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Four-Point Hybrid Block Method for direct Solution of Third-Order Ordinary Differential Equations

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Abstract

This article presents a four-point hybrid block method for directly solving third-order ordinary differential equations. The method was derived by adopting interpolation and collocation techniques using the Chebyshev polynomial of the first kind as a basis function. The developed method was implemented in block mode. The fundamental properties of the method were investigated to confirm its usability. To evaluate its performance, the method was tested by solving linear and nonlinear initial value problems of third-order ordinary differential equations. The numerical results are compared with existing methods to determine their accuracy. The results show a better accuracy over the existing methods.

Keywords: Hybrid, ordinary differential equations, collocation, interpolation, basic Function.
MSC 2000: 65E05, 65L05, 65L06.

1. Introduction

The work provides numerical solution to third order initial value problems (IVPs) of ordinary differential equations(ODEs) of this form;

$$y''' = f(x, y, y', y''); \quad y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y''(x_0) = y_0'', \quad x \in [a, b]. \tag{1.1}$$

Equations of type (1.1) are commonly solved using the reduction approach, which involves transforming the higher-order equation into an equivalent system of first-order initial value problems (IVPs). Once reduced, suitable numerical methods for solving first-order IVPs are applied to obtain the desired solutions [1], [2]. Despite its widespread use, the reduction approach has significant shortcomings. For instance, converting a single higher-order equation into a system of first-order equations increases the number of equations to solve, which can lead to increased computational complexity and sensitivity to numerical instabilities, particularly for stiff equations.

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To address these limitations, researchers such as([4], [5], [8], [9], [11], and [15]) have developed direct numerical methods tailored for higher-order ordinary differential equations (ODEs). Recent publication [17] proposed a multi-derivative multi-step method with one hybrid point for solving both linear and nonlinear third order ordinary differential equations. The method was implemented in a block mode of order 12.These methods avoid the intermediate step of reduction, providing a more efficient and robust framework for solving such equations.

Numerical methods for approximating ODEs are broadly classified into explicit and implicit methods. Explicit methods do not require starting values, making them straightforward to implement. In contrast, implicit methods rely on starting values and are often implemented using either predictor-corrector mode ([7]) or block mode ([14]).
Several authors, including [4], [5], [8], [9], [10], [6],[2], [18], [19], [20], [21], [22], [23], [24],[25] and [26] have contributed to the development of various numerical approaches for solving equations of type (1.1). These approaches range from reduction techniques and predictor-corrector schemes to hybrid methods. For example, [15] proposed a hybrid method with a block extension for the direct solution of third-order ODEs. This method employs power series as the basis function for solving IVPs. Similarly, [7] introduced a continuous linear multi-step method by combining interpolation of the approximate solution with collocation of the differential system. However, these methods have notable drawbacks, such as sub-optimal accuracy, a lack of self-starting capability, and reduced stability when applied to stiff equations.

This work focuses on the development of a sixth-order linear multi-step method for direct solving of third-order ordinary differential equations. The Chebyshev polynomial of the first kind is used as the basis function in this method, while collocation and interpolation are employed to minimize the occurrences of the source function.

2. Derivation of the Block Method

In order to develop numerical method for direct solving (1.1), we allowed the solution y(x) to be approximated by the Chebyshev polynomials of the first kind. Let $i, c, j \in \mathbb{R}$, the function $y(x) \in \mathbb{R}$ is taken to be an appropriate solution of problem (1.1) given by:

$$y(x) = \sum_{j=0}^{i+c-1} a_j T_j(x), \tag{2.1}$$

where i and c are the number of interpolation and collocation points respectively and a_j, $j = 0, 1...8$ denotes the coefficient of T_j, whose values are to be determined. T_j is the Chebyshev polynomial function of first kind of order 8.
Third derivative of equation (2.1) is obtained to be:

$$y'''(x) = \sum_{j=0}^{i+c-1} a_j T_j'''(x). \tag{2.2}$$

But

$$y'''(x) = f(x, y, y', y''), \text{in equation (1), then}$$

$$y'''(x) = f(x, y, y', y'') = \sum_{j=0}^{i+c-1} a_j T_j'''(x) \tag{2.3}$$

collocate equation (2.3) at $x = x_{n+r}$, $r = 0, 2, 4$ and interpolate equation (2.1) at $x = x_{n+s}$, $s = 0, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}$ to have $i = 6$, $c = 3$ and giving equation (2.1) and (2.3) as $y(x) = \sum_{j=0}^{8} a_j T_j(x)$ and $y'''(x) = f(x, y, y', y'') = \sum_{j=0}^{8} a_j T_j'''(x)$ respectively,

$$y_{n+s} = \sum_{j=0}^{8} a_j T_j(x), \quad s = 0, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}. \tag{2.4}$$

$$y'''_{n+r} = \sum_{j=0}^{8} a_j T_j'''(x) = f(x_{n+r}, y_{n+r}, y'_{n+r}, y''_{n+r}), \quad r = 0, 2, 4. \tag{2.5}$$

Solving for $a_i^{'}s$ in equations (2.4) and (2.5) using Maple 13, we have;

$$\begin{aligned}
a_0 = -\frac{1}{10330306560} \cdot \frac{1}{h^8} \Big(& 505099683\, h^9 f_n + 2473569594\, h^9 f_{n+2} + 7777443\, h^9 f_{n+4} - 10330306560\, h^8 y_n \\
& + 932470200\, h^7 f_n - 4919704800\, h^7 f_{n+2} - 20189400\, h^7 f_{n+4} - 19614564096\, h^6 y_n \\
& + 73851109120\, h^6 y_{n+2} + 44052558400\, h^6 y_{n+\frac{1}{2}} - 76620699840\, h^6 y_{n+\frac{3}{2}} - 21041964864\, h^6 y_{n+\frac{5}{2}} \\
& - 626438720\, h^6 y_{n+\frac{7}{2}} + 136714314\, h^5 f_n - 2081163348\, h^5 f_{n+2} - 12572406\, h^5 f_{n+4} \\
& + 16342425600\, h^4 y_n - 120846131200\, h^4 y_{n+2} - 47309363200\, h^4 y_{n+\frac{1}{2}} + 115747276800\, h^4 y_{n+\frac{3}{2}} \\
& + 34504243200\, h^4 y_{n+\frac{5}{2}} + 1561548800\, h^4 y_{n+\frac{7}{2}} + 1193010\, h^3 f_n - 27130740\, h^3 f_{n+2} \\
& - 270270\, h^3 f_{n+4} + 4506619392\, h^2 y_n - 39838092800\, h^2 y_{n+2} - 14305132160\, h^2 y_{n+\frac{1}{2}} \\
& + 38138221440\, h^2 y_{n+\frac{3}{2}} + 10622924928\, h^2 y_{n+\frac{5}{2}} + 875459200\, h^2 y_{n+\frac{7}{2}} + 46824960\, y_n \\
& - 421424640\, y_{n+2} - 154976640\, y_{n+\frac{1}{2}} + 418104960\, y_{n+\frac{3}{2}} + 96969600\, y_{n+\frac{5}{2}} \\
& + 14501760\, y_{n+\frac{7}{2}} \Big),
\end{aligned}$$

$$
\begin{aligned}
a_1 = \frac{1}{860858880} \cdot \frac{1}{h^7} \Big(&\\
&18472545\, h^9 f_n + 156389310\, h^9 f_{n+2} + 462945\, h^9 f_{n+4} + 107607360\, h^7 f_n \\
&- 3170478592\, h^6 y_n + 4856588800\, h^6 y_{n+2} + 4867576000\, h^6 y_{n+\frac{1}{2}} - 5135681600\, h^6 y_{n+\frac{3}{2}} \\
&- 1380431808\, h^6 y_{n+\frac{5}{2}} - 37572800\, h^6 y_{n+\frac{7}{2}} + 37727450\, h^5 f_n - 394382900\, h^5 f_{n+2} \\
&- 1936550\, h^5 f_{n+4} + 1307250\, h^3 f_n - 25316900\, h^3 f_{n+2} - 193550\, h^3 f_{n+4} \\
&+ 1023288320\, h^2 y_n - 8553702400\, h^2 y_{n+2} - 3131641600\, h^2 y_{n+\frac{1}{2}} + 8125676800\, h^2 y_{n+\frac{3}{2}} \\
&+ 2392615680\, h^2 y_{n+\frac{5}{2}} + 143763200\, h^2 y_{n+\frac{7}{2}} + 48025600\, y_n - 432230400\, y_{n+2} \\
&- 156273600\, y_{n+\frac{1}{2}} + 420795200\, y_{n+\frac{3}{2}} + 107486400\, y_{n+\frac{5}{2}} + 12196800\, y_{n+\frac{7}{2}} \Big),
\end{aligned}
$$

$$
\begin{aligned}
a_2 = -\frac{1}{10330306560} \cdot \frac{1}{h^8} \Big(&\\
&505099683\, h^9 f_n + 2473569594\, h^9 f_{n+2} + 7777443\, h^9 f_{n+4} \\
&+ 745976160\, h^7 f_n - 3935763840\, h^7 f_{n+2} - 16151520\, h^7 f_{n+4} - 19614564096\, h^6 y_n \\
&+ 73851109120\, h^6 y_{n+2} + 44052558400\, h^6 y_{n+\frac{1}{2}} - 76620699840\, h^6 y_{n+\frac{3}{2}} \\
&- 21041964864\, h^6 y_{n+\frac{5}{2}} - 626438720\, h^6 y_{n+\frac{7}{2}} + 93214305\, h^5 f_n \\
&- 1418975010\, h^5 f_{n+2} - 8572095\, h^5 f_{n+4} + 13073940480\, h^4 y_n - 96676904960\, h^4 y_{n+2} \\
&- 37847490560\, h^4 y_{n+\frac{1}{2}} + 92597821440\, h^4 y_{n+\frac{3}{2}} + 27603394560\, h^4 y_{n+\frac{5}{2}} \\
&+ 1249239040\, h^4 y_{n+\frac{7}{2}} + 734160\, h^3 f_n - 16695840\, h^3 f_{n+2} - 166320\, h^3 f_{n+4} \\
&+ 3072695040\, h^2 y_n - 27162336000\, h^2 y_{n+2} - 9753499200\, h^2 y_{n+\frac{1}{2}} + 26003332800\, h^2 y_{n+\frac{3}{2}} \\
&+ 7242903360\, h^2 y_{n+\frac{5}{2}} + 596904000\, h^2 y_{n+\frac{7}{2}} + 28815360\, y_n - 259338240\, y_{n+2} \\
&- 95370240\, y_{n+\frac{1}{2}} + 257295360\, y_{n+\frac{3}{2}} + 59673600\, y_{n+\frac{5}{2}} + 8924160\, y_{n+\frac{7}{2}} \Big),
\end{aligned}
$$

$$
\begin{aligned}
a_3 = \frac{1}{172171776} \cdot \frac{1}{h^7} \Big(&\\
&7173824\, h^7 f_n + 3772745\, h^5 f_n + 39438290\, h^5 f_{n+2} \\
&- 193655\, h^5 f_{n+4} + 156870\, h^3 f_n - 3038028\, h^3 f_{n+2} - 23226\, h^3 f_{n+4} \\
&+ 102328832\, h^2 y_n - 855370240\, h^2 y_{n+2} - 313164160\, h^2 y_{n+\frac{1}{2}} + 812567680\, h^2 y_{n+\frac{3}{2}} \\
&+ 239261568\, h^2 y_{n+\frac{5}{2}} + 14376320\, h^2 y_{n+\frac{7}{2}} + 5763072\, y_n - 51867648\, y_{n+2} \\
&- 18752832\, y_{n+\frac{1}{2}} + 50495424\, y_{n+\frac{3}{2}} + 12898368\, y_{n+\frac{5}{2}} + 1463616\, y_{n+\frac{7}{2}} \Big),
\end{aligned}
$$

$$a_4 = -\frac{1}{5165153280}\cdot\frac{1}{h^8}\Big(93247020\,h^7 f_n - 491970480\,h^7 f_{n+2} - 2018940\,h^7 f_{n+4} + 18642861\,h^5 f_n - 283795002\,h^5 f_{n+2} - 1714419\,h^5 f_{n+4} + 1634242560\,h^4 y_n - 12084613120\,h^4 y_{n+2} - 4730936320\,h^4 y_{n+\frac{1}{2}} + 11574727680\,h^4 y_{n+\frac{3}{2}} + 3450424320\,h^4 y_{n+\frac{5}{2}} + 156154880\,h^4 y_{n+\frac{7}{2}} + 183540\,h^3 f_n - 4173960\,h^3 f_{n+2} - 41580\,h^3 f_{n+4} + 614539008\,h^2 y_n - 5432467200\,h^2 y_{n+2} - 1950699840\,h^2 y_{n+\frac{1}{2}} + 5200666560\,h^2 y_{n+\frac{3}{2}} + 1448580672\,h^2 y_{n+\frac{5}{2}} + 119380800\,h^2 y_{n+\frac{7}{2}} + 7203840\,y_n - 64834560\,y_{n+2} - 23842560\,y_{n+\frac{1}{2}} + 64323840\,y_{n+\frac{3}{2}} + 14918400\,y_{n+\frac{5}{2}} + 2231040\,y_{n+\frac{7}{2}}\Big),$$

$$a_5 = \frac{1}{860858880}\cdot\frac{1}{h^7}\Big(3772745\,h^5 f_n - 39438290\,h^5 f_{n+2} - 193655\,h^5 f_{n+4} + 261450\,h^3 f_n - 5063380\,h^3 f_{n+2} - 38710\,h^3 f_{n+4} + 102328832\,h^2 y_n - 855370240\,h^2 y_{n+2} - 313164160\,h^2 y_{n+\frac{1}{2}} + 812567680\,h^2 y_{n+\frac{3}{2}} + 239261568\,h^2 y_{n+\frac{5}{2}} + 14376320\,h^2 y_{n+\frac{7}{2}} + 9605120\,y_n - 86446080\,y_{n+2} - 31254720\,y_{n+\frac{1}{2}} + 84159040\,y_{n+\frac{3}{2}} + 21497280\,y_{n+\frac{5}{2}} + 2439360\,y_{n+\frac{7}{2}}\Big),$$

$$a_6 = -\frac{1}{10330306560}\cdot\frac{1}{h^8}\Big(6214287\,h^5 f_n - 94598334\,h^5 f_{n+2} - 571473\,h^5 f_{n+4} + 104880\,h^3 f_n - 2385120\,h^3 f_{n+2} - 23760\,h^3 f_{n+4} + 204846336\,h^2 y_n - 1810822400\,h^2 y_{n+2} - 650233280\,h^2 y_{n+\frac{1}{2}} + 1733555520\,h^2 y_{n+\frac{3}{2}} + 482860224\,h^2 y_{n+\frac{5}{2}} + 39793600\,h^2 y_{n+\frac{7}{2}} + 4116480\,y_n - 37048320\,y_{n+2} - 13624320\,y_{n+\frac{1}{2}} + 36756480\,y_{n+\frac{3}{2}} + 8524800\,y_{n+\frac{5}{2}} + 1274880\,y_{n+\frac{7}{2}}\Big),$$

$$a_7 = \frac{1}{86085888}\cdot\frac{1}{h^7}\Big(3735\,h^3 f_n - 72334\,h^3 f_{n+2} - 553\,h^3 f_{n+4} + 137216\,y_n - 1234944\,y_{n+2} - 446496\,y_{n+\frac{1}{2}} + 1202272\,y_{n+\frac{3}{2}} + 307104\,y_{n+\frac{5}{2}} + 34848\,y_{n+\frac{7}{2}}\Big),$$

and

$$a_8 = -\frac{1}{344343552}\cdot\frac{1}{h^8}\Big(437\,h^3 f_n - 9938\,h^3 f_{n+2} - 99\,h^3 f_{n+4} + 17152\,y_n - 154368\,y_{n+2} - 56768\,y_{n+\frac{1}{2}} + 153152\,y_{n+\frac{3}{2}} + 35520\,y_{n+\frac{5}{2}} + 5312\,y_{n+\frac{7}{2}}\Big).$$

Putting $a_i^{'}s$ into (2.1) and simplifying using $x_{n+t} = x_n + th$, where $x_n = 0$ and t takes the values of i, yields the continuous scheme.

$$y(t) = \begin{bmatrix} \Big(\frac{4409}{8198656}h^3t + \frac{4807}{1537248}h^3t^4 - \frac{79}{192156}h^3t^7 - \frac{2592481}{1721717760}h^3t^2 + \frac{33}{896728}h^3t^8 \\ + \frac{9071}{5124160}h^3t^6 - \frac{5533}{1537248}h^3t^5\Big)f_{n+4} \\ + \Big(\frac{744711}{4099328}h^3t + \frac{2049877}{2690184}h^3t^4 - \frac{36167}{672546}h^3t^7 - \frac{137420533}{286952960}h^3t^2 \\ + \frac{4969}{1345092}h^3t^8 + \frac{5255463}{17934560}h^3t^6 - \frac{3943829}{5380368}h^3t^5\Big)f_{n+2} \\ + \Big(\frac{175929}{8198656}h^3t - \frac{518039}{3586912}h^3t^4 + \frac{1245}{448364}h^3t^7 + \frac{1}{6}t^3h^3 - \frac{56122187}{573905920}h^3t^2 \\ - \frac{437}{2690184}h^3t^8 - \frac{2071429}{107607360}h^3t^6 + \frac{754549}{10760736}h^3t^5\Big)f_n \\ + \Big(\frac{12836}{48039}t^5 - \frac{117415}{2690184}t - \frac{34856}{144117}t^4 + \frac{2904}{112091}t^7 + \frac{1957621}{16141104}t^2 \\ - \frac{664}{336273}t^8 - \frac{17765}{144117}t^6\Big)y_{n+\frac{7}{2}} \\ + \Big(-\frac{1027107}{640520}t - \frac{85576}{16013}t^4 + \frac{3656}{16013}t^7 + \frac{36531189}{8967280}t^2 - \frac{1480}{112091}t^8 \\ - \frac{119757}{80065}t^6 + \frac{356044}{80065}t^5\Big)y_{n+\frac{5}{2}} \\ + \Big(\frac{4045}{717}t + \frac{281824}{15057}t^4 - \frac{1536}{1673}t^7 - \frac{861137}{60228}t^2 + \frac{96}{1673}t^8 \\ + \frac{84460}{15057}t^6 - \frac{79792}{5019}t^5\Big)y_{n+2} \\ + \Big(-\frac{2292715}{384312}t - \frac{6028504}{336273}t^4 + \frac{300568}{336273}t^7 + \frac{79813229}{5380368}t^2 - \frac{19144}{336273}t^8 \\ - \frac{601929}{112091}t^6 + \frac{5078548}{336273}t^5\Big)y_{n+\frac{3}{2}} \\ + \Big(\frac{2173025}{384312}t + \frac{7392088}{1008819}t^4 - \frac{37208}{112091}t^7 - \frac{137664245}{16141104}t^2 + \frac{7096}{336273}t^8 \\ + \frac{2031979}{1008819}t^6 - \frac{1957276}{336273}t^5\Big)y_{n+\frac{1}{2}} \\ + \Big(-\frac{92423}{25095}t - \frac{12704}{5019}t^4 + \frac{512}{5019}t^7 + \frac{381191}{100380}t^2 - \frac{32}{5019}t^8 \\ - \frac{5308}{8365}t^6 + \frac{47728}{25095}t^5 + 1\Big)y_n \end{bmatrix} \tag{2.6}$$

Evaluating the continuous scheme(2.6) at $t = 4$, we obtained the required method as;

$$y_{n+4} = \frac{169}{8576}h^3 f_{n+4} - \frac{4969}{4288}h^3 f_{n+2} + \frac{169}{8576}h^3 f_n + \frac{485}{134}y_{n+\frac{7}{2}} - \frac{919}{134}y_{n+\frac{5}{2}} + \frac{919}{134}y_{n+\frac{3}{2}} - \frac{485}{134}y_{n+\frac{1}{2}} + y_n \quad (2.7)$$

$$y'_{n+4} = \frac{80950493}{860858880}h^3 f_{n+4} - \frac{1758804893}{430429440}h^3 f_{n+2} + \frac{62940893}{860858880}h^3 f_n + \frac{6882983}{896728}y_{n+\frac{7}{2}} - \frac{86500641}{4483640}y_{n+\frac{5}{2}} - \frac{4045}{717}y_{n+2} + \frac{72263239}{2690184}y_{n+\frac{3}{2}} - \frac{35742709}{2690184}y_{n+\frac{1}{2}} + \frac{92423}{25095}y_n$$

$$y''_{n+4} = \frac{4435957}{12848640}h^3 f_{n+4} - \frac{50388277}{6424320}h^3 f_{n+2} + \frac{1961717}{12848640}h^3 f_n + \frac{1256551}{120456}y_{n+\frac{7}{2}} - \frac{1500313}{66920}y_{n+\frac{5}{2}} - \frac{861137}{30114}y_{n+2} + \frac{2418575}{40152}y_{n+\frac{3}{2}} - \frac{3282023}{120456}y_{n+\frac{1}{2}} + \frac{381191}{50190}y_n \quad (2.8)$$

3. Evaluation of properties of the method

3.1. Local Truncation Error and Order of the method

The linear operator below, where $y(t)$ is arbitrary continuously differentiable function; The linear multistep method is of order p if, the constants $c_0 = c_1 = ... = c_p = 0, c_{p+1} = 0, c_{p+2} = 0, c_{p+3} \neq 0$ and the corresponding error constant is c_{p+3}.
Where:

$$c_0 = \sum_{j=0}^{k} \alpha_j, \text{and } c_i = (-1)^i [\frac{1}{i!} \sum_{j=0}^{k} j^i \alpha_j + \frac{1}{i-1} \sum_{j=0}^{k} j^{i-1} \beta j], i = 1, 2, 3, ... \quad (3.1)$$

The method will be of order *p* if:
$c_0 = c_1 = ... = c_p = 0$, $c_{p+1} = 0$, $c_{p+2} = 0$,$c_{p+3} \neq 0$ and c_{p+3} is the error constant of the method. Expanding each of the terms of (2.7) using Taylor's series expansion, and comparing the coefficients, the values of c_i is computed as follows:

$$c_0 = 1 - \left(\frac{485}{134} - \frac{919}{134} + \frac{919}{134} - \frac{485}{134} + 1 \right) = 0 \quad (3.2)$$

$$c_1 = 4 - \left(\frac{3395}{268} - \frac{4595}{268} - \frac{485}{268} + \frac{2757}{268} \right) = 0 \quad (3.3)$$

$$c_2 = 8 - \left(\frac{23765}{1072} - \frac{22975}{1072} - \frac{485}{1072} + \frac{8271}{1072} \right) = 0 \quad (3.4)$$

$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$
$c_9 = -\frac{250391}{172892160} = -1.45 \times 10^{-03}$
it was found that $c_0 = c_1 = ...c_p = 0, c_{p+1} = c_{p+2} = 0, c_{p+3}, \neq 0$.
Hence,it was deduced that the method (2.7) is of order 6 and the corresponding error constant is $c_9 = -1.45 \times 10^{-03}$

3.2. Consistency

A numerical method of order p is said to be consistent if it satisfies the following conditions:

i The order p must be equal to or greater than 1 (i.e $p \geqslant 1$)
ii $\sum_{r=0}^{n} \alpha_r = 0$ where $\alpha_r's$ are the coefficient of y_{n+r} in the method.
iii The first characteristic polynomial $\rho(r) = \rho'(r) = 0$ at $r = 1$
iv $\rho^n(r) = n!\sigma(r)$ at $r = 1$

The following list tests for the conditions above to determine whether the method is consistent or not:

i The order of the method is $p = 6 \geqslant 1$, hence the first condition is satisfied.
ii From the method (2.7),

$$\alpha_0 = 1, \alpha_{7/2} = \frac{485}{134}, \alpha_{1/2} = -\frac{485}{134}, \alpha_{3/2} = \frac{919}{134}, \alpha_{5/2} = -\frac{919}{134}, \alpha_4 = 1$$

Therefore, with subtraction all the $\alpha_r's$, then;

$$1 + \frac{485}{134} - 1 - \frac{919}{134} - \frac{485}{134} + \frac{919}{134} = 0 \tag{3.5}$$

Hence, the second condition is satisfied.

iii The first characteristic polynomial of the method (2.7) is denoted by $\rho(r)$. Thus:

$$\rho(r) = r^4 - \frac{485}{134} r^{\frac{7}{2}} - r^0 + \frac{485}{134} r^{\frac{1}{2}} - \frac{919}{134} r^{\frac{3}{2}} + \frac{919}{134} r^{\frac{5}{2}} \tag{3.6}$$

when r=1, we have that;

$$\rho(1) = 1 + \frac{485}{134} - 1 - \frac{919}{134} - \frac{485}{134} + \frac{919}{134} = 0 \tag{3.7}$$

Differentiating $\rho(r)$ with respect to r, it gives;

$$\rho'(1) = 4r^3 - \frac{3395}{268} r^{\frac{5}{2}} + \frac{485}{268\sqrt{r}} - \frac{2757}{268}\sqrt{r} + \frac{4595}{268} r^{\frac{3}{2}} \tag{3.8}$$

when $r = 1$, we have that;
Since $\rho(r) = \rho'(1) = 0$, the method satisfies the third condition

iii Third derivative of $\rho(r)$ with respect to r, is given by;

$$\rho'''(r) = 24r - \frac{50925}{1072}\sqrt{r} + \frac{1455}{1072 r^{\frac{5}{2}}} + \frac{2757}{1072 r^{\frac{3}{2}}} + \frac{13785}{1072\sqrt{r}} \tag{3.9}$$

when $r = 1$, we have that;

$\rho^{'''}(1) = -\frac{450}{67}$
Considering the second characteristic equation,

$$\sigma(r) = \frac{169}{8576}r^4 - \frac{4969}{4288}r^2 + \frac{169}{8576} \tag{3.10}$$

when $r = 1$, we have that;

$$\sigma(1) = -\frac{75}{67} \tag{3.11}$$

Clearly $\rho^{'''}(1) = 3!\sigma(1) = 6$,therefore the fourth condition is satisfied.

Since all four conditions are satisfied, then the scheme is consistent.

3.3. *Zero Stability*

A linear multi-step method is said to be zero-stable, if no root of the first characteristic polynomial $\rho(r)$ has modulus greater than one and if every root with modulus one is simple (i.e if all the roots of $\rho(r)$ lie in or on the unit circle)
To obtain the root of (2.7):

$$\rho(r) = r^4 - \frac{485}{134}r^{\frac{7}{2}} - r^0 + \frac{485}{134}r^{\frac{1}{2}} - \frac{919}{134}r^{\frac{3}{2}} + \frac{919}{134}r^{\frac{5}{2}} \tag{3.12}$$

$$(r-1),\ (r-1),\ (r-1), (r-1) = 0$$

Hence, the equation is zero-stable, as all its roots satisfy $|r| \leqslant 1$.

Since the roots are within a unit circle, this satisfies the definition of zero stability of linear multi-step method, it follows that the method (2.7) is zero stable.

3.4. *Convergence*

The method (2.7) is convergent since it satisfies the sufficient conditions of zero stability and consistency.

3.5. *Interval of Absolute Stability*

From (2.7), the first and second characteristic polynomial are as follows:

$$\rho(r) = r^4 - r^0 + \frac{485}{134}\, r^{1/2} - \frac{919}{134}\, r^{3/2} + \frac{919}{134}\, r^{5/2} - \frac{485}{134}\, r^{7/2} \tag{3.13}$$

$$\sigma(r) = -\frac{1}{8576}\, \left(169\, r^0 - 9938\, r^2 + 169\, r^4\right) \tag{3.14}$$

Subtracting the second characteristic from the first and multiplying with z using boundary locus formula:

$$-z\left(-\frac{169}{8576}\, r^4 + \frac{4969}{4288}\, r^2 - \frac{169}{8576}\right) + r^4 - 1 + \frac{485}{134}\, \sqrt{r} - \frac{919}{134}\, r^{3/2} + \frac{919}{134}\, r^{5/2} - \frac{485}{134}\, r^{7/2} \tag{3.15}$$

Simplifying and equating the equation (3.15) to zero , we have;

$$\frac{169}{8576}\,z\,r^4-\frac{4969}{4288}\,z\,r^2+\frac{169}{8576}\,z+r^4-1+\frac{485}{134}\sqrt{r}-\frac{919}{134}\,r^{3/2}+\frac{919}{134}\,r^{5/2}-\frac{485}{134}\,r^{7/2}=0 \tag{3.16}$$

Substituting $(r=e^{i\theta})$

$$\frac{169}{8576}\,z\,\left(e^{i\theta}\right)^4-\frac{4969}{4288}\,z\,\left(e^{i\theta}\right)^2+\frac{169}{8576}\,z+\left(e^{i\theta}\right)^4-1+\frac{485}{134}\sqrt{e^{i\theta}}-\frac{919}{134}\,\left(e^{i\theta}\right)^{3/2}+\frac{919}{134}\,\left(e^{i\theta}\right)^{5/2}-\frac{485}{134}\,\left(e^{i\theta}\right)^{7/2}=0. \tag{3.17}$$

Solving for z in equation (3.17) and simplifying the whole equation

$$R:=-\frac{64\left(-485\,\left(e^{i\theta}\right)^{7/2}+919\,\left(e^{i\theta}\right)^{5/2}+134\,\left(e^{i\theta}\right)^{4}-919\,\left(e^{i\theta}\right)^{3/2}+485\sqrt{e^{i\theta}}-134\right)}{169\,\left(e^{i\theta}\right)^4-9938\,\left(e^{i\theta}\right)^2+169} \tag{3.18}$$

Using the above equation (3.18), a graph over a circular range (from 0 to 2π) in the complex plane, with the region filled with a grey color, and labels the axes as the real (Re) and imaginary (Im) parts shown below.

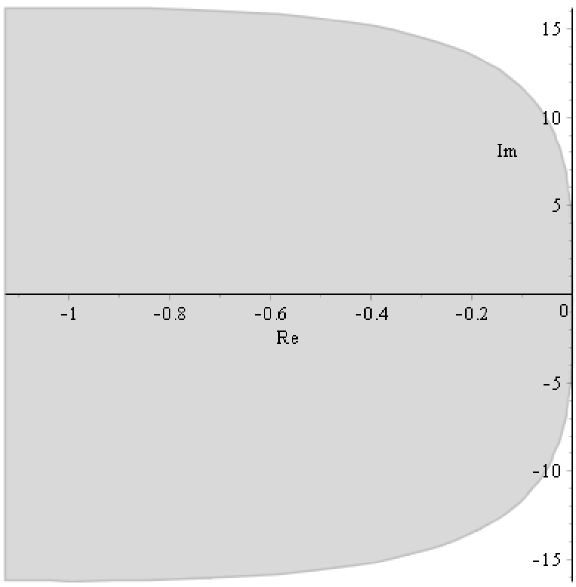

Figure 1: REGION OF ABSOLUTE STABILITY

The region is P- stable i.e the region of absolute stability covers the entire left half of the complex plane which satisfies the condition for absolute stability.

4. Numerical Application

The derived method was used for the solution of initial value problems of third order differential equations. MAPLE codes were written for the implementation of the methods. A graphical representation of the trends between the exact and computed solutions of the considered problems was also provided. the following problems solved below;

Problem 1

$$y''' = 3\sin x \tag{4.1}$$

$$y(0) = 1, y'(0) = 0, y''(0) = -2,$$

Exact solution: $y(x) = 3\cos(x) + \frac{x^2}{2} - 2$ [Source [8]]

x	Exact Solution	Computed Solution	Error	Error [8]
0.1	0.99001249583407729110	0.9900124958340772983	$7.20e-18$	$1.743050e-14$
0.2	0.96019973352372485952	0.9601997335237248934	$3.388e-17$	$1.082467e-13$
0.3	0.91100946737681797886	0.9110094673768180589	$8.004e-17$	$2.711165e-13$
0.4	0.84318298200865510264	0.8431829820086552484	$1.4576e-16$	$5.079270e-13$
0.5	0.75774768567111792351	0.7577476856711181484	$2.2489e-16$	$8.164580e-13$
0.6	0.65600684472903457111	0.6560068447290348917	$3.2059e-16$	$1.199707e-12$
0.7	0.53952656185346484563	0.5395265618534652788	$4.3317e-16$	$1.654343e-12$
0.8	0.41012012804149570040	0.4101201280414962628	$5.6240e-16$	$1.674639e-10$
0.9	0.26982990481199266729	0.2698299048119933694	$7.0211e-16$	$3.336392e-10$
1.0	0.12090691760441829866	0.1209069176044191522	$8.5354e-16$	$5.001723e-10$

Table 1: Numerical results of **Problem 1** using the new method

The graph (Figure 2) below further illustrates a strong relationship between the exact and computed solutions, demonstrating that the solution is highly accurate.

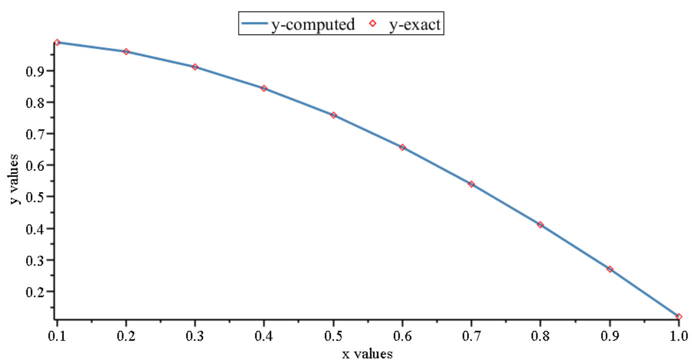

Figure 2: Graph of Errors Problem 1 Using The New Method

Problem 2

$$y''' = e^x \tag{4.2}$$

$$y(0) = 3, y'(0) = 1, y''(0) = 5,$$

Exact solution: $y(x) = 2 + 2x^2 + e^x$ [Source [5]]

x	Exact Solution	Computed Solution	New Error	Error in [5]	Error in [4]
0.1	3.1251709180756476219	3.1251709180756476248	2.9e − 18	1.5673e − 15	2.531308e − 14
0.2	3.3014027581601698201	3.3014027581601698339	1.38e − 17	7.3074e − 15	1.612044e − 13
0.3	3.5298588075760030714	3.5298588075760031040	3.26e − 17	1.72198e − 14	4.023448e − 13
0.4	3.8118246976412702585	3.8118246976412703178	5.93e − 17	3.07224e − 14	7.536194e − 13
0.5	4.1487212707001280537	4.1487212707001281468	9.31e − 17	4.97118e − 14	1.212364e − 12
0.6	4.5421188003905088364	4.5421188003905089749	1.385e − 16	7.41871e − 14	1.780798e − 12
0.7	4.9937527074704763262	4.9937527074704765216	1.954e − 16	1.033627e − 13	2.456702e − 12
0.8	5.5055409284924673407	5.5055409284924676046	2.639e − 16	1.397988e − 13	2.212097e − 11
0.9	6.0796031111569493212	6.0796031111569496638	3.426e − 16	1.834940e − 13	5.231993e − 11
1.0	6.7182818284590447971	6.7182818284590452354	4.383e − 16	2.333884e − 13	8.86011e − 11

Table 2: Numerical results of **Problem 2** using the new method

Problem 3

$$y^{'''} + y^{'} = 0 \tag{4.3}$$

$$y(0) = 0, y^{'}(0) = 1, y^{''}(0) = 2, h = 0.01$$

Exact solution: $y(x) = 2(1 - cosx) + sinx$ [Source [11]]

x	Exact Solution	Computed Solution	New Error	Error in [11]	Error in [12]
0.1	0.109825086090776624465862956724	0.109825086090776620115690222801	$4.35e-18$	$2.330e-10$	$1.61e-09$
0.2	0.238536175112577973649670434599	0.238536175112577953211019593618	$2.04e-17$	$1.467e-09$	$1.04e-08$
0.3	0.384847228410127583971745906429	0.384847228410127535820700290545	$4.82e-17$	$4.80e-09$	$2.96e-08$
0.4	0.547296354302880413235191767241	0.547296354302880326069258292696	$8.72e-17$	$1.1233e-08$	$2.32e-07$
0.5	0.724260414823457700648939858390	0.724260414823457568040724770006	$1.33e-16$	$2.1767e-08$	$4.54e-07$
0.6	0.913971243575678947184152123249	0.913971243575678762719040447749	$1.84e-16$	$3.75e-08$	1.48e?06
0.7	1.11453331266871444340870003417	1.11453331266871420116089437102	$2.42e-16$	$6.3733e-08$	$2.87e-06$
0.8	1.32394267220519222513790580900	1.32394267220519191978567464730	$3.05e-16$	$9.2767e-08$	$4.68e-06$
0.9	1.54010697308615484495817246080	1.54010697308615447549195001290	$3.69e-16$	$1.291e-07$	$6.92e-06$
1.0	1.76086637307161750441086285455	1.76086637307161707185062910674	$4.33e-16$	$1.7573e-07$	$9.60e-06$

Table 3: Numerical results of **Problem 3** using the new method

The graph[Figure 3] below further illustrates a strong relationship between the exact and computed solutions, demonstrating that the solution is highly accurate.

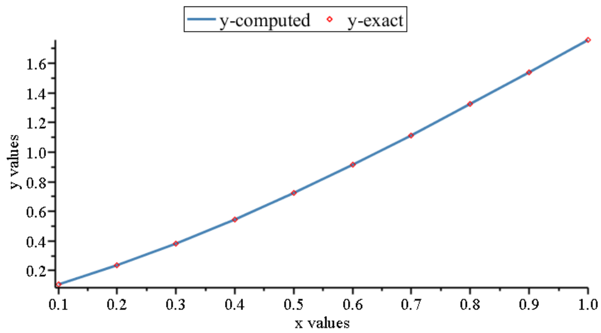

Figure 3: Graph of Errors Problem 3 Using the New Method

Problem 4

$$y''' = -e^x \tag{4.4}$$

$$y(0) = 1, y'(0) = -1, y''(0) = 3, h = 0.01$$

Exact solution: $y(x) = 2 + 2x^2 - e^x$ [Source [13]]

x	Exact Solution	Computed Solution	New Error	Error in [14]	Error in [13]
0.1	0.91482908192435237812	0.9148290819243523752	2.92e − 18	7.56477e − 11	1.82410e − 13
0.2	0.85859724183983017986	0.8585972418398301661	1.376e − 17	2.60170e − 10	1.67078e − 12
0.3	0.83014119242399692860	0.8301411924239968960	3.260e − 17	5.76003e − 10	6.00142e − 12
0.4	0.82817530235872974149	0.8281753023587296822	5.929e − 17	8.41270e − 10	1.48598e − 11
0.5	0.85127872929987194629	0.8512787292998718532	9.309e − 17	1.00013e − 09	3.01205e − 15
0.6	0.89788119960949116362	0.8978811996094910251	1.3852e − 16	1.09051e?09	5.38418e − 11
0.7	0.96624729252952367378	0.9662472925295234784	1.9538e − 16	1.07048e − 09	8.83157e − 11
0.8	1.0544590715075326593	1.0544590715075323954	2.639e − 16	1.49247e?09	1.36060e − 10
0.9	1.1603968888430506788	1.1603968888430503362	3.426e − 16	3.15695e − 09	1.99870e − 10
1.0	1.2817181715409552029	1.2817181715409547646	4.383e − 16	4.45905e − 09	2.82814e − 10

Table 4: Numerical results of **Problem 4** using the new method

Problem 5

$$y + 4y' - x = 0, \quad h = 0.1$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad h = \frac{1}{10}$$

Exact solution: $y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$ [6]

x	Exact Solution	Computed Solution	New Error	Errors in [6]	Error in [10]
0.1	0.0049875166547689346175	0.0049875166547671941600	1.7404575e − 15	2.526352800e − 13	1.6654e − 08
0.2	0.019801063624467168897	0.019801063624459046980	8.121917e − 15	1.167586530e − 12	3.8095e − 07
0.3	0.043999572204454227834	0.043999572204435319270	1.8908564e − 14	2.708435928e − 12	1.5664e − 06
0.4	0.076867491997440136161	0.076867491997406483580	3.3652581e − 14	4.572491739e − 12	3.9865e − 06
0.5	0.11744331764977343002	0.11744331764972380299	4.962703e − 14	6.72003248e − 12	7.9597e − 06
0.6	0.16455792103568947325	0.16455792103562370419	6.576906e − 14	9.06559734e?12	1.3680e − 05
0.7	0.21688116070628626730	0.21688116070620482401	8.144329e − 14	1.128454740e − 11	2.1195e − 05
0.8	0.27297491043158765475	0.27297491043149163616	9.601859e − 14	1.317840390e?11	3.0396e − 05
0.9	0.33135039275506098555	0.33135039275495382287	1.0716268e − 13	1.467186192e − 11	4.10008e − 05
1.0	0.39052753185270180582	0.39052753185258919756	1.1260826e − 13	1.556512757e − 11	5.2605e − 05

Table 5: Numerical results of **Problem 5** using the new method

5. Discussion

The paper focused on solving third-order differential equations using collocation and interpolation techniques, with the Chebyshev polynomial of the first kind as the basis function. The error constant was calculated to be $C_{p+3} = -1.558 \times 10^{-3}$, with a method order of $p = 6 \geqslant 1$. Evaluation of the scheme confirmed its consistency, as it has an order $p = 6 \geqslant 1$, zero-stability and convergence. Five examples were solved in Tables 1-5 which confirmed the method good performance and accuracy in comparison to other cited methods.

6. Conclusion

The numerical results gotten with the derived method showed very little errors as shown above with the Tables [1]-[5] when compared with the corresponding cited proposed methods in the references results. The method is usable for solving linear and non-linear initial-value problems of third-order ordinary differential equations. It's also provide reasonable approximations for the third order ordinary differential equation. Hence, it can be deduced from the errors that the method is a more accurate and that the it is effective for solving third order initial value problems.

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