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Mathematical Analysis of a Malaria model with vaccination, treatment and vector control using Sterile-insect technique

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Abstract

In this work, a mathematical model for malaria transmission is developed using a system of nonlinear ordinary differential equations. The model incorporates three control strategies: vaccination, treatment, and sterile insect technique. Analytical results demonstrate that the malaria-free equilibrium is both locally and globally asymptotically stable when the basic reproduction number, R_0 is less than one, and unstable when $R_0 > 1$. The existence of an endemic equilibrium is investigated, and conditions for the occurrence of forward or backward bifurcation are derived. Numerical simulations and graphical illustrations are provided to demonstrate the disease dynamics under various control scenarios. The findings reveal that the combined application of all three control measures is more effective in reducing malaria transmission than any individual or pairwise combination of interventions.

Keywords: Malaria, endemic equilibrium, vaccination, bifurcation, sterile-insect technology.
2010 MSC: 34A34, 92D30, 92B05.

1. INTRODUCTION

Malaria is an infectious disease transmitted to humans through the bite of the female Anopheles mosquito [1]. It is caused by the Plasmodium parasite which enters into the blood stream of humans through mosquitoes bites. Malaria is highly endemic in Asian and sub-Saharan African Countries [2]. It was reported that there were about 244 million cases of malaria in 2021 with about 619,000 deaths [3]. The number of reported cases rose to 249 million globally in 2022 while deaths due to the disease was 608,000 [4]. Vaccination against malaria has received more attention recently with two vaccines, R21/Matrix-M and RTS,S/AS01 recommended by World Health Organization for use mostly in places where malaria is endemic. Both vaccines reduce cases of malaria by 75% and are given

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in repeated doses over a period to achieve the required immunity [5]. Some of the symptoms of malaria are headache, muscle pain, fever, nausea, shaking chills, infant mortality, abdominal pain, morbidity, severe anaemia, profuse sweating, vomiting, diarrhea, convulsions, etc [6, 7, 8, 9]. Due to the increase in resistance to insecticides by mosquitoes as well as resistance by the malaria parasites to drugs, there is a growing need to explore better ways of controlling the disease than this physical methods. one of the recent propositions is the use of sterile insect technology.

The sterile insect technology (SIT) is a biological method to reduce the population of pests and insects [10]. In this method, a large number of sterile male are released to mate with the females in the wild [11]. In the case of controlling mosquitoes, using females as part of the control will be dangerous as they can feed on human blood and transmit plasmodium. A female mosquito that mates with a sterile male can only lay eggs but can not hatch them [12],[13]. This ensures that the mosquito population is reduced with time. The method is currently proposed to eradicate mosquitoes causing various illnesses [14, 15].

Modelling of diseases which are infectious helps in understanding the dynamics of disease transmission which enables qualitative prognosis in disease management and control. Nwankwo and Okounghae [16] developed a new non-autonomous model for Malaria that incorporates diurnal temperature fluctuations. Their results showed that the malaria-free equilibrium is globally asymptotically stable when the reproduction number is less than unity, otherwise the disease persists. The transmission intensity of Malaria was seen to be a function of resting place for mosquitoes and duration of rest. Atokolo and Mbah [12] worked on using sterile insect technique to reduce Aedes mosquitoes that causes zika virus disease and they discovered that introducction of more sterile males into the wild will reduce the population of the mosquitoes with time. Fatmawati et al. [17] proposed an optimal control model for malaria with seasonal factor and otherwise. Seasonal factor was shown to influence human and mosquito population dynamics in hot climatic regions. Keno et al. [18] studied an optimal control model for malaria with climate variability. The disease-free equilibrium in their work was locally and globally asymptotically stable when the reproduction number was less than one with the existence of forward and backward bifurcation in the model. Collins and Duffy [19] formulated an optimal control malaria model with drug resistance, treatment and the use of mosquito nets as controls in Nigeria. It was shown in their work that the endemicity of the disease will persists if the controls are not adequately employed as well as controlling the dominant resistant strain more effectively. Mohamed [20] developed a malaria model which showed the point of occurence of forward bifurcation in the system. Their Sensitivity analysis showed that reducing the contact rate of humans and mosquitoes, the amount of mosquitoes in circulation and increasing rate of treating infectious humans are recommended for effective control of the spread of Malaria.

This paper looked at a mathematical model for malaria that incorporates treatment, vaccination and use of SIT as controls for the disease. It is expected that vaccinating more humans, improving treatment rate of infectious humans and reducing the mosquito population by use of sterile insect technology, the disease will be controlled. The rest of this paper is arranged as follows; the new model is presented in section two. Also, the malaria-free equilibrium and the malaria reproduction number are obtained in section two. The stability analysis of the MFE is done in section three while the stability analysis of the endemic equilibrium is done in section four. The numerical simulations are carried out in section five and the work was concluded in section 6.

2. Model Development

The malaria model studied in this work has eleven (11) compartments made up of the Susceptible humans S_h, Vaccinated humans S_{hv}, Unvaccinated humans S_{hu}, Exposed humans E_{hm}, Infectious humans I_{hm}, Infectious humans undergoing treatment I_{hmT}, Infectious humans not undergoing treatment I_{hmU}, Recovered humans R_h, Susceptible mosquitoes S_{mv}, Exposed mosquitoes E_{mv} and Infectious Mosquitoes I_{mv} respectively.

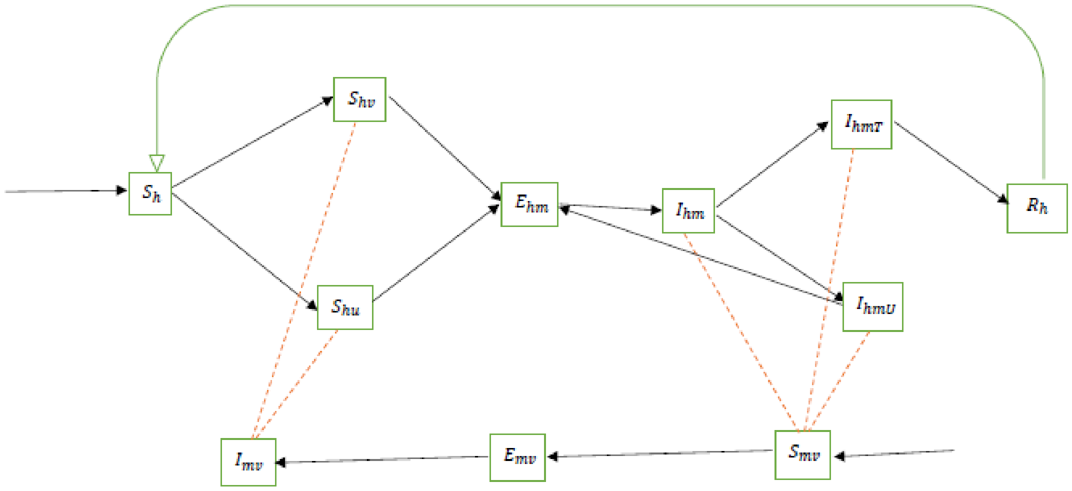

Figure 1: Flow diagram for the disease transmission.

The susceptible humans, S_h is increased by level of recruitment through births and migration denoted by Λ_h and rate of loss of immunity to the disease after recovery, θ. It is reduced as humans choose to be vaccinated or not and move to their respective compartments. S_h is partitioned into two (2), those who accept to be vaccinated against the disease and those who refused vaccination. It is assumed that vaccinated humans lose their immunity with time hence, can become infected with time as the vaccines wanes. The exposed human population, E_{hm} is increased by effective contacts of mosquitoes with susceptible humans and the proportion of untreated humans with latent stage of infection. It is reduced by rate of development of infectiousness, δ_1. The infectious human population, I_{hm} is reduced by disease-induced death rate τ_2, rate at which infectious humans accepts or reject treatment ε_1 and ε_2 respectively. The infectious human population undergoing treatment, I_{hmT} increases as more humans accept treatment, ε_1 and is reduced as rate of recovery from malaria γ_1 increases as well as by disease-induced death rate τ_2. We assume that those who accept to go for treatment are also infectious with the disease as they have not fully recovered. We also assumed that those who refuse to go for treatment and have suppressed the symptoms and the multiplication of the parasites

in their bodies return back to the exposed compartment as the parasites are latent at this stage. All human compartments accounts for natural death rate, τ_1.
The susceptible mosquito population, S_{mv} is increased by the rate of recruitment Λ_{mv}. It is reduced by rate of reduction κ_1 in mosquito population by SIT and effective contacts with humans. κ_1 is a product of mating rate and mating probability between SIT mosquitoes and female Anopheles mosquitoes in the target environment. The exposed mosquito population, E_{mv} is increased as susceptible mosquitoes become infected through contacts with infectious humans and the infectious mosquito population, I_{mv} is increased by rate of development of infectiousness, ν_1. All mosquito compartments are reduced by natural death rate, μ. The malaria model is given by;

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \theta R_h - (\rho_1 + \rho_2 + \tau_1) S_h, \\
\frac{dS_{hu}}{dt} &= \rho_1 S_h - (\alpha_1 \beta_1 I_{mv} + \tau_1) S_{hu}, \\
\frac{dS_{hv}}{dt} &= \rho_2 S_h - (\alpha_1 \beta_2 \varphi I_{mv} + \tau_1) S_{hv}, \\
\frac{dE_{hm}}{dt} &= \alpha_1 \beta_1 I_{mv} S_{hu} + \alpha_1 \beta_2 \varphi I_{mv} S_{hv} + \phi_1 I_{hmU} - (\delta_1 + \tau_1) E_{hm}, \\
\frac{dI_{hm}}{dt} &= \delta_1 E_{hm} - (\tau_1 + \tau_2 + \varepsilon_1 + \varepsilon_2) I_{hm}, \\
\frac{dI_{hmT}}{dt} &= \varepsilon_1 I_{hm} - (\gamma_1 + \tau_1 + \tau_2) I_{hmT}, \\
\frac{dI_{hmU}}{dt} &= \varepsilon_2 I_{hm} - (\phi_1 + \tau_1 + \tau_2) I_{hmU}, \\
\frac{dR_h}{dt} &= \gamma_1 I_{hmT} - (\tau_1 + \theta) R_h, \\
\frac{dS_{mv}}{dt} &= \Lambda_{mv} - \alpha_1 (\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU}) S_{mv} - (\kappa_1 I_{SIT} + \mu) S_{mv}, \\
\frac{dE_{mv}}{dt} &= \alpha_1 (\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU}) S_{mv} - (\nu_1 + \mu) E_{mv}, \\
\frac{dI_{mv}}{dt} &= \nu_1 E_{mv} - \mu I_{mv}.
\end{aligned} \tag{2.1}$$

where $S_h(0) = S_h^0, S_{hu}(0) = S_{hu}^0, S_{hv}(0) = S_{hv}^0, E_{hm}(0) = E_{hm}^0, I_{hm}(0) = I_{hm}^0, I_{hmT}(0) = I_{hmT}^0, I_{hmU}(0) = I_{hmU}^0, R_h(0) = R_h^0, S_{mv}(0) = S_{mv}^0, E_{mv}(0) = E_{mv}^0, I_{mv}(0) = I_{mv}^0$ are the initial states of (2.1). The total human and mosquito populations are given by

$$N_h = S_h + S_{hu} + S_{hv} + E_{hm} + I_{hm} + I_{hmT} + I_{hmU} + R_h,$$

and

$$N_{mv} = S_{mv} + E_{mv} + I_{mv}.$$

The solution to (2.1) exists in the region $\Omega = \Omega_1 \cup \Omega_2 \subseteq \mathbb{R}_+^8 \times \mathbb{R}_+^3$ described by

$$\Omega_1 = \left\{ (S_h, S_{hu}, S_{hv}, E_{hm}, I_{hm}, I_{hmT}, I_{hmU}, R_h) \in \mathbb{R}_+^8 : N_h \leqslant \frac{\Lambda}{\tau_1} \right\} \tag{2.2}$$

and

$$\Omega_2 = \left\{ (S_{mv}, E_{mv}, I_{mv}) \in \mathbb{R}^3_+ : N_{mv} \leqslant \frac{\Lambda}{\mu} \right\}. \tag{2.3}$$

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2.1. Positivity of solutions and Invariant Region

Theorem 2.1. *Let the initial data set for the model be*

$$S_h^0 > 0, S_{hu}^0 > 0, S_{hv}^0 > 0, E_{hm}^0 > 0, I_{hm}^0 > 0, I_{hmT}^0 > 0, I_{hmU}^0 > 0, R_h^0 > 0, S_{mv}^0 > 0, E_{mv}^0 > 0, I_{mv}^0 > 0$$

. *Then, the solution set* $\{S_h(t), S_{hu}(t), S_{hv}(t), E_{hm}(t), I_{hm}(t), I_{hmT}(t), I_{hmU}(t), R_h(t), S_{mv}(t), E_{mv}(t), I_{mv}(t)\}$ *of the model with the given initial data will remain positive for all time* $t > 0$.

Proof. It can be shown from (2.1) that

$$\begin{aligned} \frac{dS_h}{dt} &\geqslant -(\rho_1 + \rho_2 + \tau_1)S_h, \\ \frac{dS_{hu}}{dt} &\geqslant -(\alpha_1 \beta_1 I_{mv} + \tau_1)S_{hu}, \\ &\vdots \\ \frac{dI_{mv}}{dt} &\geqslant -\mu I_{mv}. \end{aligned} \tag{2.4}$$

By integration, the system (2.4) becomes,

$$\begin{aligned} S_h &\geqslant S_h^0 e^{-\int(\rho_1+\rho_2+\tau_1)dt}, \\ S_{hu} &\geqslant S_{hu}^0 e^{-\int(\alpha_1\beta_1 I_{mv}+\tau_1)dt}, \\ &\vdots \\ I_{mv} &\geqslant I_{mv}^0 e^{-\int \mu dt}. \end{aligned} \tag{2.5}$$

Thus, the solution (2.5) to the model system remains positive $\forall t > 0$. ☐

Theorem 2.2. *The region* Ω *is positively invariant and is an attractor of all positive solution to the system.*

Proof. The total human and mosquito populations satisfy the differential equations

$$\begin{aligned} \frac{dN_h}{dt} &= \Lambda_h - \tau_1 N_h - \tau_2(I_{hm} + I_{hmT} + I_{hmU}), \\ \frac{dN_{mv}}{dt} &= \Lambda_{mv} - \mu I_{mv}. \end{aligned} \tag{2.6}$$

respectively. By integration, as $t \to \infty$ the system (2.6) gives $0 \leqslant N_{hm} \leqslant \frac{\Lambda_h}{\tau_1}$ and $0 \leqslant N_{mv} \leqslant \frac{\Lambda_{mv}}{\mu}$. Hence, all the solutions of the system are positive ☐

The proofs of Theorems 2.1 and 2.2 show that the malaria model is well-posed epidemiogically and thus, can be analyzed [21, 22].

Table 1: Description of Parameters

Parameters	Description
Λ_h	Rate of recruitment of humans into S_h
Λ_{mv}	Rate of recruitment of Anopheles mosquitoes into S_{mv}
φ	Rate at which S_{hv} loses immunity
α_1	Contact rate of mosquitoes with humans
β_1	Rate of transmission from I_{mv} to S_{hu}
β_2	Rate of transmission from I_{mv} to S_{hv}
β_3	Rate of transmission from I_{hm} to S_{mv}
β_4	Rate of transmission from I_{hmT} to S_{mv}
β_5	Rate of transmission from I_{hmU} to S_{mv}
δ_1	Rate at which E_{hm} become infectious
τ_1	Natural mortality rate of humans
τ_2	Disease-induced death rate
μ	Natural mortality rate of mosquitoes
ρ_1	Unvaccinated proportion of the susceptible humans
ρ_2	Vaccinated proportion of the susceptible humans
θ	Rate of loss of immunity to malaria from recovered class
ε_1	Rate at which infectious humans accept to be treated
ε_2	Rate at which infectious humans refuse to be treated
ϕ_1	Level of movement from I_{hmU} to E_{mv}
γ_1	Rate of recovery of I_{hmT}
ν_1	Rate of development of infectiousness in mosquitoes
κ_1	Rate at which SIT mosquitoes reduces S_{mv}
I_{SIT}	Population of Sterile males to control S_{mv}

2.2. *The Malaria-free Equilibrium*

The malaria-free equilibrium (MFE) of the system (2.1) is the steady-state solution where there is no malaria in the system. If we denote the MFE by E^0, then we have

$$E^0 = \left(\frac{\Lambda_h}{\rho_1 + \rho_2 + \tau_1}, \frac{\rho_1 \Lambda_h}{\tau_1(\rho_1 + \rho_2 + \tau_1)}, \frac{\rho_2 \Lambda_h}{\tau_1(\rho_1 + \rho_2 + \tau_1)}, 0, 0, 0, 0, 0, \frac{\Lambda_{mv}}{\mu}, 0, 0 \right).$$

2.3. *The Malaria Reproduction Number*

The malaria reproduction number, R_m is the average number of new cases of malaria caused by one typical infectious individual or mosquito, in an entirely susceptible population [21, 23]. The method for obtaining the malaria reproduction number is defined in [24, 25, 26]. The disease class of (2.1) corresponds to the system;

$$\begin{aligned}
\frac{dE_{hm}}{dt} &= \alpha_1\beta_1 I_{mv} S_{hu} + \alpha_1\beta_2\varphi I_{mv} S_{hv} + \phi_1 I_{hmU} - (\delta_1 + \tau_1)E_{hm} \\
\frac{dI_{hm}}{dt} &= \delta_1 E_{hm} - (\tau_1 + \tau_2 + \varepsilon_1 + \varepsilon_2) I_{hm} \\
\frac{dI_{hmT}}{dt} &= \varepsilon_1 I_{hm} - (\gamma_1 + \tau_1 + \tau_2) I_{hmT} \\
\frac{dI_{hmU}}{dt} &= \varepsilon_2 I_{hm} - (\phi_1 + \tau_1 + \tau_2) I_{hmU} \\
\frac{dE_{mv}}{dt} &= \alpha_1(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU}) S_{mv} - (\nu_1 + \mu) E_{mv} \\
\frac{dI_{mv}}{dt} &= \nu_1 E_{mv} - \mu I_{mv}
\end{aligned} \tag{2.7}$$

with $\mathcal{F} = \begin{bmatrix} \alpha_1\beta_1 I_{mv}S_{hu} + \alpha_1\beta_2\varphi I_{mv}S_{hv} \\ 0 \\ 0 \\ 0 \\ \alpha_1(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU})S_{mv} \\ 0 \end{bmatrix}$ and $\mathcal{V} = \begin{bmatrix} (\delta_1 + \tau_1)E_{hm} \\ -\delta_1 E_{hm} + (\tau_1 + \tau_2 + \varepsilon_1 + \varepsilon_2)I_{hm} \\ -\varepsilon_1 I_{hm} + (\gamma_1 + \tau_1 + \tau_2)I_{hmT} \\ -\varepsilon_2 I_{hm} + (\phi_1 + \tau_1 + \tau_2)I_{hmU} \\ (\nu_1 + \mu)E_{mv} \\ -\nu_1 E_{mv} + \mu I_{mv} \end{bmatrix}$.

$\mathcal{F}$ is the rate of appearance of new infection in the system while $\mathcal{V}$ is the rate of movement through other means [23, 27].

Thus, $F = \frac{\partial \mathcal{F}}{\partial Y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & A_3 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $V = \frac{\partial \mathcal{V}}{\partial Y} = \begin{bmatrix} B_1 & 0 & 0 & -\phi_1 & 0 & 0 \\ -\delta_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon_1 & B_3 & 0 & 0 & 0 \\ 0 & -\varepsilon_2 & 0 & B_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_5 & 0 \\ 0 & 0 & 0 & 0 & -\nu_1 & \mu \end{bmatrix}$

where $A_1 = \dfrac{\alpha_1\Lambda_h(\beta_1\rho_1 + \beta_2\varphi\rho_2)}{\tau_1(\rho_1 + \rho_2 + \tau_1)}, A_2 = \dfrac{\alpha_1\beta_3\Lambda_{mv}}{\mu}, A_3 = \dfrac{\alpha_1\beta_4\Lambda_{mv}}{\mu}, A_4 = \dfrac{\alpha_1\beta_5\Lambda_{mv}}{\mu}, B_1 = \delta_1 + \tau_1, B_2 = \tau_1 + \tau_2 + \varepsilon_1 + \varepsilon_2, B_3 = \gamma_1 + \tau_1 + \tau_2, B_4 = \phi_1 + \tau_1 + \tau_2, B_5 = \nu_1 + \mu.$
The malaria reproduction number, R_m gotten by the matrix FV^{-1} becomes

$$R_m = \left(\frac{\nu_1\delta_1 A_1(A_2B_4B_3 + A_3\varepsilon_1 B_4 + A_4\varepsilon_2 B_3)}{\mu B_3 B_5(B_1B_2B_4 - \varepsilon_2\delta_1\phi_1)}\right)^{\frac{1}{2}}.$$

3. Stability Analysis of malaria-free equilibrium

3.1. Local Stability of malaria-free equilibrium

Theorem 3.1. *The MFE is locally asymptotically stable (LAS) if $R_m < 1$ and unstable if otherwise.*

Proof. The Jacobian, J of the system evaluated at the MFE is given by

$$J(E^0) = \begin{bmatrix} -a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho_1 & -\tau_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 \\ \rho_2 & 0 & -\tau_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & -B_1 & 0 & 0 & \phi_1 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & \delta_1 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_1 & -B_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_2 & 0 & -B_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & -B_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_2 & -A_3 & -A_4 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2 & A_3 & A_4 & 0 & 0 & -B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_1 & -\mu \end{bmatrix}$$

where $a_1 = \rho_1 + \rho_2 + \tau_1, a_2 = \frac{\alpha_1\Lambda_h\beta_1\rho_1}{\tau_1(\rho_1+\rho_2+\tau_1)}, a_3 = \frac{\alpha_1\Lambda_h\beta_2\Phi\rho_2}{\tau_1(\rho_1+\rho_2+\tau_1)}$ and all the A_i's and B_i's are as described in section 2.3.

This Jacobian matrix $J(E^0)$ can be reduced to a submatrix by deleting the columns with one element and their corresponding rows to get

$$J_1(M^0) = \begin{bmatrix} -B_1 & 0 & 0 & \phi_1 & 0 & A_1 \\ \delta_1 & -B_2 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_1 & -B_3 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & -B_4 & 0 & 0 \\ 0 & A_2 & A_3 & A_4 & -B_5 & 0 \\ 0 & 0 & 0 & 0 & \nu_1 & -\mu \end{bmatrix}$$

The parameters $(-a_1, -\tau_1, -\tau_1, -\mu, -B_6)$ in the columns deleted from the Jacobian matrix $J(E^0)$ and the eigenvalues of the Jacobian submatrix, $J_1(E^0)$ are the eigenvalues of the Jacobian of the system. Thus, we need to show that all the eigenvalues of the Jacobian submatrix, $J_1(E^0)$ will be negative for the MFE of the system to be LAS. The roots of the characteristic polynomial

$$Q_6\lambda^6 + Q_5\lambda^5 + Q_4\lambda^4 + Q_3\lambda^3 + Q_2\lambda^2 + Q_1\lambda + Q_0$$

are the eigenvalues of the Jacobian submatrix, $J_1(E^0)$ where

$$\begin{aligned}
Q_6 &= 1,\\
Q_5 &= B_1 + B_2 + B_3 + B_4 + B_5 + \mu,\\
Q_4 &= B_1(B_2 + B_3 + B_4 + B_5) + B_2(B_3 + B_4 + B_5) + B_3(B_4 + B_5) + B_4B_5\\
&\quad + \mu(B_1 + B_2 + B_3 + B_4 + B_5),\\
Q_3 &= B_1B_2[\mu + B_3 + B_5] + B_3[\mu + B_4 + B_5][B_1 + B_2] + B_4[B_1 + B_2 + B_3][\mu + B_5]\\
&\quad + \mu B_5[B_1 + B_2 + B_3 + B_4] + \frac{\nu_1\delta_1A_1(A_2B_3B_4 + A_3\epsilon_1B_4 + A_4\epsilon_2B_3)}{\mu B_3B_5R_m^2},\\
Q_2 &= \mu B_1B_3[B_2 + B_4 + B_5] + \mu B_4B_5[B_1 + B_2] + B_2B_3B_5[\mu + B_1] + \mu B_2B_3B_4\\
&\quad + \mu B_1B_2B_5\phi_1\delta_1\epsilon_2R_m^2 + \frac{\nu_1\delta_1A_1(A_3\epsilon_1B_4 + A_4\epsilon_2B_3)}{B_3B_4} + \mu B_1B_2B_5[1 - R_m^2]\\
&\quad + \frac{\nu_1\delta_1A_1(A_2B_3B_4 + A_3\epsilon_1B_4 + A_4\epsilon_2B_3)[\mu + B_3 + B_5]}{\mu B_3B_5R_m^2} + B_3B_4B_5[\mu + B_1 + B_2],\\
Q_1 &= \frac{\nu_1\delta_1A_1(A_2B_3B_4 + A_3\epsilon_1B_4 + A_4\epsilon_2B_3)[\mu + B_5]}{\mu_mB_5R_m^2} + \frac{\mu B_3B_5\phi_1\delta_1\epsilon_2R_m^2}{B_4}\\
&\quad + \mu B_3B_4B_5[B_1 + B_2] + \frac{\nu_1\delta_1A_1(A_3\epsilon_1B_4 + A_4\epsilon_2B_3)[B_4 + B_3]}{B_3B_4}\\
&\quad + \frac{\nu_1\delta_1A_1(A_2B_3B_4 + A_3\epsilon_1B_4 + A_4\epsilon_2B_3)[1 - R_m^2]}{\mu_mB_3R_m^2} + \mu B_1B_2B_3B_5[1 - R_m^2],\\
Q_0 &= \mu B_1B_2B_3B_4B_5[1 - R_m^2].
\end{aligned}$$

Routh-Hurwitz criterion for stability requires that all roots of the characteristic polynomial will have negative real parts if $Q_6 > 0, Q_5 > 0, Q_4 > 0, Q_3 > 0, Q_2 > 0, Q_1 > 0, Q_0 > 0$ and the following are satisfied; $\frac{Q_1}{Q_0} > \frac{Q_3}{Q_2}, \frac{Q_4}{Q_3} > \frac{Q_6}{Q_5}$ and $\frac{Q_2}{Q_0} > \frac{Q_6}{Q_4}$ [28],[29]. Therefore, the MFE of the system will be LAS if $R_m < 1$. The LAS of the MFE signifies that an influx of a small number of malaria infectious humans or mosquitoes into the system will not lead to an outbreak if the malaria reproduction number is less than one depending on the initial sizes of the infected population. □

3.2. Global stability of the malaria-free equilibrium

Theorem 3.2. *Consider the system of differential equations*

$$\frac{dX_1}{dt} = F_1(X_1, 0) \tag{3.1}$$

$$\frac{dX_2}{dt} = F_2(X_1, X_2), F_2(X_1, 0) = 0 \tag{3.2}$$

where (3.1) is the system of differential equations, satisfied by non-diseased classes such that $X_1 = (S_h, S_{hu}, S_{hv}, R_h, S_{mv})$ *and (3.2) is the system of differential equations satisfied by the diseased classes so that* $X_2 = (E_{hm}, I_{hm}, I_{hmT}, I_{hmU}, E_{mv}, I_{mv})$. *The MFE,* E^0 *is GAS if (3.1) is GAS, and if in (3.2),* $\mathcal{B}X_2 - F_2(X_1, X_2) \geqslant 0$, *where* $\mathcal{B}$ *is the Jacobian matrix of* $F_2(X_1, X_2)$, *evaluated at* E^0.

Proof. Theorem 3.2 is adapted from Castillo-Chavez [30]. The solutions of (3.1) at the MFE gives

$$\begin{aligned}
S_h &= \frac{\Lambda_h}{\rho_1+\rho_2+\tau_1} + \frac{\theta R_h^0 e^{-(\tau_1+\theta)t}}{(\rho_1+\rho_2-\theta)} + c_1 e^{-(\rho_1+\rho_2+\tau_1)t} \\
S_{hu} &= \frac{\rho_1 \Lambda_h}{\tau_1(\rho_1+\rho_2+\tau_1)} - \frac{\rho_1 R_h^0 e^{-(\tau_1+\theta)t}}{\rho_1+\rho_2-\theta} - \frac{\rho_1 c_1 e^{(\rho_1+\rho_2+\tau_1)t}}{(\rho_1+\rho_2)} + c_2 e^{-\tau_1 t} \\
S_{hv} &= \frac{\rho_2 \Lambda_h}{\tau_1(\rho_1+\rho_2+\tau_1)} - \frac{\rho_2 R_h^0 e^{-(\tau_1+\theta)t}}{\rho_1+\rho_2-\theta} - \frac{\rho_2 c_1 e^{(\rho_1+\rho_2+\tau_1)t}}{(\rho_1+\rho_2)} + c_3 e^{-\tau_1 t} \\
R_h &= R_h^0 e^{-(\tau_1+\theta)t} \\
S_{mv} &= \frac{\Lambda_{mv}}{\mu} + \left(S_{mv}^0 - \frac{\Lambda_{mv}}{\mu}\right) e^{-\mu t}.
\end{aligned}$$

As $t \to \infty$, irrespective of the values of the $c^{'}s$, we will have $S_h \to \frac{\Lambda_h}{\rho_1+\rho_2+\tau_1}$, $S_{hu} \to \frac{\rho_1 \Lambda_h}{\tau_1(\rho_1+\rho_2+\tau_1)}$, $S_{hv} \to \frac{\rho_2 \Lambda_h}{\tau_1(\rho_1+\rho_2+\tau_1)}$, $R_h \to 0$ and $S_{mv} \to \frac{\Lambda_{mv}}{\mu}$ respectively which corresponds to the values of these state variables at the MFE. Thus, the malaria-free equilibrium $\left(\frac{\Lambda_h}{\rho_1+\rho_2+\tau_1}, \frac{\rho_1 \Lambda_h}{\tau_1(\rho_1+\rho_2+\tau_1)}, \frac{\rho_2 \Lambda_h}{\tau_1(\rho_1+\rho_2+\tau_1)}, 0, \frac{\Lambda_{mv}}{\mu}\right)$ of the system, $\frac{dX_1}{dt} = F_1(X_1, 0)$, is GAS. For the second condition, we could see that the matrix $\mathcal{B}$ is the same as the Jacobian submatrix $J_1(E^0)$. Therefore,

$$\mathcal{B}X_2 - F_2(X_1, X_2) = \begin{bmatrix} \alpha_1\beta_1\left(\frac{\Lambda_h \rho_1}{\tau_1(\rho_1+\rho_2+\tau_1)} - S_{hu}\right) + \alpha_1\beta_2\Phi\left(\frac{\Lambda_h \rho_2}{\tau_1(\rho_1+\rho_2+\tau_1)} - S_{hv}\right) \\ 0 \\ 0 \\ 0 \\ \alpha_1\left(\frac{\Lambda_{mv}}{\mu} - S_{mv}\right)(\beta_3 I_{hm} + \beta_4 I_{hmT} + \beta_5 I_{hmU}) \\ 0 \end{bmatrix}$$

It is obvious to see that the condition $\mathcal{B}X_2 - F_2(X_1, X_2) \geqslant 0$ is guaranteed thereby satisfying the conditions for GAS. The global asymptotical stability of the malaria-free equilibrium shows that no matter the initial sizes of the infected population, malaria will be controlled in the system when the $R_m < 1$. □

4. Malaria-Endemic Equilibrium

Malaria-endemic equilibrium points (MEEP) are steady-state solutions of the system where malaria persists, that is, where all state variables are positive. The malaria-endemic equilibrium points denoted in this work by E^* is gotten by changing all state variables to have the index '*', set each equation in the model to zero and find an expression for the state variables to get

$$S_h^* = \frac{\Lambda_h + \theta R_h^*}{a_1},\ S_{hu}^* = \frac{\rho_1 S_h^*}{\alpha_1\beta_1 I_{mv}^* + \tau_1},\ S_{hv}^* = \frac{\rho_2 S_h^*}{\alpha_1\beta_2\varphi I_{mv}^* + \tau_1},\ I_{hm}^* = \frac{\delta_1 E_{hm}^*}{B_2},$$

$$E_{hm}^* = \frac{\alpha_1\beta_1 I_{mv}^* S_{hu}^* + \alpha_1\beta_2\varphi I_{mv}^* S_{hv}^* + \phi_1 I_{hmU}^*}{B_1},\ I_{hmT}^* = \frac{\varepsilon_1 I_{hm}^*}{B_3},\ I_{hmU}^* = \frac{\varepsilon_2 I_{hm}^*}{B_4},$$

$$R_h^* = \frac{\gamma_1 I_{hmT}^*}{B_6},\ S_{mv}^* = \frac{\Lambda_{mv}}{\alpha_1(\beta_3 I_{hm}^* + \beta_4 I_{hmT}^* + \beta_5 I_{hmU}^*) + (\kappa_1 I_{SIT} + \mu)},\ I_{mv}^* = \frac{\nu_1 E_{mv}^*}{\mu}.$$

$$E_{mv}^* = \frac{\alpha_1(\beta_3 I_{hm}^* + \beta_4 I_{hmT}^* + \beta_5 I_{hmU}^*) S_{mv}^*}{B_5}.$$

The MEEP corresponds to the equation

$$\begin{aligned}
\Lambda_h + \theta R_h^* - (\rho_1 + \rho_2 + \tau_1) S_h^* &= 0\\
\rho_1 S_h^* - (\alpha_1\beta_1 I_{mv}^* + \tau_1) S_{hu}^* &= 0\\
\rho_2 S_h^* - (\alpha_1\beta_2\varphi I_{mv}^* + \tau_1) S_{hv}^* &= 0\\
\alpha_1\beta_1 I_{mv}^* S_{hu}^* + \alpha_1\beta_2\varphi I_{mv}^* S_{hv}^* + \phi_1 I_{hmU}^* - (\delta_1 + \tau_1) E_{hm}^* &= 0\\
\delta_1 E_{hm}^* - (\tau_1 + \tau_2 + \varepsilon_1 + \varepsilon_2) I_{hm}^* &= 0\\
\varepsilon_1 I_{hm}^* - (\gamma_1 + \tau_1 + \tau_2) I_{hmT}^* &= 0\\
\varepsilon_2 I_{hm}^* - (\phi_1 + \tau_1 + \tau_2) I_{hmU}^* &= 0\\
\gamma_1 I_{hmT}^* - (\tau_1 + \theta) R_h^* &= 0\\
\Lambda_{mv} - \alpha_1(\beta_3 I_{hm}^* + \beta_4 I_{hmT}^* + \beta_5 I_{hmU}^*) S_{mv}^* - (\kappa_1 I_{SIT}^* + \mu) S_{mv}^* &= 0\\
\alpha_1(\beta_3 I_{hm}^* + \beta_4 I_{hmT}^* + \beta_5 I_{hmU}^*) S_{mv}^* - (\nu_1 + \mu) E_{mv}^* &= 0\\
\nu_1 E_{mv}^* - \mu I_{mv}^* &= 0
\end{aligned} \tag{4.1}$$

We can rewrite all the equations in terms of I_{hm}^* to get

$$S_h^* = \frac{\Lambda_h B_3 B_6 + \epsilon_1\theta\gamma_1 I_{hm}^*}{B_3 B_6 a_1},\ S_{hu}^* = \frac{\rho_1(\Lambda_h B_3 B_6 + \epsilon_1\theta\gamma_1 I_{hm}^*)[h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5]}{B_3 B_6 a_1[\alpha_1\beta_1 h_5 \nu_1 I_{hm}^* + \tau_1[h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5]]},$$

$$S_{hv}^* = \frac{\rho_2(\Lambda_h B_3 B_6 + \epsilon_1\theta\gamma_1 I_{hm}^*)[h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5]}{B_3 B_6 a_1[\alpha_1\beta_2\Phi h_5 \nu_1 I_{hm}^* + \tau_1[h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5]]},\quad I_{hmT}^* = \frac{\epsilon_1 I_{hm}^*}{B_3},$$

$$E_{hm}^* = \frac{\alpha_1\phi\nu_1 B_4 \rho_1(\Lambda_h B_3 B_6 + \epsilon_1\theta\gamma_1 I_{hm}^*) h_5 h_2 I_{hm}^* + \phi_1 \epsilon_2 B_1 B_3 B_6 a_1 h_4 I_{hm}^*}{B_3 B_6 a_1[\alpha_1\beta_2\Phi h_5 \nu_1 I_{hm}^* + \tau_1[h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5]]},\quad I_{hmU}^* = \frac{\epsilon_2 I_{hm}^*}{B_4},$$

$$R_h = \frac{\gamma_1\epsilon_1 I_{hm}^*}{B_3 B_6},\ S_{mv}^* = \frac{\Lambda_{mv} B_3 B_4}{h_1 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4},\ E_{mv}^* = \frac{h_1 \Lambda_{mv} I_{hm}^*}{h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5},$$

$$I_{mv}^* = \frac{(A_2 B_3 B_4 + A_3 \epsilon_1 B_4 + A_4 \epsilon_2 B_3)\nu_1 I_{hm}^*}{h_1 B_5 I_{hm}^* + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5},$$

where

$$\begin{aligned}
&h_1 = \alpha_1(\beta_3 B_3 B_4 + \beta_4 \epsilon_1 B_4 + \beta_5 \epsilon_2 B_3), \quad h_2 = (\beta_1 \rho_1 + \beta_2 \rho_2), \quad h_3 = \alpha_1 \Phi \nu_1 B_4 \rho_1,\\
&h_4 = \alpha_1 \beta_2 \Phi(A_2 B_3 B_4 + A_3 \epsilon_1 B_4 + A_4 \epsilon_2 B_3)\nu_1 I^*_{hm} + \tau_1[h_1 B_5 I^*_{hm} + (\kappa_1 I_{SIT} + \mu) B_3 B_4 B_5]\\
&h_5 = A_2 B_3 B_4 + A_3 \epsilon_1 B_4 + A_4 \epsilon_2 B_3
\end{aligned}$$

If we set $I^*_{hm} = 0$ in the new equations, we obtain the malaria-free equilibrium in section (2.2) which is the point where there is no malaria in the system already stable locally and globally. When $I^*_{hm} > 0$, malaria persists in the system and the malaria-endemic equilibrium point occurs. Thus, in the presence of malaria, the system (2.1) will have a non-trivial equilibrium point which is the malaria endemic equilibrium point. To show the existence of the malaria endemic equilibrium when it is difficult to find a closed form of the endemic equilibrium state, we use the approach by [31, 32]. The procedure involves summing the equations from all the compartments and obtain a condition to show that at least one state variable is positive and not zero.
Summing all the equations in (4.1) will give

$$\begin{aligned}
\Lambda_h + \Lambda_{mv} - \tau_1(S^*_h + S^*_{hu} + S^*_{hv} + E^*_{hm} + I^*_{hm} + I^*_{hmT} + I^*_{hmU} + R^*_h)\\
- \tau_2(I^*_{hm} + I^*_{hmT} + I^*_{hmU}) - \kappa_1 I^*_{SIT} S^*_{mv} - \mu(S^*_{mv} + E^*_{mv} + I^*_{mv}) = 0
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\Lambda_h + \Lambda_{mv} = \tau_1(S^*_h + S^*_{hu} + S^*_{hv} + E^*_{hm} + I^*_{hm} + I^*_{hmT} + I^*_{hmU} + R^*_h)\\
+ \tau_2(I^*_{hm} + I^*_{hmT} + I^*_{hmU}) + \kappa_1 I^*_{SIT} S^*_{mv} + \mu(S^*_{mv} + E^*_{mv} + I^*_{mv})
\end{aligned}$$

Since $\Lambda_h + \Lambda_{mv} > 0, \tau_1 > 0, \tau_2 > 0, \mu > 0$ and $\kappa_1 > 0$ then,

$$\begin{aligned}
&\tau_1(S^*_h + S^*_{hu} + S^*_{hv} + E^*_{hm} + I^*_{hm} + I^*_{hmT} + I^*_{hmU} + R^*_h) > 0\\
&\tau_2(I^*_{hm} + I^*_{hmT} + I^*_{hmU}) > 0\\
&\mu(S^*_{mv} + E^*_{mv} + I^*_{mv}) > 0\\
&\kappa_1 I^*_{SIT} S^*_{mv} > 0
\end{aligned} \tag{4.2}$$

The second inequality in (4.2) guarantees the existence of the endemic equilibrium since one of the infectious compartment must be greater than zero for the expression to hold. Also, the compartments represented by I^*_{hmT} and I^*_{hmU} respectively, can not exist without the presence of I^*_{hm}, hence $I^*_{hm} > 0$. The existence of I^*_{hm} also guarantees the existence of E^*_{hm} which shows that S^*_h also exists at the malaria endemic equilibrium point. From the last equation in (10), $S^*_{mv} > 0$ and their interactions with I^*_{hm} guarantees the presence of the infectious class of mosquitoes, E^*_{mv} and I^*_{mv} respectively. Therefore, the malaria endemic equilibrium point has been shown to exist when $I^*_{hm} > 0$.

4.1. Local Stability of Malaria-endemic Equilibrium Point

The MEEP is locally asymptotically stable if $R_m > 1$ and unstable if otherwise. The Jacobian, J of the system evaluated at the MEEP is given by

$$J(E^0) = \begin{bmatrix} -e_1 & 0 & 0 & 0 & 0 & 0 & 0 & \theta & 0 & 0 & 0 \\ \rho_1 & -e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_3 \\ \rho_2 & 0 & -e_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e_5 \\ 0 & e_{11} & e_{12} & -B_1 & 0 & 0 & \phi_1 & 0 & 0 & 0 & e_3+e_5 \\ 0 & 0 & 0 & \delta_1 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 & -B_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 & 0 & -B_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & -B_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_6 & -e_7 & -e_8 & 0 & -e_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_6 & e_7 & e_8 & 0 & e_{10} & -B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_1 & -\mu \end{bmatrix}$$

where $e_1 = \rho_1 + \rho_2 + \tau_1, e_2 = (\alpha_1\beta_1 I^*_{mv} + \tau_1), e_3 = \alpha_1\beta_1 S^*_{hu}, e_4 = (\alpha_1\beta_2\varphi I^*_{mv} + \tau_1), e_5 = \alpha_1\beta_2\varphi S^*_{hv}, e_6 = \alpha_1\beta_3 S^*_{mv}, e_7 = \alpha_1\beta_4 S^*_{mv}, e_8 = \alpha_1\beta_5 S^*_{mv}, e_9 = (\alpha_1(\beta_3 I^*_{hm} + \beta_4 I^*_{hmT} + \beta_5 I^*_{hmU}) + \kappa_1 I^*_{SIT} + \mu), e_{10} = \alpha_1(\beta_3 I^*_{hm} + \beta_4 I^*_{hmT} + \beta_5 I^*_{hmU}), e_{11} = \alpha_1\beta_1 I^*_{mv}$ and $e_{12} = \alpha_1\beta_1\varphi I^*_{mv}$.

Theorem 4.1. *(Gershgorin). Let* $D = d_{ij}$ *be a complex* $r \times r$ *matrix with eigenvalues,* λ*. For* $i \in r$*, let* $R_i = \sum_{J \neq i}^{r} |d_{ij}|$ *be the sum of the absolute values of the non-diagonal entries in the ith row. Let* $C(d_{ii}, R_i) := |\lambda - d_{ii}| \leqslant R_i$ *be a closed disc centered at* d_{ii} *with radius* R_i*. Then, the eigenvalues of* D *belong to the union of the discs,* $C(d_{ii}, R_i)$*.*

Theorem 4.2. *The endemic equilibrium* E^* *is LAS if the Jacobian matrix* $J(E^*)$ *is strictly diagonally dominant.*

Proof. Employing Gershgorin's theorem, we will have the following;

$$\begin{aligned} -(e_1+\theta) &\leqslant \lambda \leqslant \theta - e_1 \\ -(\rho_1+e_2+e_3) &\leqslant \lambda \leqslant \rho_1 + e_3 - e_2 \\ -(\rho_2+e_4+e_5) &\leqslant \lambda \leqslant \rho_2 + e_5 - e_4 \\ -(e_3+e_5+e_{11}+e_{12}+B_1) &\leqslant \lambda \leqslant e_3+e_5+e_{11}+e_{12}-B_1 \\ -(\delta_1+B_2) &\leqslant \lambda \leqslant \delta_1 - B_2 \\ -(\varepsilon_1+B_3) &\leqslant \lambda \leqslant \varepsilon_1 - B_3 \\ -(\varepsilon_2+B_4) &\leqslant \lambda \leqslant \varepsilon_2 - B_4 \\ -(\gamma_1+B_6) &\leqslant \lambda \leqslant \gamma_1 - B_6 \\ -(e_6+e_7+e_8+e_9) &\leqslant \lambda \leqslant e_6+e_7+e_8-e_9 \\ -(e_6+e_7+e_8+e_{10}+B_5) &\leqslant \lambda \leqslant e_6+e_7+e_8+e_{10}-B_5 \\ -(\nu_1+\mu) &\leqslant \lambda \leqslant \nu_1 - \mu \end{aligned} \tag{4.3}$$

Each left-hand side in (4.3) is negative and each right-hand side is negative if $J'(E_1)$ is strictly diagonally dominant. Therefore, Gershgorin's theorem implies if $J'(E_1)$ strictly diagonally dominant, then the eigenvalues of $J'(E_1)$, $\lambda \in [-\Phi_1, -\Phi_2]$, where $-\Phi_1$ and $-\Phi_2$ are the minimum and maximum of the left-hand and right-hand sides of (4.3) respectively. Therefore, the endemic equilibrium E_1 is LAS if the Jacobian matrix, $J'(E_1)$ is strictly diagonally dominant. If the endemic equilibrium is locally asymptotically stable, it means that the disease will always persist in the population and will not reach extinction in a finite time. However, the eighth and last entries in the right-hand side of (4.3) suggest that $J'(E_1)$ may not be strictly diagonally dominant. Specifically in line 11, ν_1 which is the rate of development of infectiousness in mosquitoes is expected to be greater than μ which is the natural mortality rate of mosquitoes. Also, from line 8, it is highly possible for $B_6 < \gamma_1$. Hence, the MEEP may not be locally asymptotically stable. □

4.2. Bifurcation Analysis

Theorem 4.3. *(Castillo-Chavez and Song (2004) [33]).*
Given a system of ordinary differential equation with a bifurcation parameter ψ*;*

$$y^{'}(t) = h(y, \pi), \;\; h : \mathbb{R}^n \times \mathbb{R}, \; h \in C^2(\mathbb{R}^n \times \mathbb{R}). \tag{4.4}$$

Let $J = \frac{\partial h_k(0,0)}{dy_j}$ *be the Jacobian matrix of* $h(y, \psi)$ *evaluated at* $(0, 0)$*, where* $y = 0$ *is the equilibrium point of the system. Assuming the following hold;*

1. *zero is a simple eigenvalue of the Jacobian matrix, and all other eigenvalues of* J *have negative real part.*
2. J *has a nonnegative right eigenvector* w *and a left eigenvector* ν *corresponding to the zero eigenvalue.*

Let $h_k(y, \pi)$ *denote the* $k\text{th}$ *component of* $h(y, \pi)$ *and*

$$q = \sum_{k,i,j}^{n} \nu_k w_i w_j \frac{\partial^2 h_k(0,0)}{\partial y_i \partial y_j}, \quad r = \sum_{k,i}^{n} \nu_k w_i \frac{\partial^2 h_k(0,0)}{\partial y_i \partial \pi}$$

then

1. *If* $q > 0$*,* $r > 0$*, then when* $\pi < 0$ *with* $|\pi| \ll 1$*,* $y = 0$ *is locally asymptotically stable and* $\exists$ *a positive equilibrium, and when* $0 < |\pi| \ll 1$*,* $y = 0$ *is unstable, and* $\exists$ *a negative locally asymptotically stable equilibrium.*
2. *If* $q < 0$ *and* $r > 0$*, when* π *changes from negative to positive,* $y = 0$ *changes from stable to unstable. correspondingly, a negative unstable equilibrium becomes positive and locally asymptotically stable.*

Particularly, if $q < 0$ *and* $r > 0$*, then a forward bifurcation occurs at* $\pi = 0$ *and if* $q > 0$*,* $r > 0$*, a backward bifurcation occurs at* $\pi = 0$*.*

Proof. Let β_3 be the bifurcation parameter, then $R_{0m} = 1$ implies that

$$\beta_3 = \beta_3^* = \frac{\mu^2 B_3 B_5 (B_1 B_2 B_4 - \epsilon_2 \phi_1 \delta_1) - A_1 \nu_1 \delta_1 (A_3 \epsilon_1 B_4 + A_4 \epsilon_2 B_3)\mu}{\alpha_1 \Lambda_{m\nu} A_1 \nu_1 \delta_1 B_3 B_4},$$

Let $J(E^0, \beta_3^*)$ be the Jacobian matrix of $f(x, \beta_3^*)$ at the malaria-free equilibrium E^0 where β_3 is now replaced with β_3^* which is our bifurcation parameter. The matrix $J(E^0, \beta_3^*)$ possesses one zero eigenvalue, while the remaining eigenvalues have negative real part. Therefore, $J(E^0, \beta_3^*)$ is non-hyperbolic, and we can apply the center manifold theorem to analyze the

dynamics of the model around the bifurcation parameter value β_3^*.
The right eigenvector $\bar{w} = (w_1, w_2, w_3, ..., w_{11})^T$ and the left eigenvector $\bar{v} = (v_1, v_2.v_3, ..., v_{11})^T$ corresponding to the zero eigenvalues satisfy the systems

$$J(E^0, \beta_3^*)\bar{w} = 0$$

and

$$\bar{v}J(E^0, \beta_3^*) = 0,$$

respectively.
The Jacobian matrix $J(E^0, \beta_3^*)$ and $J(E^0)$ are the same, only that in A_2, β_3 is replaced with β_3^*. Using the Jacobian matrix, $J(E^0, \beta_3^*)$,

$$J(E^0, \beta_3^*) = \begin{bmatrix} -a_1 & 0 & 0 & 0 & 0 & 0 & 0 & \theta & 0 & 0 & 0 \\ \rho_1 & -\tau_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 \\ \rho_2 & 0 & -\tau_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & -B_1 & 0 & 0 & \phi_1 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & \delta_1 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_1 & -B_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_2 & 0 & -B_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & -B_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_2^* & -A_3 & -A_4 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2^* & A_3 & A_4 & 0 & 0 & -B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_1 & -\mu \end{bmatrix}$$

the right eigenvectors with $w_5 = 1 > 0$ are given by

$$w_1 = \frac{\theta\gamma_1\epsilon_1 w_5}{a_1 B_3 B_6}, \quad w_2 = \frac{\rho_1\theta\gamma_1\epsilon_1\mu B_4 B_5 - a_1 a_2 B_6 \nu_1 (A_2^* B_3 B_4 + A_3\epsilon_1 B_4 + A_4\epsilon_2 B_3) w_5}{\tau_1 a_1 \mu B_3 B_4 B_5 B_6}$$

$$w_3 = \frac{\rho_2\theta\gamma_1\epsilon_1\mu B_4 B_5 - a_1 a_3 B_6 \nu_1 (A_2^* B_3 B_4 + A_3\epsilon_1 B_4 + A_4\epsilon_2 B_3) w_5}{\tau_1 a_1 \mu B_3 B_4 B_5 B_6}, \quad w_4 = \frac{B_2 w_5}{\delta_1},$$

$$w_6 = \frac{\epsilon_1 w_5}{B_3}, \quad w_7 = \frac{\epsilon_2 w_5}{B_4}, \quad w_8 = \frac{\gamma_1\epsilon_1 w_5}{B_3 B_6}, \quad w_9 = \frac{-(A_2^* B_3 B_4 + A_3\epsilon_1 B_4 + A_4\epsilon_2 B_3) w_5}{\mu B_3 B_4},$$

$$w_{10} = \frac{(A_2^* B_3 B_4 + A_3\epsilon_1 B_4 + A_4\epsilon_2 B_3) w_5}{B_3 B_4 B_5}, \quad w_{11} = \frac{\nu_1 (A_2^* B_3 B_4 + A_3\epsilon_1 B_4 + A_4\epsilon_2 B_3) w_5}{\mu B_3 B_4 B_5}$$

while the left eigenvectors with $v_4 = 1 > 0$ are given by

$$v_1 = v_2 = v_3 = v_8 = v_9 = 0, v_5 = \frac{B_1 v_4}{\delta_1}, \quad v_6 = \frac{A_1 \nu_1 A_3 v_4}{\mu B_3 B_5}, \quad v_7 = \frac{\mu B_5 \phi_1 + A_1 \nu_1 A_4 v_4}{\mu B_4 B_5},$$

$$v_{10} = \frac{\nu_1}{B_5}, \quad v_{11} = \frac{A_1 \nu_1 v_4}{\mu B_5}, \quad v_{11} = \frac{A_1 v_4}{\mu}.$$

The second partial derivatives arising from (**??**) and corresponding to q and r are
$\frac{\partial^2 f_2}{\partial x_2 \partial x_{11}} = -2\alpha_1\beta_1$, $\frac{\partial^2 f_3}{\partial x_{11} \partial x_3} = -2\alpha_1\beta_2\Phi$ $\frac{\partial^2 f_4}{\partial x_3 \partial x_{11}} = 2\alpha_1\beta_2\Phi$ $\frac{\partial^2 f_4}{\partial x_{11} \partial x_2} = 2\alpha_1\beta_1$,
$\frac{\partial^2 f_9}{\partial x_5 \partial x_9} = -2\alpha_1\beta_3^*$, $\frac{\partial^2 f_9}{\partial x_6 x_9} = -2\alpha_1\beta_4$, $\frac{\partial^2 f_9}{\partial x_7 \partial x_9} = -2\alpha_1\beta_5$, $\frac{\partial^2 f_{10}}{\partial x_9 \partial x_5} = 2\alpha_1\beta_3^*$,

$\frac{\partial^2 f_{10}}{\partial x_9 \partial x_6} = 2\alpha_1\beta_4$, $\frac{\partial^2 f_{10}}{\partial x_7 \partial x_9} = 2\alpha_1\beta_5$, $\frac{\partial f_{10}}{\partial x_5} = \alpha_1\beta_3^* x_9$, $\frac{\partial^2 f_{10}}{\partial x_5 \partial \beta_3^*} = \alpha_1 x_9$, $\frac{\partial f_{10}}{\partial x_9} = \alpha_1\beta_3^* x_5$, $\frac{\partial^2 f_{10}}{\partial x_9 \partial \beta_3^*} = \alpha_1 x_5$. Substituting the values of the $v_i^{'}s, w_i^{'}s$ and $f_i^{'}s$ into q and r, we have

$$\begin{aligned}
q &= 2v_4w_2w_{11}\frac{\partial^2 f_4(0,0)}{\partial x_2\partial x_{11}} + 2v_4w_3w_{11}\frac{\partial^2 f_4(0,0)}{\partial x_3\partial x_{11}} + 2v_{10}w_5w_9\frac{\partial^2 f_{10}(0,0)}{\partial x_5\partial x_9} + 2v_{10}w_6w_9\frac{\partial^2 f_{10}(0,0)}{\partial x_6\partial x_9} \\
&\quad + 2v_{10}w_7w_9\frac{\partial^2 f_{10}(0,0)}{\partial x_7\partial x_9} \\
&= \frac{2\alpha_1\left[\beta_1[\rho_1\theta\gamma_1\epsilon_1\mu B_4B_5 - a_1a_2B_6h_5\nu_1] + \beta_2\Phi[\rho_2\theta\gamma_1\epsilon_1\mu B_4B_5 - a_1a_3B_6h_5\nu_1]\right]h_5\nu_1v_4w_5^2}{\tau_1a_1\mu^2B_3^2B_4^2B_5^2B_6} \\
&\quad - \frac{2\alpha_1A_1h_5\nu_1v_4[\beta_3B_3B_4 + \beta_4\epsilon_1B_4 + \beta_5\epsilon_2B_3]w_5^2}{\mu^2B_3^2B_4^2B_5}
\end{aligned}$$

$$r = v_{10}\left(w_5\frac{\partial^2 f_{10}(0,0)}{\partial x_5\partial\beta_3^*} + w_9\frac{\partial^2 f_{10}(0,0)}{\partial x_9\partial\beta_3^*}\right) = \frac{A_1\nu_1\alpha_1v_4w_5}{\mu B_5}\left(x_9 - \frac{h_5^*x_5}{\mu B_3B_4}\right)$$

The terms, $\rho_1\theta\gamma_1\epsilon_1\mu B_4B_5 < a_1a_2B_6c_4$ and $\rho_2\theta\gamma_1\epsilon_1\mu B_4B_5 < a_1a_3B_6c_4$ since every member of $\rho_1\theta\gamma_1\epsilon_1\mu B_4B_5 \in a_1a_2B_6c_4$ and $\rho_2\theta\gamma_1\epsilon_1\mu B_4B_5 \in a_1a_3B_6c_4$. Thus, $q < 0$. The parameter, $r > 0$ if $x_9 > \frac{h_5x_5}{\mu B_3B_4}$. Therefore, the malaria endemic equilibrium point will be LAS if $r > 0$ since $q < 0$. In this case, forward bifurcation occurs. But, if $r < 0$, backward bifurcation occurs and the malaria endemic equilibrium point is not locally stable. □

4.3. *Global stability of the malaria endemic equilibrium*

The global stability of the malaria endemic equilibrium is ascertained by means of Lyapunov function as demonstrated in the works of [34, 35].
Consider the Lyapunov function given by

$$\mathcal{V} = \frac{1}{2}\bar{X}^2, \tag{4.5}$$

where $\bar{X} = (S_h + S_{hu} + S_{hv} + E_{hm} + I_{hm} + I_{hmT} + I_{hmU} + R_h + S_{mv} + E_{mv} + I_{mv})$.
Then,

$$\frac{d\mathcal{V}}{dt} = \bar{X}\dot{\bar{X}}, \tag{4.6}$$

where $\dot{\bar{X}} = (\dot{S_h} + \dot{S_{hu}} + \dot{S_{hv}} + \dot{E}_{hm} + \dot{I}_{hm} + \dot{I}_{hmT} + \dot{I}_{hmU} + \dot{R}_h + \dot{S}_{mv} + \dot{E}_{mv} + \dot{I}_{mv})$.
Equation (4.6) becomes

$$\frac{d\mathcal{V}}{dt} = \bar{X}(\dot{S_h} + \dot{S_{hu}} + \dot{S_{hv}} + \dot{E}_{hm} + \dot{I}_{hm} + \dot{I}_{hmT} + \dot{I}_{hmU} + \dot{R}_h + \dot{S}_{mv} + \dot{E}_{mv} + \dot{I}_{mv})$$

$$\begin{aligned}
&= \bar{X}(\Lambda_h - \tau_1(S_h + S_{hu} + S_{hv} + E_{hm} + I_{hm} + I_{hmT} + I_{hmU} + R_h) \\
&\quad - \tau_2(I_{hm} + I_{hmT} + I_{hmU}) + \Lambda_{mv} - \mu_m(S_{mv} + E_{mv} + I_{mv}) - \kappa_1 I_{SIT}S_{mv}) \\
&= \bar{X}(\Lambda_h - \tau_1N_h - \tau_2(I_{hm} + I_{hmT} + I_{hmU}) + \Lambda_{mv} - \mu N_{mv} - \kappa_1 I_{SIT}S_{mv})
\end{aligned}$$

Clearly, $0 \leqslant N_h \leqslant \frac{\Lambda_h}{\tau_1}$ and $0 \leqslant N_{mv} \leqslant \frac{\Lambda_{mv}}{\mu}$ as shown in section 4.2.3, which means that $\tau_1 N_h \leqslant \Lambda_h$ and $\mu N_{mv} \leqslant \Lambda_{mv}$.
Using $\tau_1 N_h = \Lambda_h$ and $\mu N_{mv} = \Lambda_{mv}$, then

$$\frac{d\mathcal{V}}{dt} = -\left(\tau_2(I_{hm} + I_{hmT} + I_{hmU}) + \kappa_1 I_{SIT} S_{mv}\right)\bar{X} < 0$$

This shows that the malaria endemic endemic equilibrium point is GAS since $\frac{d\mathcal{V}}{dt} < 0$. The global stability of the malaria endemic equilibrium shows that malaria will persist in the system at a stable level and converge to the malaria endemic equilibrium point with time irrespective of the size of the initial infected population. It also means that the number of infected humans with malaria can never decay to zero.

5. Numerical Simulations

Numerical simulations on the malaria model is performed here using the following initial conditions of the state variables; $S_h = 500, S_{hu} = 320, S_{hv} = 90, E_{hm} = 48, I_{hm} = 35, I_{hmU} = 8, I_{hmT} = 21, R_h = 10, S_{mv} = 2000, E_{mv} = 626, I_{mv} = 275, I_{SIT} = 300$ and the parameter values in Table 2.

Table 2: Parameters and Values

Parameters	Values	Sources	Parameters	Values	Sources
Λ_h	30	Assumed	β_2	0.12	Assumed
Λ_{mv}	100	Assumed	β_3	0.24	[36][38]
Φ	0.0125	Assumed	β_4	0.12	Assumed
θ	0.0005275	[36]	β_5	0.24	[36][38]
ρ_1	0.65	Assumed	δ_1	0.0833	[41]
ρ_2	0.28	Assumed	ν_1	0.1	[41]
τ_1	0.0000432	[1][36][37]	ϵ_1	0.62	Assumed
τ_2	0.0003454	[36][38]	ϵ_2	0.21	Assumed
μ	0.00556	[39]	γ_1	0.142	[37]
α_1	0.5	[38][40]	ϕ_1	0.13	Assumed
β_1	0.321	[36][38]	κ_1	0.5	Assumed

5.1. *Effects of treatment only*

The results of the simulations here shown in Figures 2– shows the effectiveness of treatment as a control measure. The exposed and infectious human populations with malaria reduced significantly under treatment than when there was no treatment. Similarly, the exposed and infectious mosquitoes also reduced when treatment of humans was employed. The reduction in the mosquito population is attributed to reduced infectious human population which can infect the mosquitoes by treatment.

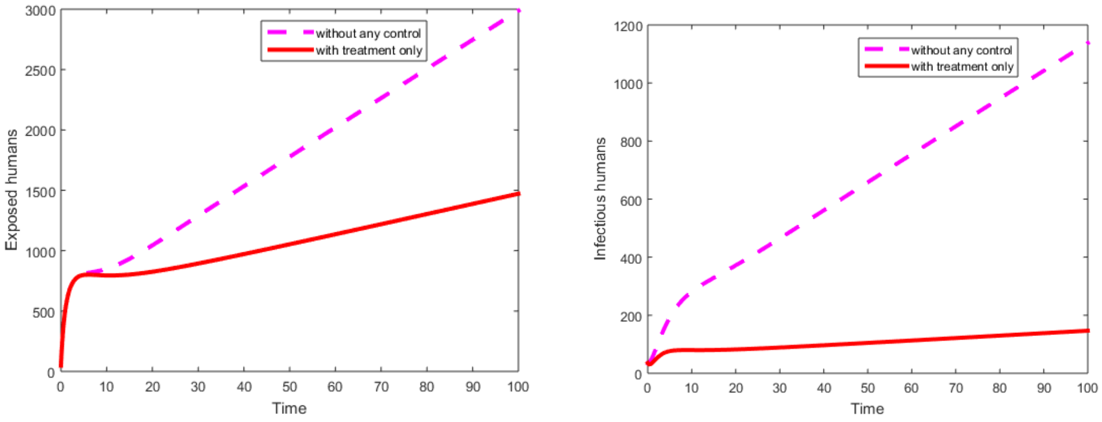

Figure 2: Exposed and Infectious humans with malaria

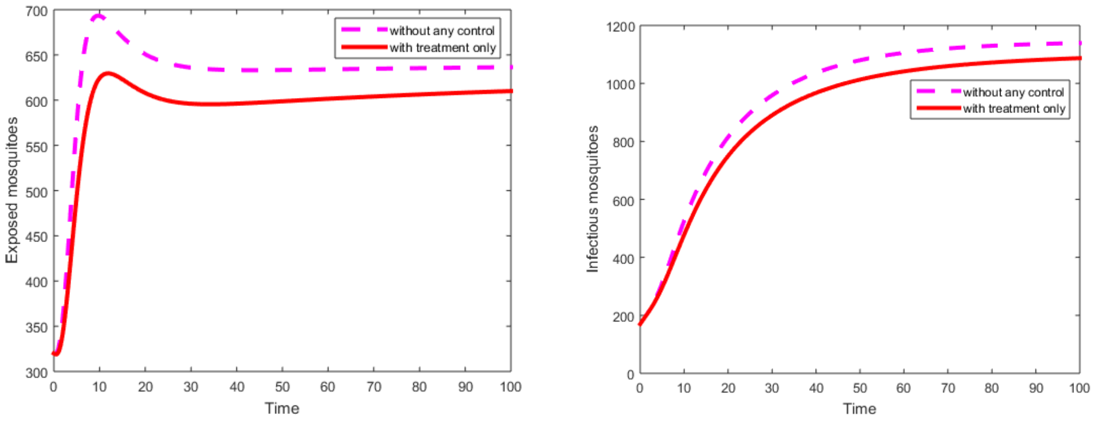

Figure 3: Exposed and Infectious mosquitoes

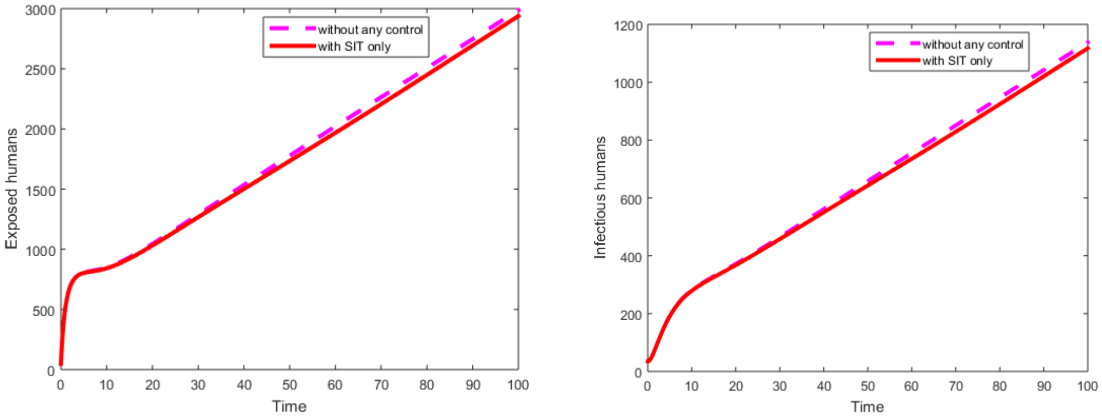

Figure 4: Exposed and Infectious humans with malaria

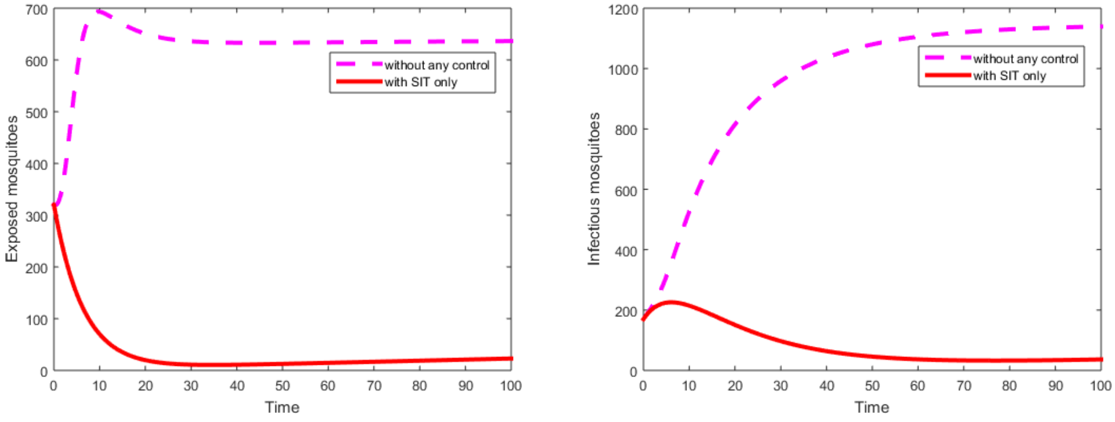

Figure 5: Exposed and Infectious mosquitoes

5.2. *Effects of SIT only*

The effects of SIT on the infectious classes of humans and mosquitoes are shown in Figures 4 – 5. The exposed and infectious human populations with malaria did not reduce significantly under use of SIT. But, the exposed and infectious mosquitoes reduced greatly under the application of SIT. The reduction in the mosquito population is attributed to reduced number of mosquitoes in circulation.

5.3. *Effects of Vaccination only*

The effects of employing only vaccination is shown in Figures 6 – 7. It is seen that vaccination reduced the number of humans susceptible to malaria and consequently, the exposed and infectious humans with malaria reduce too. The infectious mosquito population also reduced slightly as the number of humans who can infect of be infected by the mosquitoes are reduced by vaccination.

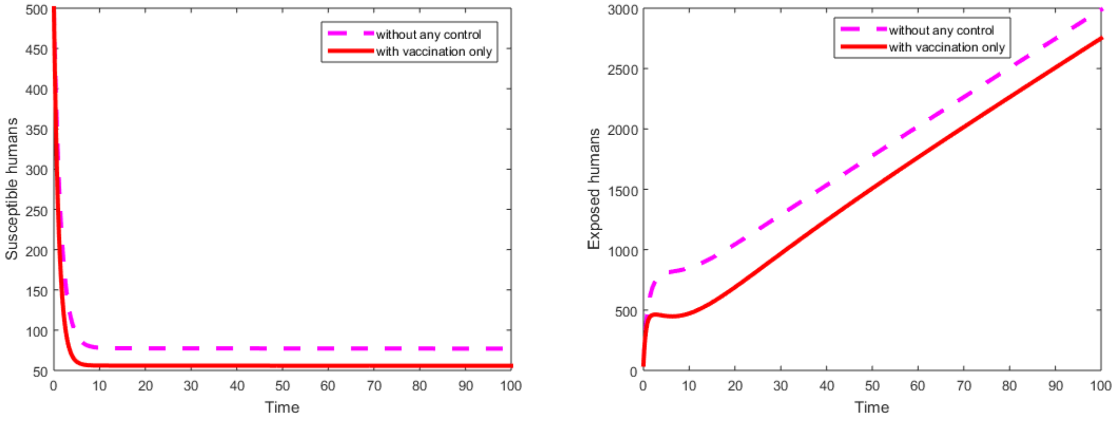

Figure 6: Susceptible and Exposed humans

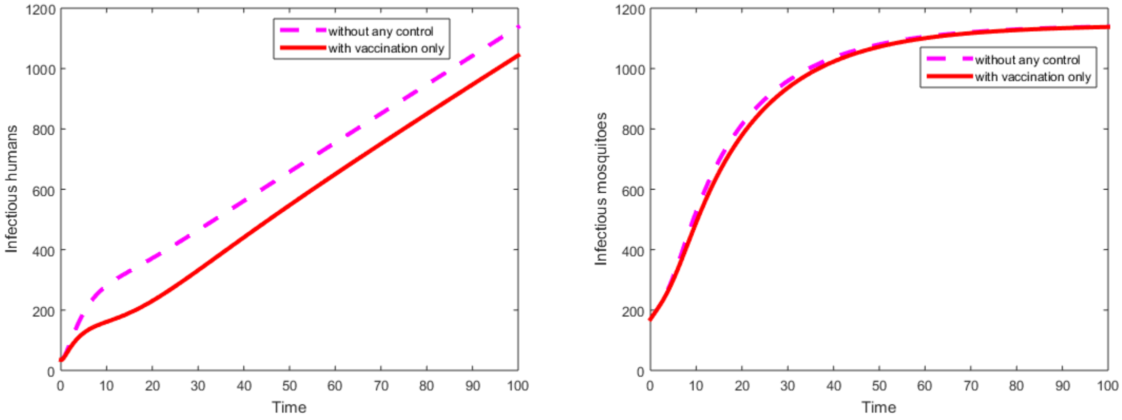

Figure 7: Infectious humans and Infectious mosquitoes

5.4. *Effects of Vaccination and treatment only*

The effects of employing vaccination and treatment as control measures for malaria are shown in Figures 8 – 9. In this simulation, it is seen that combining vaccination and treatment reduced the infectious classes in human and mosquitoes significantly than when vaccination or treatment is employed singly. This shows that combining both measures produced better result than using them individually.

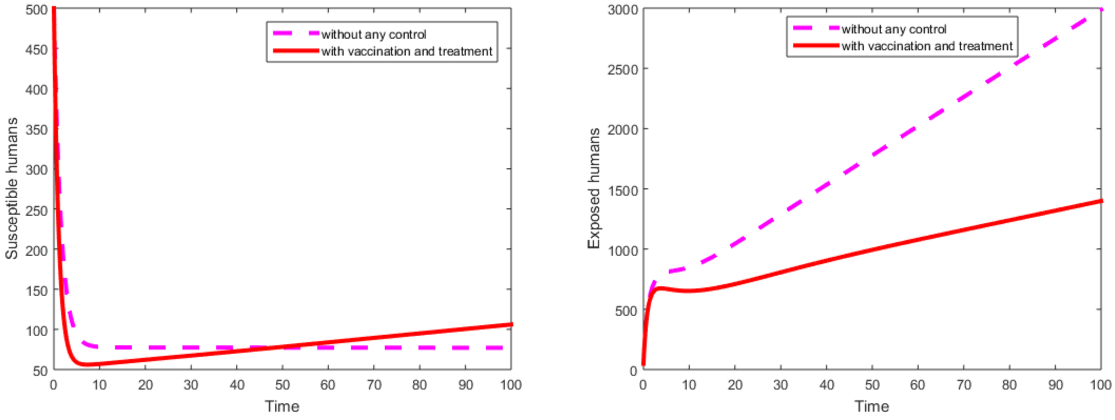

Figure 8: Susceptible and Exposed humans

5.5. *Effects of Vaccination and SIT only*

The effects of employing vaccination and SIT as control measures for malaria are shown in Figures 10 – 11. It is seen that combining vaccination and SIT reduced the infectious classes in human and mosquitoes significantly than when only vaccination or only SIT is employed. This shows that combining both measures produced better result than using them individually. Also, this strategy performed better in mosquito population than in human population when compared with using vaccination and treatment. Hence, the need to combine the three controls and harness the merits of each control.

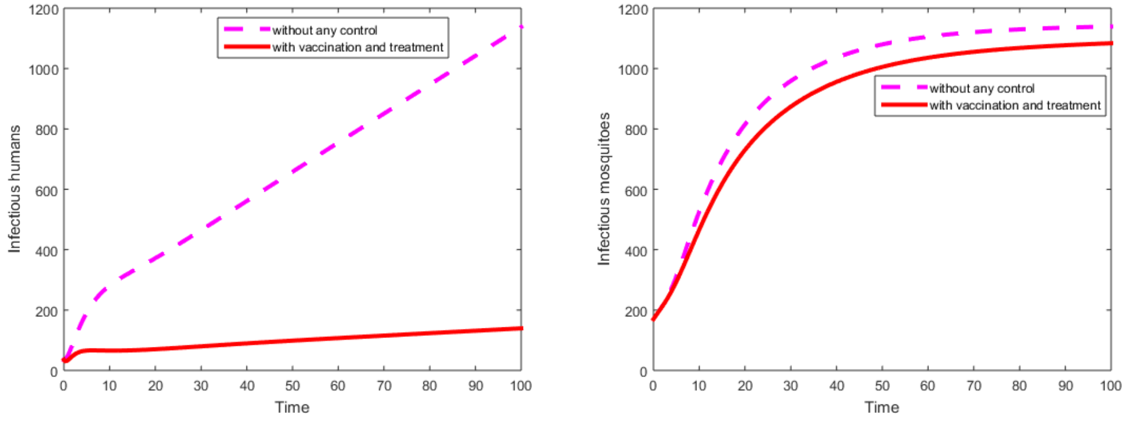

Figure 9: Infectious humans and Infectious mosquitoes

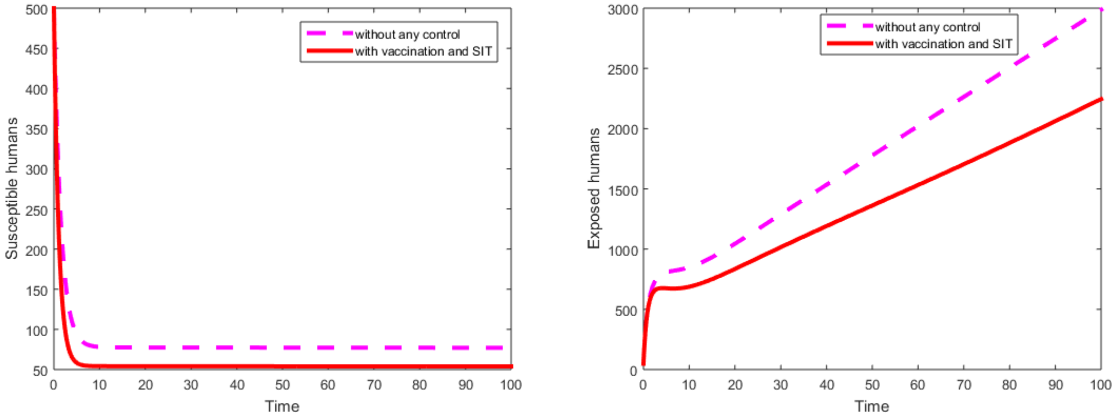

Figure 10: Susceptible and Exposed humans

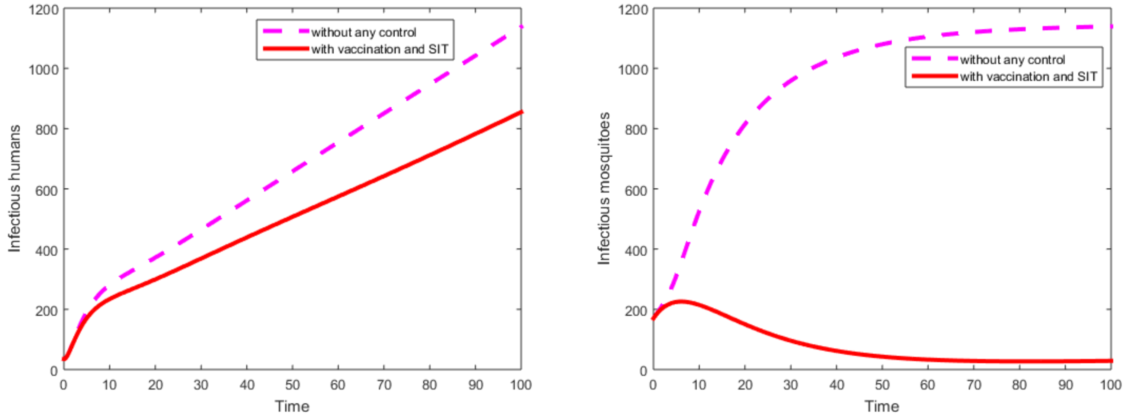

Figure 11: Infectious humans and Infectious mosquitoes

5.6. *Effects of treatment and SIT only*

The effects of employing treatment and SIT as control measures for malaria are shown in Figures 12 – 13. The simulations showed that combining treatment and SIT reduced the infectious classes in human and mosquitoes significantly than when only treatment or only SIT is employed. This shows that combining both measures produced better result than using them individually. Also, this strategy performed better in both mosquito population and human population when compared with using vaccination and treatment or vaccination and SIT.

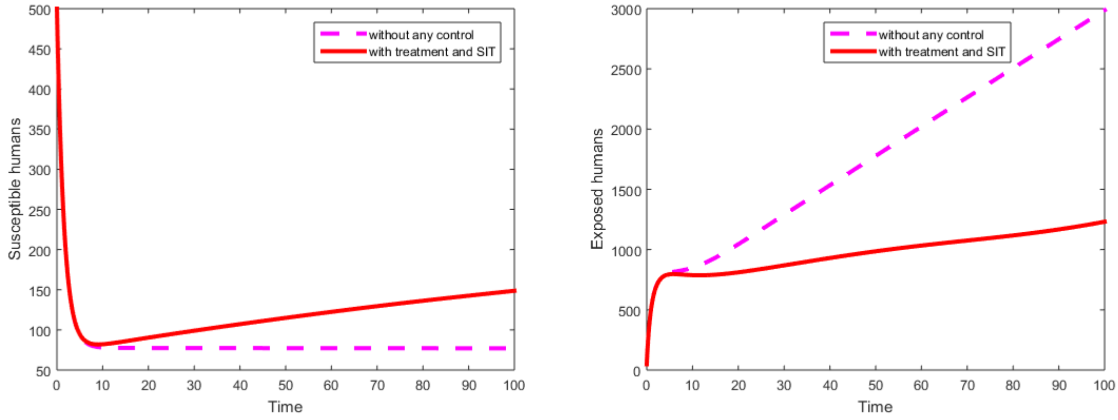

Figure 12: Susceptible and Exposed humans

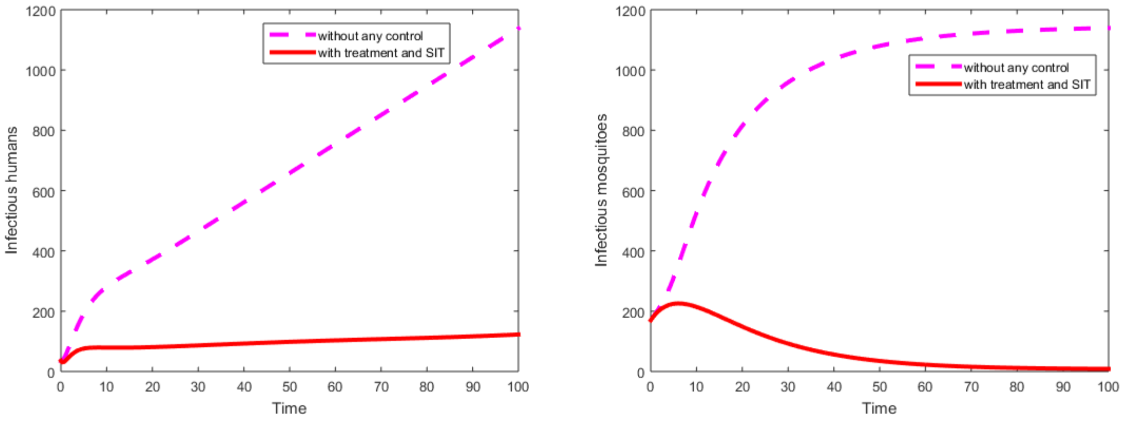

Figure 13: Infectious humans and Infectious mosquitoes

5.7. *Effects of treatment, vaccination and SIT*

The result of the simulation combining treatment, vaccination and SIT are shown in Figures 14 – 16. The combination of the controls are seen to significantly reduce the number of humans susceptible to malaria. The infectious classes in human and mosquito population were also reduced greatly. The combination of the three controls performed better than in any of the previous strategies. Vaccination and treatment are controls targeted directly on human population while SIT is applied to mosquitoes. It could be seen that in using the controls individually, only those populations they are directly applied to were significantly affected. Using the three controls simultaneously has helped to draw from each of their respective strength, showing the need that malaria will be effecively controlled if the intervention measures will focus on both the human and vector population.

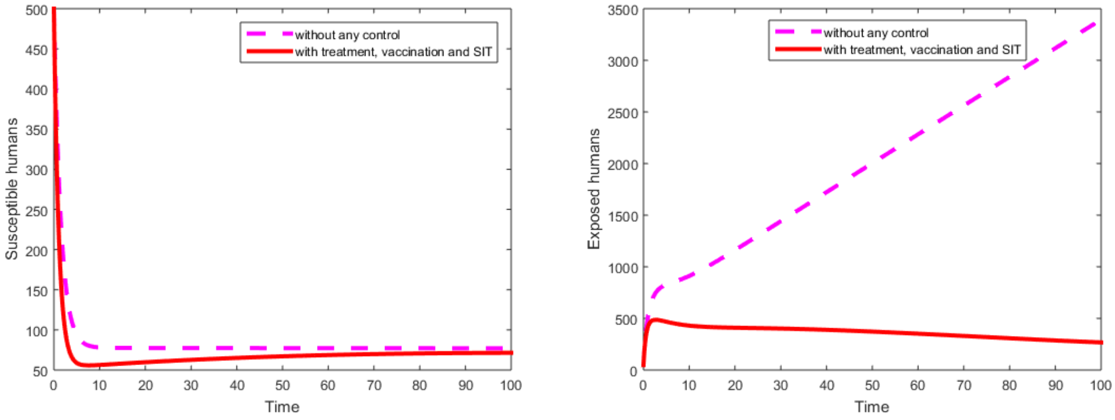

Figure 14: Susceptible and Exposed humans

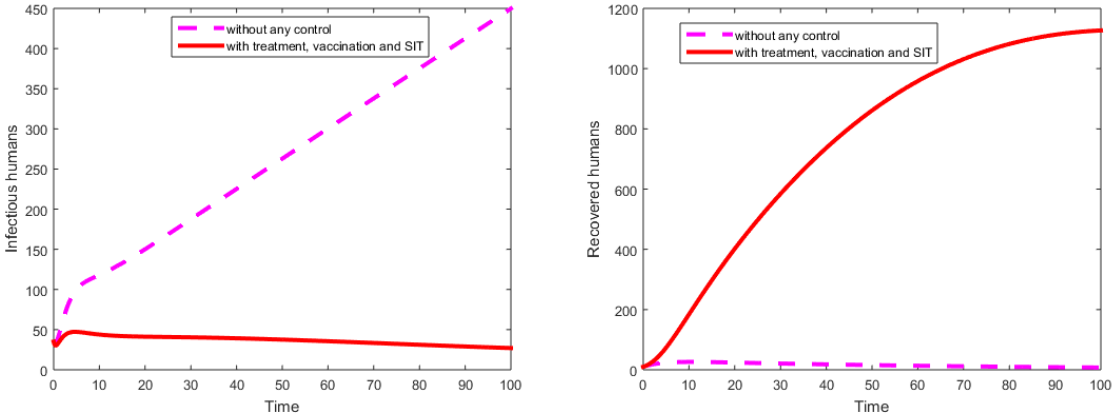

Figure 15: Infectious and Recovered humans

6. Conclusion

In this paper, a new malaria model was presented using nonlinear ordinary differential equation. The model incorporated vaccination, treatment and use of sterile insect technique as control measures. Stability analysis showed that the malaria-free equilibrium is locally and globally asymptotically stable when the malaria control number is less

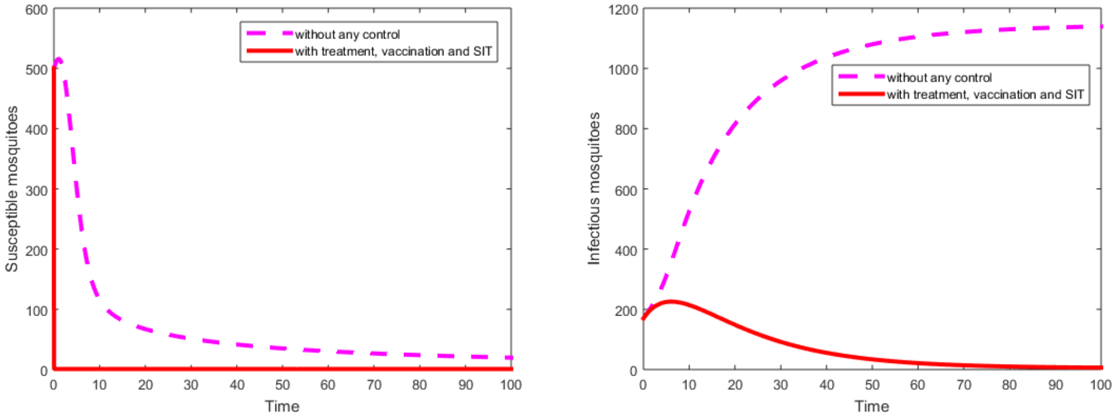

Figure 16: Susceptible and Infectious mosquitoes

than one. The existence of endemic equilibrium was shown and the conditions for local and global stability were obtained. Numerical simulation showed that combining the three controls performed better than using them individually or combining any two of the control. Hence, to effectively control diseases caused by vectors, the control measures should be targeted on both the human and vector population as demonstrated here with mosquitoes.

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