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Solvability of Generalized Fractional Hybrid Differential Inclusions in Banach Algebras

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Abstract

This research paper study the solvability of hybrid fractional differential inclusions involving generalized Caputo fractional derivative with boundary conditions under certain conditions. The existence theorems are proved by using hybrid fixed–point approach in Banach algebras of Dhage, which he presented in 2006. An example, lastly, is proposed to check the efficiency of the above-mentioned theorems. The results are novel and provide extensions to some of the findings known in the literature.

Keywords: differential inclusions, boundary conditions, fixed–point, generalized Caputo derivative, Banach algebra.
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1. Introduction

Fractional calculus is one of the greatest branches in pure and applied mathematics. It is started from some 1695 with the ideas of Leibniz [1]. During the recent decades, fractional differential equations have attracted the attention of many researchers in different fields, such as engineering, physics, finance, biology, chemistry, economy, etc. [2, 3, 4, 5]. The reason is that fractional order differential equations can model mathematically the real life events more accurately than these restricted to integer order [6, 7].

On the other hand, fractional calculus is important because it extends classical differentiation and integration to non–integer orders, offering powerful tools for modeling complex systems in various fields including: finance, control theory , signal processing, Physics, and Engineering [8].

Recently, fractional calculus play an important role in developing deep machine learning techniques. Recent literature highlights the growing interest in this interdisciplinary field. For example, in a survey paper [9], the authors provided a comprehensive review

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of the use of fractional calculus in various fractional artificial neural network (FANN) architectures, training algorithms, stability analysis, and real-world applications. In paper [10], the authors considered the application of fractional calculus in computer vision tasks, including denoising using the Fractional Optimal Control Network (FOCNet) architecture. These studies demonstrate the potential of fractional calculus to enhance the performance and capabilities of deep machine learning techniques in handling complex, memory-dependent, and non-local phenomena.

Differential inclusion is a mathematical concept that generalizes ordinary differential equations to handle problems with discontinuous or multi-valued functions, making it important for modeling complex systems and providing solutions where a single-valued function is insufficient. It plays a crucial role in areas like control theory, dynamical systems, and optimization, allowing for the analysis of systems where a transition between states can be abrupt or uncertain [11, 12, 13, 14]. For example, differential inclusions are used in flight dynamics to model systems with non-smooth characteristics or in economics to represent macrosystems with discontinuous behaviors [15].

In recent years , many researchers studied the solvability of fractional differential inclusions. In [16], Ahmad and Ntouyas studied the solvability of Hadamard fractional differential inclusion, namely

$$\begin{cases} D_H^{\alpha}[\frac{\eta(t)}{f(t,\eta(t))}] \in F(t,\eta(t)),\ 1 < t < e,\ 1 < \alpha \leqslant 2, \\ \eta(1) = \eta(e) = 0, \end{cases} \tag{1.1}$$

where D_H^{α} is the Hadamard fractional derivative, $f : [1, e] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $F : I \times \mathbb{R} \to P(\mathbb{R})$ is several–valued mapping, and $P(\mathbb{R}) = \{Z \subset \mathbb{R} : Z \neq \mathbb{R}\}$. In 2017, Abbas and Benchohra [17] investigated the solvability of Hilf–Hadamard random fractional differential inclusions by using multi-valued random fixed–point of Dhage [18]. In 2019, Alaidarous et al. [19] used semigroup theory together with the Bohnenblust–Karlin fixed–point principle to study the solvability and stability of neutral functional ordinary differential inclusions with delay. In the same year, Samei et al. [20] explored the existence results for hybrid Caputo—Hadamard fractional differential inclusions in Banach algebra by using Dhage hybrid fixed–point approach [21]. In 2020, El-Sayed and Al-Issa [22] used Schauder fixed–point approach to study the solvability of integro–fractional differential inclusions.

In the above cited articles, classic Caputo and Hadamard derivatives were utilized. Nowadays, Almeida [23] proposed a Caputo kind fractional derivative operator with respect to another function, called generalized Caputo derivative. This kind of differentiation depends on a kernel ϕ, and by particular choosing ϕ, one of the well-known fractional operators like Riemann-Liouville, Caputo, Hadamard, Hadamard-Caputo, Erdely Kober is obtained. For some recent results on generalized Caputo type fractional differential equations, see [24, 25, 26, 27, 28, 29, 30].

In a recent work [31], Belmor et al. considered the solvability of fractional differential inclusion including generalized Caputo derivative, given by

$$\begin{cases} {}^*D_{0^+,\phi}^{\alpha}\ \rho(t) \in W(t,\ \rho(t)), \\ \alpha \in (1,2),\ t \in [0,T],\ T > 0, \end{cases} \tag{1.2}$$

equipped with integral boundary value conditions, where ${}^*D_{0^+,\phi}^{\alpha}$ is a generalized Caputo derivative w.r.t. the function ϕ and $W : I \times \mathbb{R} \to P(\mathbb{R})$. They applied endpoint theory to obtain the existence results.

Motivated by these papers, in particular systems (1.1) and (2.1), we consider the hybrid generalized fractional differential with boundary conditions, given by

$$\begin{cases} {}^{*}D^{\alpha}_{0^+,\phi}\left(\frac{\rho(t)}{V(t,\rho(t))}\right) \in W(t, \rho(t), \ a.e., \ t \in I = [0, a], \ a > 0, \\ \rho(0) = \rho(a) = 0, \end{cases} \tag{1.3}$$

where ${}^{*}D^{\alpha}_{0^+,\phi}$ is a generalized Caputo derivative w.r.t. the function ϕ such that $\alpha \in (1, 2)$, $V \in C(I \times \mathbb{R}, \ \mathbb{R} \setminus \{0\})$ and $W : I \times \mathbb{R} \to P(\mathbb{R})$ is several–valued mapping. We obtain the existence results for a system (1.3) by using the hybrid fixed-point approach in Banach algebras due to Dhage [21].

An outline of this manuscript is as follows. In Section 2, some preliminaries on multi-valued analysis and fractional operators are presented. In Section 3, we study the solvability of the generalized system (1.3). In Section 4, we provide an example to illustrate the abstract results.

2. Preliminaries

In this section, we will recall some concepts and facts that will be needed in the sequel. Let $(X, \|.\|)$ be a Banach space and $P(X) = \{Z \subset X : Z \neq \emptyset\}$. Now, throughout this manuscript, let

$$P_{cl}(X) = \{F \in P(X) : F \text{ is closed}\},$$

$$P_{b}(X) = \{F \in P(X) : F \text{ is bounded}\},$$

$$P_{cp}(X) = \{F \in P(X) : F \text{ is compact}\},$$

$$P_{cpv}(X) = \{F \in P(X) : F \text{ is a convex and compact}\}.$$

The multi–valued map $\Psi : X \to P(X)$ is convex (closed) valued if $\Psi(\rho)$ is convex (closed) for each $\rho \in X$. Furthermore, Ψ is bounded on bounded sets if $\Psi(U) = \bigcup_{\rho \in U} \Psi(\rho)$ is bounded in X for all $U \in P_b(X)$, (i.e. $\sup_{\rho \in U}\{\sup\{|\Lambda| : \Lambda \in \Psi(\rho)\}\} < \infty$). The map Ψ is called upper semi-continuous (u.s.c.) on X if for each $\rho \in X$, the set $\Psi(\rho)$ is nonempty closed subset of X and if each open set N of X containing $\Psi(\rho)$, there exists an open neighborhood N_0 of ρ such that $\Psi(N_0) \subseteq N$. The map Ψ is called completely continuous if $\Psi(U)$ is relatively compact for every $U \in P_b(X)$. If Ψ is completely continuous with nonempty compact values, then Ψ is u.s.c. if Ψ has a closed graph, that is, $\rho_n \to \rho^*$, $\xi_n \to \xi^*$ and $\xi_n \in \Psi(\rho_n)$ implies that $\xi^* \in \Psi(\rho^*)$. Ψ has a fixed point if there exists $\rho \in X$ such that $\rho \in \Psi(\rho)$. The fixed point set of a multi–valued operator Ψ will be denoted by Fix Ψ. A multi–valued map $\Psi : I \to P_{cl}(\mathbb{R})$ is called measurable if for each $\rho \in \mathbb{R}$, the map $t \mapsto d(\rho, \Psi(t)) = \inf\{| \rho - \xi | : \xi \in \Psi(t)\}$ is measurable.

Let $L^1(I, X) = \{\rho : I \to X \mid \|\rho\| : I \to R_+ \text{ be Lebesgue integrable}\}$, then $L^1(I, X)$ is Banach with the norm $\|\rho\|_{L^1} = \int_I \|\rho(t)\| \, dt$.

Definition 2.1. [32] A multi–valued function $\Psi : I \times \mathbb{R} \to P(\mathbb{R})$ is called Carathéodory if
(i) $t \to \Psi(t, \rho)$ is measurable for each $\rho \in \mathbb{R}$,
(ii) $\rho \to \Psi(t, \rho)$ is u.s.c. for almost all $t \in I$.
Furthermore, a Carathéodory map Ψ is called L^1-carathéodory if
(iii) there exists a function $\Theta \in L^1(I, \mathbb{R}_+)$ such that

$$\|\Psi(t, \rho)\| = \sup\{|\ \xi\ |: \xi \in \Psi(t, \rho)\} \leqslant \Theta(t)\ ,\ \text{for each}\ \ \rho \in \mathbb{R} \text{ and for a.e. } t \in I.$$

Definition 2.2. [33] Let $C(I, E)$ be the Banach space of all continuous functions $\rho : I \to E$ with the norm $\|\rho\|_\infty = \sup_{t \in I} \|\rho(t)\|_E$. Therefore, $C^n(I, E)$ is the Banach space of all $n-$differentiable maps $\rho : I \to E$ with $\rho^{(n)}(t) \in C(I, E)$, $n \in \mathbb{N}$.

Definition 2.3. [34] For each $\rho \in C(I, \mathbb{R})$, define the set of selection of $\Psi : I \times \mathbb{R} \to P_{cpv}(\mathbb{R})$ as

$$S_{\Psi,\rho} = \{\psi \in L^1(I, \mathbb{R}) : \psi(t) \in \Psi(t, \rho(t)) \text{ for a.e. } t \in I\} \neq \emptyset.$$

Lemma 2.4. *[34] Let $\Psi : I \times \mathbb{R} \to P_{cpv}(X)$ be an L^1-Carathéodory-multi-valued function and $\Phi : L^1(I, X) \to C(I, E)$ be a continuous linear mapping. Then*

$$\Phi \circ S_\Psi : C(I, X) \to P_{cpv}(C(I, X)), \rho \mapsto \Phi(S^1_{\Psi,\rho}),$$

is a closed graph map in $C(I, X) \times C(I, X)$.

Fixed–point principle plays a major role to investigate different types of differential equations. The existence results of the proposed system are based on the following fixed–point theorem of Dhage.

Theorem 2.5. *[21] Let X be a Banach algebra, $\vartheta : X \to X$ be a single-valued and $\Psi : X \to P_{cpv}$ be a multi–valued mapping, satisfying*
(1) ϑ is single value Lipschitz with a Lipschitz constant ℓ;
(2) Ψ is u.s.c. and compact;
(3) $2\ell M < 1$ where $M = \|\Psi(X)\|$.
Then either
(a) the operator inclusion $\rho \in \vartheta\rho\Psi\rho$ has a solution, or
(b) the set $\Upsilon = \{\xi \in X : \lambda\xi \in \vartheta\xi\Psi\zeta,\ \lambda > 1\}$ is unbounded.

Definition 2.6. [23] Let $\alpha \in (n, n+1)$ where $n \in \{0, 1, 2, 3, ...\}$, $\eta \in C([a, b], \mathbb{R})$ and $\varphi \in C^1([a, b], \mathbb{R})$ be an increasing function with $\varphi'(t) \neq 0$ for all $t \in [a, b]$. The left Riemann-Liouville fractional integral of order α of a function η w.r.t. a function φ is defined as

$$J^{\alpha}_{a^+,\varphi}\ \eta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(\zeta)(\varphi(t) - \varphi(\zeta))^{\alpha-1}\ \eta(\zeta)\ d\zeta. \tag{2.1}$$

It is clear that when $\varphi(\zeta) = \zeta$, (2.1) is the classic Riemann-Liouville fractional integral and when $\varphi(\zeta) = \ln \zeta$, (2.1) is the Hadamarad fractional integral [35].

Therefore, the left Riemann-Liouville fractional derivative of order α of a function η w.r.t. a function $\varphi \in C^n([a, b], \mathbb{R})$ is given by

$$D^{\alpha}_{a^+,\varphi}\ \eta(t) = \frac{1}{\Gamma(n-\alpha)}\ \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \int_a^t \varphi'(\zeta)(\varphi(t) - \varphi(\zeta))^{n-\alpha-1}\ \eta(\zeta)\ d\zeta. \tag{2.2}$$

Putting $\phi(\zeta) = \zeta$ in (2.2), we have the classic Riemann-Liouville derivative and putting $\phi(\zeta) = \ln\zeta$ in (2.2), we have the Hadamarad fractional derivative [36].

Definition 2.7. [24] Let $\alpha \in (n, n+1)$ for some $n \in \{0, 1, 2, ...\}$, $\eta \in C^n(I, \mathbb{R})$, and $\phi \in C^n(I, \mathbb{R})$ be an increasing mapping such that $\phi'(t) \neq 0$ for all $t \in I$. The left-sided generalized Caputo fractional derivative of order α of a function η w.r.t. a function ϕ is defined as

$$ {}^*D^{\alpha}_{0^+,\phi}\ \eta(t) = D^{\alpha}_{0^+,\phi}\ [\eta(t) - \sum_{k=0}^{n-1} \frac{\eta^{[k]}_{\phi}(0)}{k!}(\phi(t) - \phi(0))^k], \tag{2.3}$$

where $\eta^{[k]}_{\phi}(t) = (\frac{1}{\phi'(t)}\frac{d}{dt})^k\ \eta(t)$. If we take $\phi(\zeta) = \zeta$, (2.3) is the classic Caputo derivative and if we take $\phi(\zeta) = \ln\zeta$, (2.3) is the Caputo–Hadamarad fractional derivative [36, 37, 38].

Lemma 2.8. *[24] Let $\vartheta : I \to \mathbb{R}$, then*
(1) if $\vartheta \in C(I, \mathbb{R})$, then ${}^*D^{\alpha}_{0^+,\phi}\ J^{\alpha}_{0^+,\phi}\ \vartheta(t) = \vartheta(t)$,
(2) if $\vartheta \in C^n(I, \mathbb{R})$ then

$$J^{\alpha}_{0^+,\phi}\ {}^*D^{\alpha}_{0^+,\phi}\ \vartheta(t) = \vartheta(t) - \sum_{k=0}^{n-1} \frac{\vartheta^{[k]}_{\phi}(0)}{k!}(\phi(t) - \phi(0))^k.$$

3. Main results

The existence of the solution of the proposed system (1.3) is given in this partition. A function $\rho \in AC^2(I, \mathbb{R})$ is called a solution of the proposed system (1.3) if there exists a mapping $\psi \in L^1(I, \mathbb{R})$ with $\psi(t) \in W(t, \rho(t))$ a.e. on I such that

$$\begin{cases} {}^*D^{\alpha}_{0^+,\phi}\ (\frac{\rho(t)}{V(t, \rho(t))}) = \psi(t,\ \rho(t)),\ \text{a.e. on } I, \\ \rho(0) = \rho(a) = 0,\ \alpha \in (1, 2). \end{cases} \tag{3.1}$$

Lemma 3.1. *The integral solution of system (3.1) is given by*

$$\begin{aligned} \rho(t) = V(t, \rho(t)).[&\frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}\psi(s)ds \\ &- \frac{(\phi(t) - \phi(0))}{(\phi(a) - \phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}\psi(s)ds]. \end{aligned} \tag{3.2}$$

Proof. Applying the operator $J^{\alpha}_{0^+,\phi}$ to both sides of (3.1) and using Lemma 2.8, we get

$$\rho(t) = V(t, \rho(t)).[J^{\alpha}_{0^+,\phi}\psi(t) + \zeta_0 + \zeta_1(\phi(t) - \phi(0))], \tag{3.3}$$

where ζ_0 and ζ_1 are constants. Since $\rho(0) = 0$, we get $\zeta_0 = 0$. Additionally from $\rho(a) = 0$, we get

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}\psi(s)ds + \zeta_1(\phi(a) - \phi(0)).$$

Hence, we have

$$\zeta_1 = -\frac{1}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds.$$

Substituting ζ_0 and ζ_1 in equation (3.3), we obtain equation (3.2).□ □

The existence results for the system (1.3) will be obtain under the following assumptions:

(A1) The function $V : I \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous and there exists a bounded function ϑ, with bound $\|\vartheta\|$, such that $\vartheta(t) > 0$ a.e. $t \in I$ and

$$|V(t,\rho(t)) - V(t,\eta(t))| \leqslant \vartheta(t)|\rho(t)-\eta(t)|, \text{ a.e. } t \in I, \ \forall \rho,\eta \in \mathbb{R}.$$

(A2) The multi–valued function $W : I \times \mathbb{R} \to P(\mathbb{R})$ is L^1–Carathéodory and has nonempty compact and convex values.

(A3) There exists a function $h \in L^1(I,\mathbb{R}_+)$ and a constant $M > 0$ such that $\|W(t,\eta(t))\| = \sup\{|\psi| : \psi \in W(t,\rho)\} \leqslant h(t)$ a.e. $t \in I$ and

$$M = \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}h(s)ds.$$

(A4) There exists $K \in \mathbb{R}_+$ such that

$$K > \frac{(2\omega_0/\Gamma(\alpha))M}{1-(2\|\vartheta\|_\infty/\Gamma(\alpha))M},$$

where $(2\|\vartheta\|_\infty/\Gamma(\alpha))M < \frac{1}{2}$ and $\omega_0 = \sup_{t\in I}|V(t,0)|$.

Theorem 3.2. *Under the assumptions (A1)-(A4), system (1.3) has at least one solution on I.*

Proof. Let $X = C(I,\mathbb{R})$. For all $\rho \in X$, the set of selection of W is defined as

$$S_{W,\rho} = \{\psi \in L^1(I,\mathbb{R}) : \psi(t) \in W(t,\rho(t)) \text{ for a.e. } t \in I\}.$$

Define the operator $H : X \to P(\mathbb{R})$ as

$$\begin{aligned} H\rho(t) =& \{\eta \in C(I,\mathbb{R}) : \eta(t) = V(t,\rho(t)).[\frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds \\ &- \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds] : \psi \in S_{W,\rho}\}. \end{aligned} \tag{3.4}$$

Define also $\mathcal{A} : X \to X$ as $\mathcal{A}\ \rho(t) = V(t, \rho(t)),\ t \in I$ and define $\mathcal{B} : X \to P(\mathbb{R})$ as

$$\begin{aligned} \mathcal{B}\ \rho(t) = \{\eta \in X : \eta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}\psi(s)ds - \\ \frac{(\phi(t) - \phi(0))}{(\phi(a) - \phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}\psi(s)ds : \psi \in S_{W,\rho}\}. \end{aligned} \tag{3.5}$$

Hence $H(\rho) = \mathcal{A}\ \rho\ \mathcal{B}\ \rho$. The proof is splitted into several steps.

Step 1. $\mathcal{A}$ **is Lipschitz on** X. Let $\rho, \eta \in C(I, \mathbb{R})$. From (A1), we get

$$|\mathcal{A}\ \rho(t) - \mathcal{A}\ \eta(t)| \leqslant \vartheta(t)|\rho(t) - \eta(t)| \leqslant \|\vartheta\|_\infty \|\rho - \eta\|_\infty, \tag{3.6}$$

for each $t \in I$. Therefore, we have

$$\|\mathcal{A}\ \rho(t) - \mathcal{A}\ \eta(t)\|_\infty \leqslant \|\vartheta\|_\infty \|\rho - \eta\|_\infty. \tag{3.7}$$

Thus, $\mathcal{A}$ is Lipschitz on $C(I, \mathbb{R})$ with Lipschitz constant $\|\vartheta\|_\infty$.

Step 2. $\mathcal{B}$ **is compact and u.s.c. on** X. First, we prove that $\mathcal{B}$ has convex value. Let $\rho_1, \rho_2 \in \mathcal{B}\ \rho$. Then there exists $\psi_1, \psi_2 \in S_{W,\rho}$ such that

$$\begin{aligned} \rho_i(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}\psi_i(s)ds - \tfrac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}\psi_i(s)ds, \\ i = 1, 2,\ t \in I. \end{aligned} \tag{3.8}$$

For any $\lambda \in [0, 1]$, we get

$$\begin{aligned} \lambda\rho_1(t) + (1-\lambda)\rho_2(t) &= \tfrac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}[\lambda\psi_1(s) + (1-\lambda)\psi_2(s)]ds \\ &\quad - \tfrac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}[\lambda\psi_1(s) + (1-\lambda)\psi_2(s)]ds \\ &= \tfrac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}\bar{\psi}(s)ds \\ &\quad - \tfrac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}\bar{\psi}(s)ds, \end{aligned} \tag{3.9}$$

where $\bar{\psi}(s) = [\lambda\psi_1(s) + (1-\lambda)\psi_2(s)] \in W(t, \rho(t))$ for each $t \in I$. Thus, $\lambda\rho_1(t) + (1-\lambda)\rho_2(t) \in \mathcal{B}\ \rho$. Hence $\mathcal{B}\ \rho$ is convex for each $\rho \in C(I, \mathbb{R})$ and $\mathcal{B}$ define as a multi–valued operator $\mathcal{B} : C(I, \mathbb{R}) \to P_{cv}(\mathbb{R})$. Next, we prove that $\mathcal{B}$ maps each bounded set into bounded set. Let $r > 0$ and $Q_r = \{\rho \in C(I, \mathbb{R}) : \|\rho\| \leqslant r\}$. Let $\eta \in B\ \rho$, then there exists $\psi \in S_{W,\rho}$ such that

$$\eta(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}\psi(s)ds - \tfrac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a) - \phi(s))^{\alpha-1}\psi(s)ds. \tag{3.10}$$

Then, for all $t \in I$, we get

$$\begin{aligned} |\mathcal{B}\ \rho(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds - \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds\Big| \\ &\leqslant \frac{2}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}h(s)ds \leqslant \frac{2M}{\Gamma(\alpha)}. \end{aligned} \tag{3.11}$$

Therefore, we have $\|\eta\|_\infty \leqslant \frac{2M}{\Gamma(\alpha)}$. Next, we prove that $\mathcal{B}$ maps bounded sets into equicontinuous sets. For all $t_1, t_2 \in I$ such that $t_1 < t_2$ we get

$$\begin{aligned} |\eta(t_2)-\eta(t_1)| &\leqslant \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \phi'(s)(\phi(t_1)-\phi(s))^{\alpha-1}\psi(s)ds \\ &-\frac{(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds\Big| \\ &\leqslant \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \phi'(s)(\phi(t_1)-\phi(s))^{\alpha-1}\psi(s)ds\Big| \\ &+\Big|\frac{(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds\Big| \\ &\leqslant \Big|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds\Big| \\ &+\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \phi'(s)|(\phi(t_2)-\phi(s))^{\alpha-1}-(\phi(t_1)-\phi(s))^{\alpha-1}||\psi(s)|ds \\ &+\frac{|\phi(t_2)-\phi(t_1)|}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|h(s)|ds. \end{aligned} \tag{3.12}$$

From Lagrange mean value theorem, we get

$$\begin{aligned} |\eta(t_2)-\eta(t_1)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} |\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)|ds+ \\ &\frac{(\alpha-1)(t_2-t_1)\phi'(\hat{t})}{\Gamma(\alpha)}\int_0^{t_1} \phi'(s)(\phi(\hat{t})-\phi(s))^{\alpha-2}|\psi(s)|ds \\ &+\frac{|\phi(t_2)-\phi(t_1)|}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|h(s)|ds, \end{aligned} \tag{3.13}$$

where $t_1 < \hat{t} < t_2$. As $t_1 \to t_2$, then $|\eta(t_2)-\eta(t_1)| \to 0$. Thus, $\mathcal{B}(Q_r)$ is equi-continuous. Hence, $\mathcal{B}$ is completely continuous.

Now, we prove that $\mathcal{B}$ has a closed graph. Let $\rho_n \to \rho^*$, $\eta_n \in \mathcal{B}(\rho_n)$ and $\eta_n \to \eta^*$. Then we need to prove $\eta^* \in \mathcal{B}(\rho^*)$. Associated with $\eta_n \in \mathcal{B}(\rho_n)$, there exists $\psi_n \in S_{W,\rho_n}$ such that, for each $t \in I$

$$\eta_n(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi_n(s)ds - \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi_n(s)ds. \tag{3.14}$$

This it suffices to prove that there exists $\psi^* \in S_{W,\rho^*}$ such that for each $t \in I$,

$$\eta^*(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi^*(s)ds - \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi^*(s)ds. \tag{3.15}$$

Consider the linear operator $\Phi : L^1(I,\mathbb{R}) \to C(I,\mathbb{R})$ given by

$$\begin{aligned} \eta \mapsto \Phi(\psi)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds \\ &- \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds. \end{aligned}$$

Observe that

$$\begin{aligned} \|\eta_n(t)-\eta^*(t)\|_\infty &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\|\psi_n(s)-\psi^*(s)\|_\infty ds \\ &+ \frac{\|\phi(t)-\phi(0)\|_\infty}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\|\psi_n(s)-\psi^*(s)\|_\infty ds. \end{aligned} \tag{3.16}$$

Hence, $\|\eta_n(t)-\eta^*(t)\|_\infty \to 0$ as $n \to \infty$. Thus, $\Phi \circ S_W$ is a closed graph operator. Therefore $\eta_n(t) \in \Phi \circ S_W$. Since $\rho_n \to \rho^*$, then we get

$$\eta^*(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi^*(s)ds - \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi^*(s)ds, \tag{3.17}$$

for some $\psi^* \in S_{W,\rho^*}$. Thus, $\mathcal{B}$ is compact and u.s.c. on X.

Step 3. Prove that $2N\ell < 1$**.** Since $N = \|\mathcal{B}(X)\| = \sup\{|\mathcal{B}\ \rho| : \rho \in X\} \leqslant \frac{2M}{\Gamma(\alpha)}$ and $\ell = \|\vartheta\|_\infty$, we prove that the conclusion (b) of Theorem 2.5 is not possible. Let $\eta \in H\rho$, $\lambda > 1$ and $\lambda\eta \in \mathcal{A}\ \eta(t)\ \mathcal{B}\ \eta(t)$. Then there exists $\psi \in S_{W,\rho}$ such that for any $\lambda > 1$

$$\begin{aligned} \eta(t) &= \lambda^{-1}[V(t,\eta(t))][\frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds \\ &- \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds], \end{aligned} \tag{3.18}$$

for all $t \in I$. Then we get

$$\begin{aligned} |\eta(t)| &\leqslant \lambda^{-1}|V(t,\eta(t))|[\frac{1}{\Gamma(\alpha)} \int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}|\psi(s)|ds \\ &+ \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)} \int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|\psi(s)|ds] \\ &\leqslant \lambda^{-1}[\|\vartheta\|_\infty\|\eta\|_\infty + \omega_0]\frac{2M}{\Gamma(\alpha)}. \end{aligned} \tag{3.19}$$

Let $\|\eta\|_\infty = K$, then we have

$$K \leqslant \lambda^{-1}[\|\vartheta\|_\infty K + \omega_0]\frac{2M}{\Gamma(\alpha)} \leqslant [\|\vartheta\|_\infty K + \omega_0]\frac{2M}{\Gamma(\alpha)}. \tag{3.20}$$

Therefore, we get

$$K \leqslant \frac{2\omega_0 M/\Gamma(\alpha)}{1-(2\|\vartheta\|_\infty M/\Gamma(\alpha))}. \quad (3.21)$$

Thus, condition (b) of Theorem 2.5 does not hold. Thus, the operator H and therefore problem (1.3) have the solution on I. □

□

Theorem 3.3. *Let assumptions (A1)and (A2) hold. In addition, one assumes that*

(A5) There exists a nondecreasing continuous function $\Theta : [0,\infty) \to [0,\infty)$ *and* $p \in C(I, \mathbb{R}_+)$,

$$\|W(t,\rho)\| = \sup\{|\eta| : \eta \in W(t,\rho)\} \leqslant p(t)\Theta(|\rho|),$$

for all $(t,\rho) \in I \times \mathbb{R}$.

(A6) There exists $R > 0$ *such that*

$$R > \frac{\omega_0[\frac{2\|p\|_\infty \Theta(R)}{\Gamma(\alpha+1)}](\phi(a)-\phi(0))^\alpha}{1-\|\vartheta\|_\infty(\frac{2\|p\|_\infty\Theta(R)}{\Gamma(\alpha+1)}(\phi(a)-\phi(0))^\alpha)},$$

where

$$\|\vartheta\|_\infty(\frac{2\|p\|_\infty\Theta(R)}{\Gamma(\alpha+1)})(\phi(a)-\phi(0))^\alpha < \frac{1}{2}.$$

Then system (1.3) has at least one solution on I.

Proof. Define the operators H, $\mathcal{A}$ and $\mathcal{B}$ as in Theorem 3.2.

Step 1. $\mathcal{A}$ **is Lipschitz on** $C(I,\mathbb{R})$**.** By doing the same steps as in Theorem 10, we obtain that $\mathcal{A}$ is Lipschitz on $C(I,\mathbb{R})$ with Lipschitz constant $\|\vartheta\|$.

Step 2. $\mathcal{B}$ **is compact and u.s.c. on** $C(I,\mathbb{R})$**.** By doing the same steps as in Theorem 3.2, we obtain that

$\mathcal{B}$ has convex value. Next, we prove that $\mathcal{B}$ maps each bounded set into a bounded sets. Let $r > 0$ and $Q_r = \{\rho \in C(I,\mathbb{R}) : \|\rho\| \leqslant r\}$. Let $\eta \in B\ \rho$, then there exists $\psi \in S_{W,\rho}$ such that

$$\eta(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds - \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds. \quad (3.22)$$

Then, for all $t \in I$, we have

$$\begin{aligned}
&|\mathcal{B}\ \rho(t)| \\
&= |\frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds - \frac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}|\psi(s)|ds + \frac{(|\phi(t)-\phi(0)|)}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|\psi(s)|ds \\
&\leqslant \frac{\Theta(|\rho|)}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}|p(s)|ds + \frac{\Theta(|\rho|)(|\phi(t)-\phi(0)|)}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|p(s)|ds \\
&\leqslant \frac{2\Theta(|\rho|)\|p\|_\infty}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}ds \leqslant \frac{2\Theta(|\rho|)\|p\|_\infty}{\Gamma(\alpha+1)}(\phi(a)-\phi(0))^\alpha.
\end{aligned} \quad (3.23)$$

Therefore, we have $\|\eta\|_\infty \leqslant \frac{2\Theta(|\rho|)\|p\|_\infty}{\Gamma(\alpha+1)}(\phi(a)-\phi(0))^\alpha$. Next, we prove that $\mathcal{B}$ maps bounded sets into equicontinuous sets. For all $t_1, t_2 \in I$ such that $t_1 < t_2$ we get

$$\begin{aligned}
|\eta(t_2)-\eta(t_1)| &\leqslant |\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds - \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\phi'(s)(\phi(t_1)-\phi(s))^{\alpha-1}\psi(s)ds - \\
&\tfrac{(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^{a}\phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds| \\
&\leqslant |\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds - \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds \\
&+\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}\psi(s)ds - \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\phi'(s)(\phi(t_1)-\phi(s))^{\alpha-1}\psi(s)ds| \\
&+|\tfrac{(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^{a}\phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds| \\
&\leqslant \tfrac{\Theta(|\rho|)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}|p(s)|ds \\
&+\tfrac{\Theta(|\rho|)}{\Gamma(\alpha)}\int_0^{t_1}\phi'(s)|(\phi(t_2)-\phi(s))^{\alpha-1}-(\phi(t_1)-\phi(s))^{\alpha-1}||p(s)|ds \\
&+|\tfrac{\Theta(|\rho|)(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))\Gamma(\alpha)}|\int_0^{a}\phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|p(s)|ds \\
&\leqslant \tfrac{\Theta(|\rho|)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}|p(s)|ds \\
&+\tfrac{\|p\|_\infty\Theta(|\rho|)}{\Gamma(\alpha+1)}[(\phi(t_2)^\alpha-\phi(t_1)^\alpha)-(\phi(t_2)-\phi(t_1))^\alpha] \\
&+|\tfrac{\Theta(|\rho|)(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))\Gamma(\alpha)}|\int_0^{a}\phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|p(s)|ds \\
&\leqslant \tfrac{\Theta(|\rho|)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha-1}|p(s)|ds \\
&+\tfrac{\|p\|_\infty\Theta(|\rho|)}{\Gamma(\alpha+1)}[\phi(t_2)^\alpha-\phi(t_1)^\alpha] \\
&+|\tfrac{\Theta(|\rho|)(\phi(t_2)-\phi(t_1))}{(\phi(a)-\phi(0))|\Gamma(\alpha)}|\int_0^{a}\phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\mid p(s)|ds,
\end{aligned} \tag{3.24}$$

where $t_1 < \hat{t} < t_2$. As $t_1 \to t_2$, then $|\eta(t_2)-\eta(t_1)| \to 0$. Thus, $\mathcal{B}(Q_r)$ is equicontinuous. Hence, $\mathcal{B}$ is completely continuous.

Now, we prove that $\mathcal{B}$ has a closed graph. By doing the same steps as in Theorem 3.2, we obtain that $\mathcal{B}$ is compact and u.s.c. on X.

Step 3. Prove that $2N\ell < 1$. Since $N = \|\mathcal{B}(X)\| = \sup\{|\mathcal{B}\ \rho| : \rho \in X\} \leqslant \frac{2\Theta(|\rho|)\|p\|_\infty}{\Gamma(\alpha+1)}(\phi(a)-\phi(0))^\alpha$ and $\ell = \|\vartheta\|_\infty$, we prove that the conclusion (b) of Theorem 2.5 is not possible. Let $\eta \in H\rho$, $\lambda > 1$ and $\lambda\eta \in \mathcal{A}\ \eta(t)\ \mathcal{B}\ \eta(t)$. Then there exists $\psi \in S_{W,\rho}$ such that for any $\lambda > 1$

$$\begin{aligned}
\eta(t) &= \lambda^{-1}[V(t,\eta(t))][\tfrac{1}{\Gamma(\alpha)}\int_0^{t}\phi'(s)(\phi(t)-\phi(s))^{\alpha-1}\psi(s)ds \\
&-\tfrac{(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^{a}\phi'(s)(\phi(a)-\phi(s))^{\alpha-1}\psi(s)ds],
\end{aligned} \tag{3.25}$$

for all $t \in I$. Then, we get

$$\begin{aligned} |\eta(t)| &\leqslant \lambda^{-1}|V(t,\eta(t))|[\tfrac{\Theta(|\eta|)}{\Gamma(\alpha)}\int_0^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1}|p(s)|ds \\ &+\tfrac{\Theta(|\eta|)(\phi(t)-\phi(0))}{(\phi(a)-\phi(0))\Gamma(\alpha)}\int_0^a \phi'(s)(\phi(a)-\phi(s))^{\alpha-1}|p(s)|ds] \\ &\leqslant \lambda^{-1}[\|\vartheta\|_\infty\|\eta\|_\infty+\omega_0]\tfrac{2\|p\|_\infty\Theta(|\eta|)}{\Gamma(\alpha+1)}(\phi(a)-\phi(0))^\alpha. \end{aligned} \tag{3.26}$$

Let $\|\eta\|_\infty = R$, then we have

$$\begin{aligned} R &\leqslant \lambda^{-1}[\|\vartheta\|_\infty R+\omega_0](\tfrac{2\|p\|_\infty\Theta(R)}{\Gamma(\alpha+1)})(\phi(a)-\phi(0))^\alpha \\ &\leqslant [\|\vartheta\|_\infty R+\omega_0](\tfrac{2\|p\|_\infty\Theta(R)}{\Gamma(\alpha+1)})(\phi(a)-\phi(0))^\alpha. \end{aligned} \tag{3.27}$$

Therefore, we get

$$R \leqslant \frac{\omega_0\frac{2\|p\|_\infty\Theta(R)}{\Gamma(\alpha+1)}(\phi(a)-\phi(0))^\alpha}{1-\|\vartheta\|_\infty(\frac{2\|p\|_\infty\Theta(R)}{\Gamma(\alpha+1)})(\phi(a)-\phi(0))^\alpha)} \tag{3.28}$$

Thus, condition (b) of Theorem 2.5 does not hold. Thus, the operator H and therefore problem (1.3) have the solution on I. □

□

4. Applications and special cases

The proposed system (1.3) is very general and includes some particular cases.

4.1. *Special cases*

As particular cases of our results, we can obtain the existence results for the following systems.

(1) Putting $V(t,\rho(t)) = 1$ in system (1.3), we obtain Classical Caputo fractional hybrid differential inclusion system.

(2) Putting $\phi(t) = \ln(t)$ in system (1.3) and replacing the interval I with the interval $I_e = [1,e]$ in (1.3) and in all assumptions (A1)-(A6), we obtain system (1.1) and we will have the same results in [16].

(3)Putting $V(t,\rho(t)) = 1$ in system (1.3), we obtain system (1.2).

In the following example, we point to how to apply the abstract results in particular generalized fractional differential inclusion.

4.2. *An Example*

Consider the following generalized fractional differential inclusion

$$\begin{cases} {}^*D^{\frac{3}{2}}_{0^+,\phi}\,[\frac{\rho(t)}{\frac{1}{24}e^{t-1}\tan^{-1}(\rho(t)+2)}] \in W(t,\rho(t)), \ a.e.\ \ t\in[0,1], \\ \rho(0)=\rho(1)=0, \\ W:[0,1]\times\mathbb{R}\to P(\mathbb{R}), \\ t\mapsto [\frac{|\rho^3|}{10(|\rho|^3+3)}, \frac{|\sin\rho|}{4(|\sin\rho|+1)}+\frac{3}{4}]. \end{cases} \tag{4.1}$$

Let $\vartheta(t) = \frac{e^{t-1}}{24}$. Here, we get $\|\vartheta\|_\infty = \frac{1}{24}$. Therefore, for $\eta \in W$ we get

$$|\eta| \leqslant \max[\frac{|\rho^3|}{10(|\rho|^3+3)}, \frac{|\sin\rho|}{4(|\sin\rho|+1)} + \frac{3}{4}] \leqslant 1.$$

Let $\phi(t) = \frac{t^2+t}{2}$. Clearly, $M = \frac{2}{7}$, $\frac{2\|\vartheta\|_\infty M}{\Gamma(\frac{3}{2})} \cong 0.2685 < \frac{1}{2}$ and

$K > \frac{(2\omega_0/\Gamma(\alpha))M}{1-(2\|\vartheta\|_\infty/\Gamma(\alpha))M} = 2.79765835$. Hence all conditions of Theorem 3.2 are satisfied. Thus, the system (4.1) has a solution on $[0, 1]$.

5. Conclusions

We developed qualitative results for certain generalized Caputo fractional differential inclusions in Banach algebras. The results were derived some criteria for the existence of solution to the above-mentioned problem. Remarkably, the fixed point theorem in Banach algebras of Dhage type played a significant role in constructing our analysis. Our results generalized and improved some interesting fractional differential inclusions in the literature. We presented an example to illustrate the solvability results. In the future, it will be interesting if the current systems are studied in the frame of $\psi-$Atangana—Baleanu—Caputo, recently introduced in [39].

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