edge incidence), which reproduces the three adjacency rules of $T(G)$. □

Theorem 6.21 (Iterated lift of total graphs). *For every finite simple graph* G *and every* $t \geqslant 1$,

$$\left[\mathbf{T}^t(H_G)\right]_2 = T^t(G).$$

Proof. By induction on t. The case $t = 1$ is Lemma 6.20. Assume $[\mathbf{T}^{t-1}(H_G)]_2 = T^{t-1}(G)$. Then the vertices of $[\mathbf{T}^t(H_G)]_2$ are $\iota\big(V(T^{t-1}(G))\big)\dot{\cup}\mathcal{E}\big(\mathbf{T}^{t-1}(H_G)\big)$, which correspond naturally to $V(T^{t-1}(G))\dot{\cup}E(T^{t-1}(G))$. As in the base step, the three families $\mathcal{A}, \mathcal{B}, \mathcal{C}$ at stage t encode precisely the three total-graph adjacencies at stage t; hence $[\mathbf{T}^t(H_G)]_2 = T^t(G)$. □

Lemma 6.22 (Line is induced inside total; total preserves induced substructures). *For every* $\mathcal{H}^{(n)}$, $\mathbf{L}(\mathcal{H}^{(n)}) = \mathbf{T}(\mathcal{H}^{(n)})[\,\mathcal{E}\,]$. *Moreover, if* $\mathcal{K}$ *is induced in* $\mathcal{H}$, *then* $\mathbf{T}(\mathcal{K})$ *is induced in* $\mathbf{T}(\mathcal{H})$.

Proof. The induced substructure on the vertex set $\mathcal{E} \subseteq U_{n+1}$ keeps exactly the $\mathcal{B}$-family, which equals the edge-set of $\mathbf{L}(\mathcal{H}^{(n)})$. For preservation, note that the three families $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are defined by local incidence relations; restricting vertices and intersecting hyperedges commute with the construction. □

Theorem 6.23 (Iterated line superhypergraphs inside iterated total superhypergraphs). *For every* $\mathcal{H}^{(n)}$ *and* $t \geqslant 1$,

$$\mathbf{L}^t(\mathcal{H}^{(n)}) \text{ is an induced sub(super)hypergraph of } \mathbf{T}^t(\mathcal{H}^{(n)}).$$

Proof. For $t = 1$, Lemma 6.22. Assume the claim holds for $t - 1$. Applying Lemma 6.22 to $\mathbf{L}^{t-1}(\mathcal{H}^{(n)}) \subseteq_{\text{ind}} \mathbf{T}^{t-1}(\mathcal{H}^{(n)})$ and then the functoriality of $\mathbf{T}$ on induced substructures yields

$$\mathbf{L}\big(\mathbf{L}^{t-1}\big) \subseteq_{\text{ind}} \mathbf{T}\big(\mathbf{L}^{t-1}\big) \subseteq_{\text{ind}} \mathbf{T}\big(\mathbf{T}^{t-1}\big) = \mathbf{T}^t,$$

i.e. $\mathbf{L}^t \subseteq_{\text{ind}} \mathbf{T}^t$. □

7. Conclusion

This paper developed hypergraph and superhypergraph analogues of line, total, iterated line, and iterated total graph constructions, establishing a theoretical basis for modeling hierarchical, incidence-rich, and self-referential network structures. Superhypergraph analogues provide hierarchical, higher-order views of adjacency and incidence, lifting classical constructions, preserving multilevel structure, revealing shared substructures, modeling teams-of-teams, and enabling iterative analyses.

It is anticipated that future research will advance the study of extended models based on various graph-theoretic frameworks, including Fuzzy Graphs[51, 52, 53], Intuitionistic Fuzzy Graphs [54, 55, 56], Neutrosophic Graphs [57, 58, 59], Fuzzy HyperGraphs[60, 61], Plithogenic Graphs[62, 63, 64, 65], Directed Graphs[66, 67], Neutrosophic Directed

Graphs[68, 69, 70], Bidirected Graphs[71, 72], and Multidirected Graphs[73, 74]. Moreover, further progress in the development of algorithms for these structures is also expected.

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Conflicts of Interest

The authors declare that they have no conflicts of interest related to this publication.

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Data Availability

This is a theoretical study; no empirical data were generated or analyzed. We encourage future work to apply and test these concepts in practical settings.

Ethical Approval

No human participants or animals were involved; therefore, ethics approval was not required.

Code Availability

No software or code was produced for this study.

Clinical Trial

This study did not include any clinical trials.

Consent to Participate

Not applicable.

Disclaimer

The ideas presented are theoretical and have not yet been validated empirically. While care has been taken to ensure accuracy and proper citation, inadvertent errors may remain; readers should independently verify referenced material. The views expressed are those of the authors and do not necessarily reflect the positions of their institutions.

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