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Investigation of a Class of Implicit Anti-Periodic Boundary Value Problems

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Abstract

This research is devoted to studying a class of implicit fractional order differential equations (FODEs) under anti-periodic boundary conditions (APBCs). With the help of classical fixed point theory due to Schauder and Banach, we derive some adequate results about the existence of at least one solution. Moreover, this manuscript discusses the concept of stability results including Ulam-Hyers (HU) stability, generalized Hyers-Ulam (GHU) stability, Hyers-Ulam Rassias (HUR) stability, and generalized Hyers-Ulam- Rassias (GHUR)stability. Finally, we give three examples to illustrate our results.

Keywords: Anti-periodic boundary value problem, Fixed point theorem, Stability results.
2010 MSC: 26A33, 34A08, 35R11.

1. Introduction

In previous years the area of FODEs has been considered a powerful procedure for solving practical problems that arise in several fields such as biological science, control theory, heat conduction, viscoelasticity, chemical physics, economics, ecology, aerodynamics [1, 2, 3, 4, 5, 6], etc. A comprehensive study in the form of a book has been given in 1999 about FODEs and their applications, we provide reference as [7].

In present time, the study of nonlinear differential and integral equations have received much attention from mathematicians due to its worldwide applications in several fields of engineering and technologies. Since using integer order differential operators for modeling various dynamical systems, the hereditary process and memory description cannot be well explained in many situations. Therefore researchers brilliantly have applied the fractional differential operators to describe memory and hereditary precoces in more accurate way than integer order derivatives. This fact motivated researchers to take interest in FODEs. So far we know considerable amount of work has been done in this area. The said area has been investigated from different direction including qualitative theory,

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stability theory, optimization and numerical simulations. Abundant of work in this regard can be founded about existence theory of solutions, we refer some as [8, 9, 10, 11, 12]. On the other hand the area devoted to establish procedure for numerical solutions has been investigated very well. Therefore for this purposes plenty of research papers have been formed in literature which address very good investigations, for instance (we give references as [13, 14, 15, 16, 17, 18]). Since it is necessary for numerical procedure to be stable to produce good results which are highly acceptable in applications. Therefore another aspect has been considered which is known as stability analysis. Various kinds of stability like exponential, Mittag-Leffler and Lyapunov type have been investigated for differential equations of integer order. In last few years the mentioned stabilities have been upgraded for linear and nonlinear FODEs and their systems, see detail as [19, 20, 21]. Establishing these stabilities for nonlinear systems have merits and de-merits in constructions. Some of them need a pre-defined Lyapunov function which often is very difficult and time consuming to construct on trail basis. on other hand the exponential and Mittag-Lefller stability involving exponential functions which often create difficulties in treating during numerical analysis of problems. In this regard another kind stability has been given proper attention by the mathematicians known as HU stability. Ulam in 1940 was the first man who pointed this stability during a talk. After that in 1940 Hyers very nicely explained for functional equations, for detail we refer [22, 23]. Onward the said stability was further modified to more general form by other researchers for functional equations, ordinary differential equations. Some very fruitful results were formed in this regard which can be traced in [24, 25, 26, 27, 28], etc. In last two decades the said stability theory has been considered very well for FODEs and their systems, see [29, 30, 31, 32, 33, 34].

Inspired from the above mentioned work, in this research article we are considered the following class of antiperiodic boundary value problem (ABVP) in implicit nature

$$\begin{cases} {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t) = \hbar(t, \mathbf{w}(t), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t)), \ t \in \mathscr{J} \ \tau > 0, \ 2 < \delta \leqslant 3, \\ \mathbf{w}(0) = -\mathbf{w}(\tau), \ {}^{c}\mathbf{D}^{r}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{r}\mathbf{w}(\tau), \ {}^{c}\mathbf{D}^{s}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{s}\mathbf{w}(\tau), \end{cases} \tag{1.1}$$

where $0 < r < 1, \ 1 < s < 2, \ \mathscr{J} = [0, \tau]$ and $\hbar : \mathscr{J} \times \mathscr{R}^2 \to \mathscr{R}$ is continous. We investigate qualitative theory as well as different kinds of stability including HU, GHU , HUR and GHUR stability for the considered problem. For qualitative theory we utilize classical fixed point theorem due to Schauder and Banach while for the stability theory nonlinear functional analysis is used. In last, this work is strengthened by providing examples and short conclusion.

2. Preliminaries

The space $\mathscr{M} = C^3(\mathscr{J}, \mathscr{R})$ is a Banach space with respect to the norm defined by

$$||\mathbf{w}||_{\mathscr{M}} = \max_{t \in \mathscr{J}}\{|\mathbf{w}(t)| : t \in \mathscr{J}\}. \tag{2.1}$$

Definition 2.1. [35] Integral of a function $\mathbf{w} \in L(\mathscr{J}, \mathscr{R}^+)$ with fractional order $\delta > 0$ is defined by

$$\mathbf{I}_a^{\delta}\mathbf{w}(t) = \int_a^t \frac{(t-\eta)^{\delta-1}}{\Gamma(\delta)}\mathbf{w}(\eta)d\eta, \tag{2.2}$$

provided that integral on the right exists.

Definition 2.2. [35] The Caputo derivative of $\mathbf{w} \in \mathscr{M}$ corresponding to fractional order $\delta > 0$ is expressed as

$$ {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t) = \frac{1}{\Gamma(n-\delta)} \int_{a}^{t} (t-\eta)^{n-\delta-1}\mathbf{w}^{(n)}(\eta)d\eta, \tag{2.3} $$

where $n = [\delta] + 1$.

Lemma 2.3. *[36] For $\delta > 0$, the given result holds*

$$ \mathbf{I}^{\delta}[{}^{c}\mathbf{D}^{\delta}\mathbf{w}(t)] = \mathbf{w}(t) - \sum_{l=0}^{n-1} \frac{\mathbf{w}^{(l)}(0)}{l!} t^{l}, \text{ where } n = [\delta] + 1. $$

Definition 2.4. [27] The fractional order ABVP (1.1) become HU stable if for any constant $\mathcal{C}_{\hbar} > 0$ and for any solution $\bar{\mathbf{w}} \in \mathscr{M}$ there exists $\epsilon > 0$ such that for the relation

$$ |{}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t) - \hbar(t, \bar{\mathbf{w}}(t), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t))| \leqslant \epsilon, \text{ for } t \in \mathscr{J}, \tag{2.4} $$

there is a unique solution $\mathbf{w} \in \mathscr{M}$ of the considered problem (1.1), such that

$$ \|\bar{\mathbf{w}} - \mathbf{w}\|_{\mathscr{M}} \leqslant \mathcal{C}_{\hbar}\epsilon, \text{ for } t \in \mathscr{J}. $$

Definition 2.5. [27] The fractional order ABVP (1.1) becomes GHU stable if there exist $\Psi \in C((0,1), \mathscr{R}^{+})$, $\Psi(0) = 0$, such that for any solution $\bar{\mathbf{w}} \in \mathscr{M}$ of the relation (2.4), there is at most solution $\mathbf{w} \in \mathscr{M}$ of the considered problem (1.1) such that

$$ \|\bar{\mathbf{w}} - \mathbf{w}\|_{\mathscr{M}} \leqslant \mathcal{C}_{\hbar}\Psi(\epsilon), \text{ with } t \in \mathscr{J}. $$

Definition 2.6. [27] The fractional order ABVP (1.1) is called HUR stable corresponding to $\phi \in C((0,1), \mathscr{R}^{+})$, with a constant $\mathcal{C}_{\hbar} > 0$, such that for $\epsilon > 0$ and for any solution $\bar{\mathbf{w}} \in \mathscr{M}$ of the relation

$$ |{}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t) - \hbar(t, \bar{\mathbf{w}}(t), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t))| \leqslant \phi(t)\epsilon, \text{with } t \in \mathscr{J}, \tag{2.5} $$

there exists at most one solution $\mathbf{w} \in \mathscr{M}$ of problem (1.1), such that

$$ \|\bar{\mathbf{w}} - \mathbf{w}\|_{\mathscr{M}} \leqslant \mathcal{C}_{\hbar}\epsilon\phi(t), \text{for } t \in \mathscr{J}. \tag{2.6} $$

Definition 2.7. [27] The fractional order ABVP (1.1) is called to be GHUR stable with respect to $\phi \in \mathscr{M}$, if the relation

$$ |{}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t) - \hbar(t, \bar{\mathbf{w}}(t), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t))| \leqslant \phi(t), \text{ with } t \in \mathscr{J}, \tag{2.7} $$

there is $\mathcal{C}_{\hbar} \in \mathscr{R}^{+}$, such that for any solution $\bar{\mathbf{w}} \in \mathscr{M}$ of the relation (2.5) there at most one solution $\mathbf{w} \in \mathscr{M}$ of problem (1.1), such that

$$ \|\bar{\mathbf{w}} - \mathbf{w}\| \leqslant \mathcal{C}_{\hbar}\phi(t), \text{ for all } t \in \mathscr{J}. \tag{2.8} $$

Remark 2.8. A function $\bar{\mathbf{w}} \in \mathscr{M}$ is a solution of (2.4) if there is a function $\psi(t) \in \mathscr{M}$ such that

(i) ${}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t) = \hbar(t, \bar{\mathbf{w}}(t), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t)) + \psi(t)$, with $t \in \mathscr{J}$;

(ii) $|\psi(t)| \leqslant \epsilon$ for all $t \in \mathscr{J}$.

Remark 2.9. A function $\bar{\mathbf{w}} \in \mathscr{M}$ is a solution of (2.5), if there exists a function $\psi(t) \in \mathscr{M}$ depend on $\bar{\mathbf{w}}$ only with

(i) ${}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t) = \hbar(t, \bar{\mathbf{w}}(t), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(t)) + \psi(t)$, with $t \in \mathscr{J}$;

(ii) $|\psi(t)| \leqslant \epsilon\phi(t)$, for all $t \in \mathscr{J}$.

3. Main work

Theorem 3.1. *let $y \in L(\mathscr{J})$, the solution*

$$\begin{cases} {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t) = y(t), \text{ for } t \in [0,\tau], \text{ and } \tau > 0,\ 2 < \delta \leqslant 3, \\ \mathbf{w}(0) = -\mathbf{w}(\tau),\ {}^{c}\mathbf{D}^{r}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{r}\mathbf{w}(\tau),\ {}^{c}\mathbf{D}^{s}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{s}\mathbf{w}(\tau), \end{cases} \tag{3.1}$$

is given by

$$\mathbf{w}(t) = \int_0^t \mathscr{G}(t,\eta)y(\eta)d\eta, \tag{3.2}$$

here $\mathscr{G}(t,\eta)$ is expressed as

$$\mathscr{G}(t,\eta) = \begin{cases} \dfrac{(t-\eta)^{\delta-1} - \frac{1}{2}(\tau-\eta)^{\delta-1}}{\Gamma(\delta)} + \dfrac{\Gamma(2-r)(\tau-2t)(\tau-\eta)^{\delta-r-1}}{2\Gamma(\delta-r)\tau^{1-r}} \\ -\dfrac{[r\tau^2 - 4\tau t + 2(2-r)t^2\Gamma(3-s)(\tau-\eta)^{\delta-s-1}]}{4(2-r)\Gamma(\delta-r)\tau^{2-s}}, & \eta \leqslant t, \\ \dfrac{(\tau-\eta)^{\delta-1}}{2\Gamma(\delta)} + \dfrac{\Gamma(2-r)(\tau-2t)(\tau-\eta)^{\delta-r-1}}{2\Gamma(\delta-r)\tau^{1-r}} \\ -\dfrac{[r\tau^2 - 4\tau t + 2(2-r)t^2\Gamma(3-s)(\tau-\eta)^{\delta-s-1}]}{4(2-r)\Gamma(\delta-r)\tau^{2-s}}, & \eta \geqslant t. \end{cases} \tag{3.3}$$

Proof. Let $\mathbf{w}$ be a solution of (3.1). Then by Lemma 2.3, we have

$$\mathbf{w}(t) = \mathbf{I}^{\delta}y(t) - c_0 - c_1 t - c_2 t^2 = \frac{1}{\Gamma(\delta)}\int_0^t (t-\eta)^{\delta-1}y(\eta) - c_0 - c_1 t - c_2 t^2, \tag{3.4}$$

with real constants c_0, c_1, c_2. Further we have ${}^{c}\mathbf{D}^{r}t = \frac{t^{1-r}}{\Gamma(2-r)}$, ${}^{c}\mathbf{D}^{r}t^2 = \frac{2t^{2-r}}{\Gamma(3-r)}$ and ${}^{c}\mathbf{D}^{r}\mathbf{I}^{\delta}y(t) = \mathbf{I}^{\delta-r}y(t)$, we get

$$^{c}\mathbf{D}^{r}\mathbf{w}(t) = \frac{1}{\Gamma(\delta-r)}\int_0^t (t-\eta)^{\delta-r-1}y(\eta)d\eta - c_1\frac{t^{1-r}}{\Gamma(2-r)} - c_2\frac{2t^{2-r}}{\Gamma(3-r)}$$

In view of ${}^{c}\mathbf{D}^{s}t = 0$ $(1 < s < 2)$ and ${}^{c}\mathbf{D}^{s}t^2 = \frac{2t^{2-s}}{\Gamma(3-s)}$, we get

$$^{c}\mathbf{D}^{s}\mathbf{w}(t) = \frac{1}{\Gamma(\delta-s)}\int_0^t (t-\eta)^{\delta-s-1}y(\eta)d\eta - c_2\frac{2t^{2-s}}{\Gamma(3-s)}.$$

Thank to the conditions due to boundary $\mathbf{w}(0) = -\mathbf{w}(\tau)$, ${}^{c}\mathbf{D}^{r}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{r}\mathbf{w}(\tau)$, ${}^{c}\mathbf{D}^{s}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{s}\mathbf{w}(\tau)$, one can find that

$$\begin{aligned} c_0 &= \int_0^{\tau} \frac{(\tau-\eta)^{\delta-1}}{2\Gamma(\delta)} y(\eta)d\eta - \frac{\Gamma(2-r)\tau^r}{2\Gamma(\delta-r)} \int_0^{\tau} (\tau-\eta)^{\delta-r-1} y(\eta)d\eta \\ &+ \frac{r\Gamma(3-s)\tau^s}{4(2-r)\Gamma(\delta-s)} \int_0^{\tau} (\tau-\eta)^{\delta-s-1} y(\eta)d\eta, \\ c_1 &= \frac{\Gamma(2-r)}{\Gamma(\delta-r)\tau^{1-r}} \int_0^{\tau} (\tau-\eta)^{\delta-r-1} y(\eta)d\eta \\ &- \frac{\Gamma(3-s)}{(2-r)\Gamma(\delta-s)\tau^{1-s}} \int_0^{\tau} (\tau-\eta)^{\delta-s-1} y(\eta)d\eta, \\ c_2 &= \frac{\Gamma(3-s)}{2\Gamma(\delta-s)\tau^{2-s}} \int_0^{\tau} (\tau-\eta)^{\delta-s-1} y(\eta)d\eta, \end{aligned}$$

Plugging the values of c_0, c_1 and c_2 in (3.4), one has the following solution

$$\begin{aligned} \mathbf{w}(t) &= \frac{1}{\Gamma(\delta)} \int_0^{t} (t-\eta)^{\delta-1} y(\eta)d\eta - \frac{1}{2\Gamma(\delta)} \int_0^{\tau} (\tau-\eta)^{\delta-1} y(\eta)d\eta \\ &+ \frac{\Gamma(2-r)(\tau-2t)}{2\Gamma(\delta-r)\tau^{1-r}} \int_0^{\tau} (\tau-\eta)^{\delta-r-1} y(\eta)d\eta \\ &- \frac{[r\tau^2 - 4\tau t + 2(2-r)t^2]\Gamma(3-s)}{4(2-r)\Gamma(\delta-s)\tau^{2-s}} \int_0^{\tau} (\tau-\eta)^{\delta-s-1} y(\eta)d\eta \\ &= \int_0^{\tau} \mathscr{G}(t,\eta) y(\eta)d\eta. \end{aligned} \tag{3.5}$$

□

Corollary 3.2. *In light of Theorem 3.1, the proposed problem (1.1) has the following solution*

$$\mathbf{w}(t) = \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(t, \mathbf{w}(t), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t))d\eta,$$

where $\mathscr{G}(t,\eta)$ is the same Green's function given in (3.3).

Lemma 3.3. *The Green function $\mathscr{G}(t,\eta)$, given in (3.3) satisfies the given relations:*

(A_1) $\mathscr{G}(t,\eta) \geqslant 0$ forall $t,\eta \in \mathscr{J}$;

(A_2) $\mathscr{G}(t,\eta)$ *is continuous over* $\mathscr{J} \times \mathscr{J}$;

(A_3) *moreover for the Green's function one has the given result*

$$\max_{t\in\mathscr{J}} \int_0^{\tau} \mathscr{G}(t,\eta)d\eta \leqslant \frac{\tau^{\delta}}{\Gamma(\delta+1)} + \frac{\Gamma(2-r)\tau^{\delta}}{2\Gamma(\delta-r+1)} + \frac{(r+2)(\Gamma(3-s)\tau^{\delta}}{2(2-r)\Gamma(\delta-s+1)}.$$

Proof. Hypothesis (A_1) and (A_2) are obvious.
(A_3): We have from the solution (3.5)

$$\begin{aligned}
\max_{t\in\mathscr{J}}\int_0^\tau \mathscr{G}(t,\eta)d\eta &= \max_{t\in\mathscr{J}}\Bigg(\frac{1}{\Gamma(\delta)}\int_0^t (t-\eta)^{\delta-1}d\eta - \frac{1}{2\Gamma(\delta)}\int_0^\tau(\tau-\eta)^{\delta-1}d\eta \\
&+\frac{\Gamma(2-r)(\tau-2t)}{2\Gamma(\delta-r)\tau^{1-r}}\int_0^\tau(\tau-\eta)^{\delta-r-1}d\eta \\
&-\frac{[r\tau^2-4\tau t+2(2-r)t^2]\Gamma(3-s)}{4(2-r)\Gamma(\delta-s)\tau^{2-s}}\int_0^\tau(\tau-\eta)^{\delta-s-1}d\eta,\Bigg) \\
&\leqslant \max_{t\in\mathscr{J}}\left(\frac{\tau^\delta}{\Gamma(\delta+1)}+\frac{\Gamma(2-r)\tau^{\delta-r+1}}{2\Gamma(\delta-r+1)\tau^{1-r}}-\frac{[r\tau^2-4\tau t+2(2-r)t^2]\Gamma(3-s)\tau^{\delta-s}}{4(2-r)\Gamma(\delta-s+1)\tau^{2-s}}\right) \\
&\leqslant \frac{\tau^\delta}{\Gamma(\delta+1)}+\frac{\Gamma(2-r)\tau^\delta}{2\Gamma(\delta-r+1)}+\frac{(r+2)(\Gamma(3-s)\tau^\delta}{2(2-r)\Gamma(\delta-s+1)}.
\end{aligned}$$

Hence this complete the proof. □

Here we note that for convince we use

$$\Delta = \frac{\tau^\delta}{\Gamma(\delta+1)}+\frac{\Gamma(2-r)\tau^\delta}{2\Gamma(\delta-r+1)}+\frac{(r+2)(\Gamma(3-s)\tau^\delta}{2(2-r)\Gamma(\delta-s+1)}. \tag{3.6}$$

To go ahead we give the assumptions bellow for $t\in\mathscr{J}$

(A_4) $\exists$ constants $0<\mathscr{L}_\hbar<1$ and $\mathscr{K}_\hbar>0$, with

$$|\hbar(t,\mathbf{w},\beta_{\mathbf{w}})-\hbar(t,\bar{\mathbf{w}},\beta_{\bar{\mathbf{w}}})\leqslant \mathscr{K}_\hbar|\mathbf{w}-\bar{\mathbf{w}}|+\mathscr{L}_\hbar|\beta_{\mathbf{w}}-\beta_{\bar{\mathbf{w}}}|,\text{ against any } \mathbf{w},\bar{\mathbf{w}}\in\mathscr{R}.$$

(A_5) $\exists$ $\alpha_1,\alpha_2,\alpha_3\in C((0,1),\mathscr{R}^+)$ with

$$|\hbar(t,\mathbf{w}(t),\beta_{\mathbf{w}}(t))|\leqslant \alpha_1(t)+\alpha_2(t)|\mathbf{w}(t)|+\alpha_3(t)|\beta_{\mathbf{w}}(t)|,\ \text{ for } \ \mathbf{w},\in\mathscr{R},$$

with

$$\alpha_1^*=\sup_{t\in\tau}\alpha_1(t),\ \alpha_2^*=\sup_{t\in\tau}\alpha_2(t),\ \alpha_3^*=\sup_{t\in\tau}\alpha_3(t)<1.$$

To convert the proposed problem into fixed point problem, we define an operator $\mathcal{N}:\mathscr{M}\to\mathscr{M}$ as

$$\mathcal{N}(\mathbf{w})(t)=\int_0^\tau \mathscr{G}(t,\eta)\hbar(t,\mathbf{w}(t),{}^c\mathbf{D}^\delta\mathbf{w}(t))d\eta, \tag{3.7}$$

where $\beta_{\mathbf{w}}(t)\in\mathscr{M}$, such that $\beta_{\mathbf{w}}(t)=\hbar(t,\mathbf{w}(t),{}^c\mathbf{D}^\delta\mathbf{w}(t))$.

Theorem 3.4. *In light of Hypothesis* (A_5), *the operator* $\mathcal{N}:\mathscr{M}\to\mathscr{M}$ *defined in* (3.7) *is completely continuous.*

Proof. The continuity of functions $\hbar$, $\mathscr{G}(t,\eta)$ implies the continuity of operator $\mathcal{N}$. Let $\mathscr{B}\subset\mathscr{M}$ be a bounded subset. Then for $t\in\mathscr{J}$ and $\mathbf{w}\in\mathscr{B}$, we have

$$\begin{aligned}
|\mathcal{N}\mathbf{w}(t)| &= \left|\int_0^\tau \mathscr{G}(t,\eta)\hbar(t,\mathbf{w}(t),\beta_{\mathbf{w}}(t))d\eta\right| \\
&\leqslant \int_0^\tau |\mathscr{G}(t,\eta)||\hbar(\eta,\mathbf{w}(\eta),\beta_{\mathbf{w}}(\eta))|d\eta.
\end{aligned} \tag{3.8}$$

By assumption (A_3), we have

$$\begin{aligned} |\beta_{\mathbf{w}}(t)| &= |\hbar(t, \mathbf{w}(t), \beta_{\mathbf{w}}(t))| \\ &\leqslant \alpha_1(t) + \alpha_2(t)|\mathbf{w}(t)| + \alpha_3(t)|\beta_{\mathbf{w}}(t)| \\ &\leqslant \alpha_1^* + \alpha_2^*\|\mathbf{w}\| + \alpha_3^*\|\beta_{\mathbf{w}}\| \end{aligned}$$

which gives on simplification

$$\|\beta_{\mathbf{w}}\|_{\mathscr{M}} \leqslant \frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}. \tag{3.9}$$

Using property (A_3) of Lemma 3.3 and the relation (3.9) in the relation (3.8), we obtain.

$$\|\mathcal{N}\mathbf{w}\|_{\mathscr{M}} \leqslant \left[\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}\right]\Delta.$$

Which shows that $\mathcal{N}$ is uniformly bounded. To derive $\mathcal{N}$ is equicontinuous, let $t_2 > t_1 \in \mathscr{J}$ and consider

$$\begin{aligned} |\mathcal{N}\mathbf{w}(t_2) - \mathcal{N}\mathbf{w}(t_1)| &\leqslant \int_0^{t_2} \frac{(t_2-\eta)^{\delta-1}}{\Gamma(\delta)}|\hbar(\eta, \mathbf{w}(\eta), \beta_{\mathbf{w}}(\eta))|d\eta \\ &- \int_0^{t_1} \frac{(t_1-\eta)^{\delta-1}}{\Gamma(\delta)}|\hbar(\eta, \mathbf{w}(\eta), \beta_{\mathbf{w}}(\eta))|d\eta \\ &+ \frac{2\Gamma(2-r)(t_2-t_1)}{2\tau^{1-r}\Gamma(\delta-r)}\int_0^{\tau}(\tau-\eta)^{\delta-r-1}|\hbar(\eta, \mathbf{w}(\eta), \beta_{\mathbf{w}}(\eta))|d\eta \\ &+ \frac{4\tau(1+2\Gamma(2-r))(t_2-t_1)}{4(2-r)\tau^{2-s}\Gamma(\delta-s)}\int_0^{\tau}(\tau-\eta)|\hbar(\eta, \mathbf{w}(\eta), \beta_{\mathbf{w}}(\eta))|d\eta \\ &\leqslant \int_0^{t_2} \frac{(t_2-\eta)^{\delta-1}}{\Gamma(\delta)}\left[\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}\right]d\eta \\ &- \int_0^{t_1} \frac{(t_1-\eta)^{\delta-1}}{\Gamma(\delta)}\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}d\eta \\ &+ \frac{2\Gamma(2-r)(t_2-t_1)}{2\tau^{1-r}\Gamma(\delta-r)}\int_0^{\tau}(\tau-\eta)^{\delta-r-1}\left[\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}\right]d\eta \\ &+ \frac{4\tau(1+2\Gamma(2-r))(t_2-t_1)}{4(2-r)\tau^{2-s}\Gamma(\delta-s)}\int_0^{\tau}(\tau-\eta)^{\delta-s}\left[\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}\right]d\eta \\ &\leqslant \left[\frac{(t_2^{\delta}-t_1^{\delta})}{\Gamma(\delta+1)} + \frac{2\tau^{\delta-r}\Gamma(2-r)(t_2-t_1)}{2\tau^{1-r}\Gamma(\delta-r+1)}\right. \\ &+ \left.\frac{4\tau^{\delta-s+1}(1+2\Gamma(2-r))(t_2-t_1)}{4(2-r)\tau^{2-s}\Gamma(\delta-s+1)}\right]\left(\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}\right). \end{aligned} \tag{3.10}$$

Since at $t_1 \to t_2$, (3.10) tends to zero in the right hand side. Therefore, operator $\mathcal{N}$ is equicontinuous and hence it is uniformly continuous. Also it is easy to show that $\mathcal{N}(\mathscr{B}) \subset \mathscr{B}$. Hence by Arzelá-Ascoli theorem $\mathcal{N}$ is completely continuous. □

Theorem 3.5. *Under the completely continuity of $\mathcal{N}$ and if the hypothesis $(A_3)-(A_5)$ hold, then the problem (1.1) has at least one solution.*

Proof. We define a set $\mathscr{E}$ as

$$\mathscr{E} = \{\mathbf{w} \in \mathscr{M} : \mathbf{w} = \rho\mathcal{N}(\mathbf{w}),\ \rho \in (0,1)\}.$$

The operator $\mathcal{N} : \bar{\mathscr{E}} \to \mathscr{M}$ as defined in (3.7) is completely continuous by Theorem 3.4. Take $\mathbf{w} \in \mathscr{E}$ then by definition of the set $\mathscr{E}$, one has by using (A_5)

$$\begin{aligned} \|\mathbf{w}\|_{\mathscr{M}} &= \|\rho\mathcal{N}(\mathbf{w})\|_{\mathscr{M}} \\ &\leqslant \max_{\mathbf{w}\in\mathscr{J}} \int_0^{\tau} |\mathscr{G}(t,\eta)||\hbar(\eta,\mathbf{w}(\eta),\beta_{\mathbf{w}}(\eta))|d\eta \\ &\leqslant \max_{\mathbf{w}\in\mathscr{J}} \int_0^{\tau} |\mathscr{G}(t,\eta)|\frac{\alpha_1^* + \alpha_2^*\|\mathbf{w}\|_{\mathscr{M}}}{1-\alpha_3^*}d\eta. \end{aligned}$$

From which we have

$$\|\mathbf{w}\|_{\mathscr{M}} \leqslant \frac{\alpha_1^*\Delta}{1-(\alpha_3^* + \alpha_2^*\Delta)} = \mu. \tag{3.11}$$

Hence the set $\mathscr{E}$ is bounded. So the operator $\mathcal{N}$ has at least one solution. Consequently the APBVP (1.1) has at least one solution. □

Theorem 3.6. *If the hypothesis* (A_3), (A_4) *and the condition* $\frac{\mathscr{K}_\hbar}{1-\mathscr{L}_\hbar}\Delta < 1$ *holds, where* Δ *is already given in* (3.6), *then the APBVP* (1.1) *has a unique solution in* $\mathscr{M}$.

Proof. Here we shall use Banach contraction principle to prove the required result. Let $\mathbf{w}, \bar{\mathbf{w}} \in \mathscr{M}$, then for $t \in \mathscr{J}$ consider

$$\begin{aligned} |\mathcal{N}\mathbf{w}(t) - \mathcal{N}\bar{\mathbf{w}}(t)| &= \left| \int_0^{\tau} \mathscr{G}(t,\eta)\Big(\hbar(\eta,\mathbf{w}(\eta),\beta_{\mathbf{w}}(\eta)) - \hbar(\eta,\bar{\mathbf{w}}(\eta),\beta_{\bar{\mathbf{w}}}(\eta))\Big)d\eta \right| \\ &\leqslant \int_0^{\tau} |\mathscr{G}(t,\eta)||\hbar(\eta,\mathbf{w}(\eta),\beta_{\mathbf{w}}(\eta)) - \hbar(\eta,\bar{\mathbf{w}}(\eta),\beta_{\bar{\mathbf{w}}}(\eta))|d\eta. \end{aligned}$$

On taking maximum of both sides and repeating the same fashion as in (3.9), we have

$$\begin{aligned} \|\mathcal{N}\mathbf{w} - \mathcal{N}\bar{\mathbf{w}}\|_{\mathscr{M}} &\leqslant \max_{t\in\mathscr{J}} \int_0^{\tau} |\mathscr{G}(t,\eta)|\frac{\mathscr{K}_\hbar}{1-\mathscr{L}_\hbar}\|\mathbf{w} - \bar{\mathbf{w}}\|_{\mathscr{M}}d\eta \\ &\leqslant \frac{\mathscr{K}_\hbar\Delta}{1-\mathscr{L}_\hbar}\|\mathbf{w} - \bar{\mathbf{w}}\|_{\mathscr{M}}. \end{aligned} \tag{3.12}$$

Since $\frac{\Delta\mathscr{K}_\hbar}{1-\mathscr{L}_\hbar} < 1$, therefore, the operator $\mathcal{N}$ is contraction. Thus by Banach contraction principle, we get that $\mathcal{N}$ has a unique fixed point. Consequently APBVP (1.1) has unique solution. □

4. Stability Analysis

In this section, we provide stability results for the corresponding problem of previous section. Here we provide an assumption needed in further analysis.

Lemma 4.1. *For the given APBVP*

$$\begin{cases} {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t) = \hbar(t, \mathbf{w}(t), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t)) + \psi(t), \ t \in \mathscr{J}, \ 2 < \delta \leqslant 3, \\ \mathbf{w}(0) = -\mathbf{w}(\tau), \ {}^{c}\mathbf{D}^{r}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{r}\mathbf{w}(\tau), \ {}^{c}\mathbf{D}^{s}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{s}\mathbf{w}(\tau), \end{cases} \tag{4.1}$$

we have the following inequality

$$\left| \mathbf{w}(t) - \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(\eta, \mathbf{w}(\eta), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta \right| \leqslant \Delta\epsilon, \ t \in \mathscr{J}. \tag{4.2}$$

Proof. Thank to Corollary 3.2 the solution of perturbed problem (4.1) is given by

$$\mathbf{w}(t) = \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(t, \mathbf{w}(t), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t))d\eta + \int_0^{\tau} \mathscr{G}(t,\eta)\psi(\eta)d\eta.$$

From which one has by using (i) of Remark 2.8

$$\begin{aligned} \left| \mathbf{w}(t) - \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(t, \mathbf{w}(t), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(t))d\eta \right| &\leqslant \int_0^{\tau} |\mathscr{G}(t,\eta)||\psi(\eta)|d\eta \\ &\leqslant \Delta\epsilon, \ t \in \mathscr{J}. \end{aligned}$$

□

Theorem 4.2. *Inview of hypothesis* (A_5) *and Lemma 4.1, the solution of the APBVP (1.1) is HU stable and consequently it is GHU stable if the condition* $\mathscr{L}_{\hbar} + \Delta\mathscr{K}_{\hbar} < 1, \ \mathscr{L}_{\hbar} < 1$ *hold.*

Proof. Let $\bar{\mathbf{w}} \in \mathscr{M}$ be unique solution of APBVP (1.1) and $\mathbf{w} \in \mathscr{M}$ be any solution of the said problem, then consider with $t \in \mathscr{J}$

$$\begin{aligned} \|\mathbf{w} - \bar{\mathbf{w}}\|_{\mathscr{M}} &= \max_{t \in \mathscr{J}} \left| \mathbf{w} - \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(\eta, \bar{\mathbf{w}}(\eta), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(\eta))d\eta \right| \\ &\leqslant \left| \mathbf{w} - \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(\eta, \mathbf{w}(\eta), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta \right| \\ &+ \left| \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(\eta, \mathbf{w}(\eta), {}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta - \int_0^{\tau} \mathscr{G}(t,\eta)\hbar(\eta, \bar{\mathbf{w}}(\eta), {}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(\eta))d\eta \right| \\ &\leqslant \Delta\epsilon + \frac{\Delta\mathscr{K}_{\hbar}}{1 - \mathscr{L}_{\hbar}} \|\mathbf{w} - \bar{\mathbf{w}}\|_{\mathscr{M}}. \end{aligned} \tag{4.3}$$

Upon simplification (4.3) yields

$$\|\mathbf{w} - \bar{\mathbf{w}}\| \leqslant \mathcal{C}_{\hbar}\epsilon, \ \mathcal{C}_{\hbar} = \frac{\Delta(1 - \mathscr{L}_{\hbar})}{1 - (\mathscr{L}_{\hbar} + \Delta\mathscr{K}_{\hbar})}. \tag{4.4}$$

Hence the APBVP (1.1) is HU stable. Further if there exist a nondecreasing function $\Psi : (0,1) \to (0,\infty)$ such that $\Psi(\epsilon) = \epsilon$ with $\Psi(0) = 0$, then from (4.4) one has

$$\|\mathbf{w} - \bar{\mathbf{w}}\| \leqslant \mathcal{C}_{\hbar}\Psi(\epsilon). \tag{4.5}$$

Thus APBVP (1.1) is GHU stable. □

Lemma 4.3. *For the given APBVP (4.1), the following inequality holds.*

$$\left|\mathbf{w}(t)-\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(\eta,\mathbf{w}(\eta),{}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta\right|\leqslant\Delta\phi(t)\epsilon,\ t\in\mathscr{J}. \tag{4.6}$$

Proof. Thank to Corollary 3.2 the solution of perturbed problem (4.1) is given by

$$\mathbf{w}(t) = \int_0^{\tau}\mathscr{G}(t,\eta)\hbar(t,\mathbf{w}(t),{}^{c}\mathbf{D}^{\delta}\mathbf{w}(t))d\eta+\int_0^{\tau}\mathscr{G}(t,\eta)\psi(\eta)d\eta.$$

From which one has by using (i) of Remark 2.9

$$\begin{aligned}\left|\mathbf{w}(t)-\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(t,\mathbf{w}(t),{}^{c}\mathbf{D}^{\delta}\mathbf{w}(t))d\eta\right| &\leqslant \int_0^{\tau}|\mathscr{G}(t,\eta)||\psi(\eta)|d\eta\\ &\leqslant \Delta\phi(t)\epsilon,\ t\in\mathscr{J}.\end{aligned}$$

□

Lemma 4.4. *The solution of the perturbed APBVP given in (4.1) satisfies the following property*

$$\left|\mathbf{w}(t)-\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(\eta,\mathbf{w}(\eta),{}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta\right|\leqslant\Delta\phi(t),\ t\in\mathscr{J}. \tag{4.7}$$

Proof. For the proof follow Lemma 4.1. □

Theorem 4.5. *If the hypothesis* (A_5) *together with the condition* $\mathscr{L}_{\hbar}+\Delta\mathscr{K}_{\hbar}<1,\ \mathscr{L}_{\hbar}<1$ *hold. Then the APBVP (1.1) is HUR stable.*

Proof. Let $\bar{\mathbf{w}}\in\mathscr{M}$ be unique solution of inequality (1.1) and $\mathbf{w}$ be the any solution of problem (1.1), then using Lemma 4.4, one has

$$\begin{aligned}\|\mathbf{w}-\bar{\mathbf{w}}\|_{\mathscr{M}} &= \max_{t\in\mathscr{J}}\left|\mathbf{w}-\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(\eta,\bar{\mathbf{w}}(\eta),{}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(\eta))d\eta\right|\\ &\leqslant \left|\mathbf{w}-\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(\eta,\mathbf{w}(\eta),{}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta\right|\\ &+\left|\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(\eta,\mathbf{w}(\eta),{}^{c}\mathbf{D}^{\delta}\mathbf{w}(\eta))d\eta-\int_0^{\tau}\mathscr{G}(t,\eta)\hbar(\eta,\bar{\mathbf{w}}(\eta),{}^{c}\mathbf{D}^{\delta}\bar{\mathbf{w}}(\eta))d\eta\right|\\ &\leqslant \Delta\phi(t)\epsilon+\frac{\Delta\mathscr{K}_{\hbar}}{1-\mathscr{L}_{\hbar}}\|\mathbf{w}-\bar{\mathbf{w}}\|_{\mathscr{M}}.\end{aligned} \tag{4.8}$$

Upon simplification (4.8) gives

$$\|\mathbf{w}-\bar{\mathbf{w}}\|_{\mathscr{M}}\leqslant\mathcal{C}_{\hbar}\phi(t)\epsilon,\ \mathcal{C}_{\hbar}=\frac{\Delta(1-\mathscr{L}_{\hbar})}{1-(\mathscr{L}_{\hbar}+\Delta\mathscr{K}_{\hbar})}. \tag{4.9}$$

Thus the APBVP (1.1) is HUR stable. □

Theorem 4.6. *Under the Hypothesis* (A_5) *and if* $\mathscr{L}_{\hbar}+\Delta\mathscr{K}_{\hbar}<1$ *holds, then the solution of (1.1) is GHUR stable.*

Proof. Just like Theorem 4.5, we have

$$\|\mathbf{w}-\bar{\mathbf{w}}\|_{\mathscr{M}} \leqslant \mathcal{C}_{\hbar}\phi(t),\ \mathcal{C}_{\hbar} = \frac{\Delta(1-\mathscr{L}_{\hbar})}{1-(\mathscr{L}_{\hbar}+\Delta\mathscr{K}_{\hbar})}. \tag{4.10}$$

□

Hence the APBVP (1.1) is GHUR stable.

5. Examples

In this section, we prove suitable examples to illustrate our analysis.

Example 5.1. Consider the APBVP as

$$\begin{cases} {}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t) = \frac{1}{150}\Big[t\cos\mathbf{w}(t) - \mathbf{w}(t)\sin(t)\Big] + \frac{{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)}{100+{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)},\ t\in\mathscr{J}=[0,1], \\ \mathbf{w}(0) = -\mathbf{w}(1),\ {}^{c}\mathbf{D}^{\frac{1}{2}}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{\frac{1}{2}}\mathbf{w}(1),\ {}^{c}\mathbf{D}^{\frac{3}{2}}\mathbf{w}(0) = -{}^{c}\mathbf{D}^{\frac{3}{2}}\mathbf{w}(1). \end{cases} \tag{5.1}$$

Here one has $\delta = \frac{5}{2}$, $r = \frac{1}{2}$, $s = \frac{3}{2}$, $\tau = 1$. Continuity of the function

$$\hbar(t,\mathbf{w}(t),{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)) = \frac{1}{150}\Big[t\cos\mathbf{w}(t) - \mathbf{w}(t)\sin(t)\Big] + \frac{{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)}{100+{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)}$$

is obvious for $\mathbf{w}\in\mathscr{M} = C^3[0,1]$. Again we have using hypothesis (A_4) for any $\mathbf{w},\bar{\mathbf{w}}\in\mathscr{R}$ that

$$\begin{aligned} |\hbar(t,\mathbf{w}(t),{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)) &- \hbar(t,\bar{\mathbf{w}}(t),{}^{c}\mathbf{D}^{\frac{5}{2}}\bar{\mathbf{w}}(t))| \leqslant \frac{1}{150}|t||\cos(\mathbf{w}(t))-\cos(\bar{\mathbf{w}}(t))| \\ &+ \frac{1}{150}|\sin(t)||\mathbf{w}(t)-\bar{\mathbf{w}}(t)| \\ &+ \left|\frac{100[{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t) - {}^{c}\mathbf{D}^{\frac{5}{2}}\bar{\mathbf{w}}(t)]}{[100+{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)][100+{}^{c}\mathbf{D}^{\frac{5}{2}}\bar{\mathbf{w}}(t)]}\right| \\ &\leqslant \frac{1}{150}\Big[|\mathbf{w}-\bar{\mathbf{w}}| + |\mathbf{w}-\bar{\mathbf{w}}|\Big] + \frac{1}{100}\left|{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t) - {}^{c}\mathbf{D}^{\frac{5}{2}}\bar{\mathbf{w}}(t)\right| \\ &= \frac{1}{75}\Big[|\mathbf{w}-\bar{\mathbf{w}}|\Big] + \frac{1}{100}\left|{}^{c}\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t) - {}^{c}\mathbf{D}^{\frac{5}{2}}\bar{\mathbf{w}}(t)\right|. \end{aligned}$$

Hence we have $\mathscr{K}_{\hbar} = \frac{1}{75}$, $\mathscr{L}_{\hbar} = \frac{1}{100}$. On computation we have $\Delta = 1.26098028$. Now thank to Theorem 3.6, we see that

$$\frac{\mathscr{K}_{\hbar}\Delta}{1-\mathscr{L}_{\hbar}} = 1.6982\times 10^{-2} < 1.$$

Thus the APBVP (5.1) has at most one solution. Further by using Theorem 4.2, we observe that

$$\mathscr{L}_{\hbar} + \Delta\mathscr{K}_{\hbar} = 0.0168130 + .01 = 0.0268130704 < 1.$$

Hence the solution is HU stable. Further it is also GHU stable. For HUR stability we thank Theorem 4.5 by taking a nondecreasing function $\phi(t) = t$ for $t \in (0,1)$. One has $\mathcal{C}_{\hbar} = \frac{\Delta(1-\mathcal{L}_{\hbar})}{1-(\mathcal{L}_{\hbar}+\Delta\mathcal{K}_{\hbar})} = 1.7103 \times 10^{-2}$. Hence we see that the results for unique solution $\bar{\mathbf{w}} \in \mathcal{M}$ and any solution $\mathbf{w} \in \mathcal{M}$ the following relation

$$\|\mathbf{w} - \bar{\mathbf{w}}\|_{\mathcal{M}} \leqslant 1.7103 \times 10^{-2}\epsilon t, \text{ for all } t \in [0,1]$$

holds true. Hence the solution of (5.1) is HUR stable. Consequently it is obviously GHUR stable on using Theorem 4.6.

Example 5.2. Let we have the following APBVP

$$\begin{cases} {}^c\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t) = \frac{e^{-\Pi t}}{10} + \frac{e^{-t}}{40+t^2}\Big(\sin(|\mathbf{w}(t)|) + \sin(|{}^c\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)|)\Big), & t \in [0,1] \\ \mathbf{w}(0) = -\mathbf{w}(1), \quad {}^c\mathbf{D}^{\frac{1}{2}}\mathbf{w}(0) = -{}^c\mathbf{D}^{\frac{1}{2}}\mathbf{w}(1), \ {}^c\mathbf{D}^{\frac{3}{2}}\mathbf{w}(0) = -{}^c\mathbf{D}^{\frac{3}{2}}\mathbf{w}(1). \end{cases} \tag{5.2}$$

Here $\delta = \frac{5}{2}$, $r = \frac{1}{2}$, $s = \frac{3}{2}$, $\tau = 1$ and

$$\hbar(t, \mathbf{w}, \beta_{\mathbf{w}}) = \frac{\exp(-\pi t)}{10} + \frac{\exp(-t)}{40+t^2}\Big(\sin(|\mathbf{w}(t)|) + \sin(|{}^c\mathbf{D}^{\frac{5}{2}}\mathbf{w}(t)|)\Big).$$

continuity of $\hbar$ is obvious.
Now for any $\mathbf{w}$, $\bar{\mathbf{w}} \in \mathcal{M}$ and $t \in [0,1]$, we have

$$\begin{aligned} & |\hbar(t, \mathbf{w}(t), \beta_{\mathbf{w}}(t)) - \hbar(t, \bar{\mathbf{w}}(t), \beta_{\bar{\mathbf{w}}}(t))| \\ \leqslant\ & \frac{\exp(-t)}{40+t^2}\Big[|\sin|\mathbf{w}(t)| - \sin|\bar{\mathbf{w}}(t)|| + |\sin|\beta_{\mathbf{w}}(t)| - \sin|\beta_{\bar{\mathbf{w}}}(t)||\Big] \\ \leqslant\ & \frac{1}{40}\Big[|\mathbf{w}(t) - \bar{\mathbf{w}}(t)| + |\beta_{\mathbf{w}}(t) - \beta_{\bar{\mathbf{w}}}(t)|\Big]. \end{aligned}$$

Hence $\hbar$ satisfies the Hypothesis (A_2) with $\mathcal{K}_{\hbar} = \mathcal{L}_{\hbar} = \frac{1}{40}$. The function $\hbar$ also satisfies the Hypothesis (A_3) with $\alpha_1(t) = \frac{\exp(-\pi t)}{10}$, $\alpha_2(t) = \alpha_3(t) = \frac{\exp(-t)}{40+t}$, where $\alpha_1^* = \frac{1}{10}$, $\alpha_2^* = \alpha_3^* = \frac{1}{40}$. Upon calculation, we get

$$\begin{aligned} \Delta &= \frac{1}{\Gamma(\frac{5}{2}+1)} + \frac{\Gamma(2-\frac{1}{2})}{2\Gamma(\frac{5}{2}-\frac{1}{2}+1)} + \frac{(\frac{1}{2}+2)(\Gamma(3-\frac{3}{2}))}{2(2-\frac{1}{2})\Gamma(\frac{5}{2}-\frac{3}{2}+1)} \\ &= 1.26098028. \end{aligned}$$

Thank to Theorem 3.5, we see that $\mu = \frac{\alpha_1^*\Delta}{1-(\alpha_3^*+\alpha_2^*\Delta)} = 0.1336526 < 1$ and therefore, the condition $\alpha_3^* + \alpha_2^*\Delta < 1$ holds true. Thus the given APBVP (5.2) has at least one solution. Further using Theorem 3.6, we see that

$$\frac{\mathcal{K}_{\hbar}\Delta}{1-\mathcal{L}_{\hbar}} = 3.23328 \times 10^{-2} < 1.$$

So criteria for unique solution has been followed. Further by using Theorem 4.2, we observe that

$$\mathcal{L}_{\hbar} + \Delta\mathcal{K}_{\hbar} = 5.6524 \times 10^{-2} < 1.$$

Hence the solution is HU stable. Further it is also GHU stable. For HUR stability we thank Theorem 4.5 by taking a nondecreasing function $\phi(t) = t$ for $t \in (0,1)$. One has $\mathcal{C}_{\hbar} = \frac{\Delta(1-\mathcal{L}_{\hbar})}{1-(\mathcal{L}_{\hbar}+\Delta\mathcal{K}_{\hbar})} = 3.25778 \times 10^{-2}$. Hence we see that the results for unique solution $\mathbf{w} \in \mathcal{M}$ and any solution $\bar{\mathbf{w}} \in \mathcal{M}$ the following relation

$$\|\mathbf{w} - \bar{\mathbf{w}}\|_{\mathcal{M}} \leqslant 1.7103 \times 10^{-2}\epsilon t, \text{ for all } t \in [0,1]$$

holds true. Hence the solution of (5.1) is HUR stable. Consequently it is obviously GHUR stable on using Theorem 4.6.

Example 5.3. Consider the APBVP of FODEs as

$$\begin{cases} {}^c\mathbf{D}^{\frac{7}{3}}\mathbf{w}(t) = \frac{t}{50} + \frac{(t^2+3)}{40}\left(\sqrt{(|\mathbf{w}(t)|)} + {}^c\mathbf{D}^{\frac{7}{3}}(\mathbf{w}(t))\right), \ t \in [0,2], \\ \mathbf{w}(0) = -\mathbf{w}(2), \ {}^c\mathbf{D}^{\frac{1}{3}}\mathbf{w}(0) = -{}^c\mathbf{D}^{\frac{1}{3}}\mathbf{w}(2), \ {}^c\mathbf{D}^{\frac{4}{3}}\mathbf{w}(0) = -{}^c\mathbf{D}^{\frac{4}{3}}\mathbf{w}(2). \end{cases} \tag{5.3}$$

Here $\delta = \frac{7}{3}$, $r = \frac{1}{3}$, $s = \frac{4}{3}$, $\tau = 2$ and

$$\hbar(t, \mathbf{w}(t), \beta_{\mathbf{w}}(t)) = \frac{t}{50} + \frac{(t^2+3)}{90}\left(\sqrt{|\mathbf{w}(t)|} + {}^c\mathbf{D}^{\frac{9}{4}}(\mathbf{w}(t))\right).$$

Let for any $\mathbf{w}, \bar{\mathbf{w}} \in \mathcal{R}$ and $t \in [0,2]$, one has

$$\begin{aligned} |\hbar(t, \mathbf{w}(t), \beta_{\mathbf{w}}(t)) - \hbar(t, \bar{\mathbf{w}}(t), \beta_{\bar{\mathbf{w}}}(t))| &= \left|\frac{t^2+1}{90}\left(\sqrt{|\mathbf{w}(t)|} - \sqrt{|\bar{\mathbf{w}}(t)|} + \beta_{\mathbf{w}}(t) - \beta_{\bar{\mathbf{w}}}(t)\right)\right| \\ &\leqslant \frac{2}{90}(|\mathbf{w} - \bar{\mathbf{w}}| + |\beta_{\mathbf{w}}(t) - \beta_{\bar{\mathbf{w}}}(t)|). \end{aligned}$$

Hence $\hbar$ satisfies the Hypothesis (A_2) with $\mathcal{K}_{\hbar} = \mathcal{L}_{\hbar} = \frac{1}{45}$. The function $\hbar$ also satisfies the Hypothesis (A_3) with $\alpha_1(t) = \frac{1}{50}$, $\alpha_2(t) = \alpha_3(t) = \frac{t^2+1}{45}$, where $\alpha_1^* = \frac{1}{50}$, $\alpha_2^* = \alpha_3^* = \frac{1}{45}$. Upon computation, we can arrive that $\alpha_3^* + \alpha_2^*\Delta < 1$ and $\frac{\mathcal{K}_{\hbar}}{1-\mathcal{L}_{\hbar}}\Delta < 1$. Thus on using Theorem 3.6, the solution of the problem is unique. Moreover it also satisfies the condition of HU stability and consequently GHU stability by computing $\mathcal{L}_{\hbar} + \mathcal{K}_{\hbar}\Delta < 1$ by using Theorem 4.2. Taking a nondecreasing function $\phi(t) = 1 + t$, and using hypothesis (A_5) the solution of the consider problem (5.3) is HUR stable and hence GHUR stable upon the application of Theorem 4.5 and Theorem 4.6 respectively.

6. Concluding remarks

A stability theory of HU type as well as existence theory of solutions has been successfully established for a class of nonlinear APBVP. The obtained results have been demonstrated by proper examples. The aforesaid analysis in future can be carried out for more general and complicated problem of applied nature.

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