Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2024)5(3) : 89-99
doi:10.48185/jmam.v5i3.1368

An efficient fourth-order method for direct integration of second-order ordinary differential equations

Opeyemi Enoch[a], Emmanuel Adeyefa[b], Catherine Alakofa[c*]

[a,b,c]Department of Mathematics, Federal University Oye-Ekiti, Ekiti State, Nigeria

• Received: 17 October 2024 • Accepted: 08 November 2024 • Published Online: 21 December 2024

Abstract

A novel fourth-order numerical algorithm for the direct integration of second-order differential equations is examined in this paper. The method is derived from a basis function that combines a third-order polynomial with the sum of sine and cosine functions. By leveraging this unique basis function, the proposed method maintains computational efficiency while achieving fourth-order accuracy. It outlines the method's derivation and analyzes its stability and accuracy properties. Numerical experiments demonstrate its effectiveness and efficiency compared to existing techniques. The results indicate that the proposed fourth-order block method offers significant advantages in accuracy and computational cost, making it promising for directly integrating second-order differential equations.

Keywords: Continuous linear multistep method, Zero stability, Collocation.

1. Introduction

Direct integration methods play a crucial role in solving differential equations across various scientific and engineering disciplines. Among these methods, block methods have gained considerable attention due to their high accuracy and computational efficiency, which have been established by various authors in the likes of [1]-[4] to mention but a few. In this paper, a novel fourth-order block method derived from a basis function that combines a third-order polynomial with the sum of sine and cosine functions is considered. This innovative approach aims to enhance the accuracy and stability of direct integration techniques for second-order differential equations.

The choice of basis function is pivotal in the design of numerical integration methods, as it directly influences the accuracy and convergence properties of the scheme. By combining a third-order polynomial with sine and cosine functions, we leverage the strengths of both polynomial approximation and trigonometric functions, offering a flexible and effective framework for capturing complex dynamics in differential equations. This method

*Corresponding author: alakofaoluwaseyi@gmail.com

allows for improved accuracy, particularly in oscillatory and periodic systems, where traditional methods may struggle to maintain stability.

The proposed fourth-order block method builds upon existing block integration techniques, which partition the integration interval into smaller blocks and apply high-order approximations within each block. By incorporating the unique basis function into the formulation, the method achieves fourth-order accuracy while retaining the computational efficiency characteristic of block methods. This balance between accuracy and efficiency is essential for practical applications, where numerical solutions must strike a compromise between precision and computational cost.

In this paper, a comprehensive derivation and analysis of the proposed fourth-order block method is established and its stability properties and conduct rigorous numerical experiments to validate its performance across a range of differential equation problems are examined. By comparing the results with existing integration techniques, we demonstrate the superiority of our approach in both accuracy and efficiency.

2. Derivation of the Method

We investigate the formulation of a three-step block technique for the solution of the form:

$$y^{''} = f(x, y, y^{'}), \;\; y(a) = \zeta_0, \; y^{'}(a) = \zeta_1 \;\; x\varepsilon[a, b] \tag{2.1}$$

In formulating this method, we combine polynomials with the sum of sine and cosine functions in the following manner:

$$y = \sum_{n=0}^{3} a_n x^n + a_4 \sin(\varpi x) + a_5 \cos(\varpi x) \tag{2.2}$$

This combination is regarded as an approximate solution to equation (1). Equation (2), when differentiated twice yields

$$y^{''} = \sum_{n=2}^{3} n(n-1)a_n x^{n-2} - a_4 \sin(\varpi x) - a_5 \cos(\varpi x) \tag{2.3}$$

equation (3) is Collocated at $x = x_{n+j}, \; j = 0(1)3$ and interpolating equation (2) at $x = x_{n+j}, \; j = 0, 1$ produces six non-singular equations that can be formulated as a system in matrix form in equation (4).

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & \sin \varpi x_n & \cos \varpi x_n \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \sin \varpi x_{n+1} & \cos \varpi x_{n+1} \\ 0 & 0 & 2 & 6x_n & -\varpi^2 \sin \varpi x_n & -\varpi^2 \cos \varpi x_n \\ 0 & 0 & 2 & 6x_{n+1} & -\varpi^2 \sin \varpi x_{n+1} & -\varpi^2 \cos \varpi x_{n+1} \\ 0 & 0 & 2 & 6x_{n+2} & -\varpi^2 \sin \varpi x_{n+2} & -\varpi^2 \cos \varpi x_{n+2} \\ 0 & 0 & 2 & 6x_{n+3} & -\varpi^2 \sin \varpi x_{n+3} & -\varpi^2 \cos \varpi x_{n+3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{2.4}$$

Then solving equation(4) using Gaussian elimination method yields values for the unknown parameters $a_n(s)$ which when substituted into (2) gives the continuous linear multistep method

$$y(t) = \alpha_0(t)y_n + \alpha_1(t)y_{n+1} + h^2\left[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3}\right] \quad (2.5)$$

Using $t = \frac{x-x_n}{h}$, $\frac{dt}{dx} = \frac{1}{h}$, The coefficients of y_{n+j} and f_{n+j} are obtained with respect to t as follows:

$$\alpha_0(t) = -(t-1)$$

$$\alpha_1(t) = t$$

$$\beta_0(t) = \frac{1}{24}\left(\frac{24\sin ht\varpi - 24t\sin h\varpi + t\varpi^2\sin h\varpi - 12h^2t^2\varpi^2\sin^3 h\varpi + 6h^2t^3\varpi^2\sin^3 h\varpi - h^2t^4\varpi^2\sin^3 h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right.$$
$$+\frac{4t\varpi^2\cos h\varpi\sin h\varpi - 26h^2t\varpi^2\sin h\varpi - 3t\varpi^2\cos^2 h\varpi\sin h\varpi + 36h^2t^2\varpi^2\sin h\varpi + 6h^2t\varpi^2\sin^3 h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}$$
$$\frac{-10h^2t^3\varpi^2\sin h\varpi + h^2t^4\varpi^2\sin h\varpi - 18h^2t\varpi^2\cos^2 h\varpi\sin h\varpi - 72h^2t^2\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}$$
$$\left.+\frac{32h^2t^3\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right)$$

$$\beta_1(t) = -\frac{1}{12}\left(\frac{36\sin ht\varpi - 36t\sin h\varpi + t\varpi^2\sin^3 h\varpi + 4h^2t^3\varpi^2\sin^3 h\varpi - h^2t^4\varpi^2\sin^3 h\varpi + 3t\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right.$$
$$\frac{-3t\varpi^2\cos^2 h\varpi\sin h\varpi - 4h^2t\varpi^2\sin^3 h\varpi + 12h^2t\varpi^2\cos^2 h\varpi\sin h\varpi + 18h^2t^3\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}$$
$$\left.\frac{-3h^2t^4\varpi^2\cos h\varpi\sin h\varpi - 12h^2t^3\varpi^2\cos^2 h\varpi\sin h\varpi + 3h^2t^4\varpi^2\cos^2 h\varpi\sin h\varpi - 18h^2t\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right)$$

$$\beta_2(t) = \frac{1}{24}\left(\frac{72\sin ht\varpi - 72t\sin h\varpi + t\varpi^2\sin^3 h\varpi + 3t\varpi^2\sin h\varpi + 2h^2t^3\varpi^2\sin^3 h\varpi - h^2t^4\varpi^2\sin^3 h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right.$$
$$\frac{-18h^2t\varpi^2\sin h\varpi - 3t\varpi^2\cos^2 h\varpi\sin h\varpi - 2h^2t\varpi^2\sin h\varpi + 18h^2t^3\varpi^2\sin h\varpi - 3h^2t^4\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}$$
$$\left.\frac{+6h^2t\varpi^2\cos^2 h\varpi\sin h\varpi + 3h^2t^4\varpi^2\cos^2 h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right)$$

$$\beta_3(t) = -\frac{1}{12}\left(\frac{12\sin ht\varpi - 12t\sin h\varpi + t\varpi^2\sin h\varpi - t\varpi^2\cos h\varpi\sin h\varpi - 4h^2t\varpi^2\sin h\varpi + 4h^2t^3\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right.$$
$$\left.\frac{-h^2t^4\varpi^2\sin h\varpi - 2h^2t^3\varpi^2\cos h\varpi\sin h\varpi + h^2t^4\varpi^2\cos h\varpi\sin h\varpi + 2h^2t\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(-6\cos h\varpi + 3\cos^2 h\varpi - \sin^2 h\varpi + 3)}\right)$$

(2.6)

Equation (6) is evaluated at the non-interpolation points i.e. at $t = 2, 3$ which is written in the series form by letting $u = \varpi h$ to give:

$$\left.\begin{aligned}
&y_{n+2} = 2y_{n+1} - y_n + h^2\left[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3}\right],\\
&\beta_0 = \frac{1}{12} + \frac{1}{240}u^2 + \frac{1}{6048}u^4 + \frac{1}{172800}u^6 + \frac{1}{5322240}u^8 + O(u^{10})\\
&\beta_1 = \frac{5}{6} - \frac{1}{120}u^2 - \frac{1}{3024}u^4 - \frac{1}{86400}u^6 - \frac{1}{2661120}u^8 + O(u^{10})\\
&\beta_2 = \frac{1}{12} + \frac{1}{240}u^2 + \frac{1}{6048}u^4 + \frac{1}{172800}u^6 + \frac{1}{5322240}u^8 + O(u^{10})\\
&\beta_3 = 0
\end{aligned}\right\} \quad (2.7)$$

$$\left.\begin{aligned}
&y_{n+3} = 3y_{n+1} - 2y_n + h^2\left[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3}\right],\\
&\beta_0 = \frac{1}{6} + \frac{1}{120}u^2 + \frac{1}{3024}u^4 + \frac{1}{86400}u^6 + \frac{1}{2661120}u^8 + O(u^{10})\\
&\beta_1 = \frac{7}{4} - \frac{1}{80}u^2 - \frac{1}{2016}u^4 - \frac{1}{57600}u^6 - \frac{1}{1774080}u^8 + O(u^{10})\\
&\beta_2 = 1 + O(u^{10})\\
&\beta_3 = \frac{1}{12} + \frac{1}{240}u^2 + \frac{1}{6048}u^4 + \frac{1}{172800}u^6 + \frac{1}{5322240}u^8 + O(u^{10})
\end{aligned}\right\} \quad (2.8)$$

Also the first derivative of (6) gives

$$
\begin{aligned}
&\alpha_0' = \frac{-1}{h}\\
&\alpha_1' = \frac{1}{h}\\
&\beta_0'(t) =\\
&\frac{1}{96}\Bigg(\frac{-48\sin h\varpi + 2\varpi^2\sin h\varpi - 6\varpi^2\cos^2 h\varpi\sin h\varpi + 8\varpi^2\sin h\varpi\cos h\varpi - 52h^2\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+126h^2t\varpi^2\sin h\varpi + 6h^2t\varpi^2\sin 3h\varpi - 36h^2\varpi^2\cos^2 h\varpi\sin h\varpi - 33h^2t^2\varpi^2\sin h\varpi - 6h^2t^3\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{-9h^2t^2\varpi^2\sin 3h\varpi + 2h^2t^3\varpi^2\sin 3h\varpi + 48h\varpi\cos ht\varpi + 8h^2t^3\varpi^2\sin h\varpi + 192h^2t^2\varpi\sin h\varpi\cos h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{-288h^2t\varpi^2\cos h\varpi\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\Bigg)\\
&\beta_1'(t) =\\
&-\frac{1}{96}\Bigg(\frac{-144\sin h\varpi + 3\varpi^2\sin h\varpi - \varpi\sin 3h\varpi - 12\varpi^2\cos^2 h\varpi\sin h\varpi - 12h^2\varpi^2\sin h\varpi + 4h^2\varpi^2\sin 3h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+12\varpi^2\sin h\varpi\cos h\varpi + 48h^2\varpi^2\cos^2 h\varpi\sin h\varpi + 36h^2t^2\varpi^2\sin h\varpi - 12h^2t^3\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{-12h^2t^2\varpi^2\sin 3h\varpi + 4h^2t^3\varpi^2\sin 3h\varpi + 144h\varpi\cos ht\varpi - 72h^2\varpi^2\sin h\varpi\cos h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+216h^2t^2\varpi^2\sin h\varpi\cos h\varpi - 48h^2t^3\varpi^2\sin h\varpi\cos h\varpi - 144h^2t^2\varpi^2\sin h\varpi\cos^2 h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+48h^2t^3\varpi^2\sin h\varpi\cos^2 h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\Bigg)\\
&\beta_2'(t) =\\
&\frac{1}{192}\Bigg(\frac{-288\sin h\varpi - 3\varpi^2\sin h\varpi - \varpi^2\sin 3h\varpi - 3\varpi^2\sin h\varpi(-2) - 3\varpi^2\sin h\varpi(+2) + 12\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{-6h^2\varpi^2\sin h\varpi + 2h^2\varpi^2\sin 3h\varpi - 66h^2\varpi^2\sin h\varpi + 6h^2\varpi^2\sin 3h\varpi + 234h^2t^2\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{-60h^2t^3\varpi^2\sin h\varpi - 6h^2t^2\varpi^2\sin 3h\varpi + 4h^2t^3\varpi^2\sin 3h\varpi + 288h\varpi\cos ht\varpi - 18h^2t^2\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+12h^2t^3\varpi^2\sin h\varpi - 18h^2t^2\varpi^2\sin 3h\varpi + 12h^2t^3\varpi^2\sin 3h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\Bigg) \qquad (2.9)\\
&\beta_3'(t) =\\
&-\frac{1}{96}\Bigg(\frac{-48\sin h\varpi + 3\varpi^2\sin h\varpi - \varpi^2\sin 3h\varpi - 4\varpi^2\cos h\varpi\sin h\varpi + 16h^2\varpi^2\sin h\varpi + 48h^2t^2\varpi^2\sin h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+48h\varpi\cos ht\varpi + 8h^2\varpi^2\sin h\varpi\cos h\varpi - 16h^2t^3\varpi^2\sin h\varpi - 24h^2t^2\varpi^2\sin h\varpi\cos h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\\
&\frac{+16h^2t^3\varpi^2\sin h\varpi\cos h\varpi}{\varpi^2(\sin h\varpi)(2\cos h\varpi - 1)(\cos h\varpi - 1)}\Bigg)
\end{aligned}
$$

Evaluating equation(9) at all points, which gives the following equations in the series form:

$$\left.\begin{aligned}
hy'_n &= y_{n+1} - y_n + h^2\left[\beta'_0(t)f_n + \beta'_1(t)f_{n+1} + \beta'_2(t)f_{n+2} + \beta'_3(t)f_{n+3}\right],\\
\beta'_0 &= -\frac{97}{180} - \frac{107}{5040}u^2 - \frac{629}{453600}u^4 - \frac{4633}{39916800}u^6 - \frac{399383}{36324288000}u^8 + O(u^{10})\\
\beta'_1 &= -\frac{19}{60} + \frac{29}{1680}u^2 + \frac{137}{151200}u^4 + \frac{859}{13305600}u^6 + \frac{(68879}{12108096000}u^8 + O(u^{10})\\
\beta'_2 &= \frac{13}{120} - \frac{3}{1120}u^2 + \frac{3}{11200}u^4 + \frac{19}{422400}u^6 + \frac{13763}{2690688000}u^8 + O(u^{10})\\
\beta'_3 &= -\frac{1}{45} - \frac{1}{252}u^2 - \frac{109}{226800}u^4 - \frac{257}{4989600}u^6 - \frac{96373}{18162144000}u^8 + O(u^{10})
\end{aligned}\right\} \quad (2.10)$$

$$\left.\begin{aligned}
hy'_{n+1} &= y_{n+1} - y_n + h^2\left[\beta'_0(t)f_n + \beta'_1(t)f_{n+1} + \beta'_2(t)f_{n+2} + \beta'_3(t)f_{n+3}\right],\\
\beta'_0 &= \frac{19}{180} + \frac{41}{5040}u^2 + \frac{293}{453600}u^4 + \frac{2287}{39916800}u^6 + \frac{199571}{36324288000}u^8 + O(u^{10})\\
\beta'_1 &= \frac{19}{40} - \frac{127}{10080}u^2 - \frac{251}{302400}u^4 - \frac{1703}{26611200}u^6 - \frac{1246753}{217945728000}u^8 + O(u^{10})\\
\beta'_2 &= -\frac{1}{10} + \frac{1}{1260}u^2 - \frac{1}{3600}u^4 - \frac{73}{1663200}u^6 - \frac{274693}{54486432000}u^8 + O(u^{10})\\
\beta'_3 &= \frac{7}{360} + \frac{37}{10080}u^2 + \frac{419}{907200}u^4 + \frac{577}{11404800}u^6 + \frac{1148099}{217945728000}u^8 + O(u^{10})
\end{aligned}\right\} \quad (2.11)$$

$$\left.\begin{aligned}
hy'_{n+2} &= y_{n+1} - y_n + h^2\left[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3}\right],\\
\beta'_0 &= \frac{23}{360} + \frac{1}{2016}u^2 - \frac{269}{907200}u^4 - \frac{511}{11404800}u^6 - \frac{1107149}{217945728000}u^8 + O(u^{10})\\
\beta'_1 &= \frac{61}{60} - \frac{5}{1008}u^2 + \frac{17}{151200}u^4 + \frac{169}{4435200}u^6 + \frac{528911}{108972864000}u^8 + O(u^{10})\\
\beta'_2 &= \frac{53}{120} + \frac{17}{2016}u^2 + \frac{67}{100800}u^4 + \frac{1549}{26611200}u^6 + \frac{1205803}{217945728000}u^8 + O(u^{10})\\
\beta'_3 &= -\frac{1}{45} - \frac{1}{252}u^2 - \frac{109}{226800}u^4 - \frac{257}{4989600}u^6 - \frac{96373}{18162144000}u^8 + O(u^{10})
\end{aligned}\right\} \quad (2.12)$$

$$\left.\begin{aligned}
hy'_{n+3} &= y_{n+1} - y_n + h^2\left[\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3}\right],\\
\beta'_0 &= \frac{19}{180} + \frac{41}{5040}u^2 + \frac{293}{453600}u^4 + \frac{2287}{39916800}u^6 + \frac{199571}{36324288000}u^8 + O(u^{10})\\
\beta'_1 &= \frac{97}{120} - \frac{1}{672}u^2 - \frac{131}{302400}u^4 - \frac{193}{3801600}u^6 - \frac{128417}{24216192000}u^8 + O(u^{10})\\
\beta'_2 &= \frac{37}{30} - \frac{3}{140}u^2 - \frac{3}{2800}u^4 - \frac{13}{184800}u^6 - \frac{3953}{672672000}u^8 + O(u^{10})\\
\beta'_3 &= \frac{127}{360} + \frac{149}{10080}u^2 + \frac{779}{907200}u^4 + \frac{1019}{15966720}u^6 + \frac{413033}{72648576000}u^8 + O(u^{10})
\end{aligned}\right\} \quad (2.13)$$

Using matrix inversion on the methods in equations $(7-8)$, yields the block method as seen below $\begin{bmatrix} y_{n+3} \\ y_{n+2} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y_n + h \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} y'_n + h^4 \begin{bmatrix} \frac{39}{40} \\ \frac{28}{45} \\ \frac{97}{360} \end{bmatrix} f_n + h^4u^2 \begin{bmatrix} \frac{9}{224} \\ \frac{8}{315} \\ \frac{107}{10080} \end{bmatrix} f_n +$

$$h^4u^4\begin{bmatrix}\frac{27}{11200}\\ \frac{22}{14175}\\ \frac{629}{907200}\end{bmatrix}f_n+h^4u^6\begin{bmatrix}\frac{183}{985600}\\ \frac{19}{155925}\\ \frac{4633}{79833600}\end{bmatrix}f_n$$
$$+h^4u^8\begin{bmatrix}\frac{15129}{89689000}\\ \frac{577}{51597000}\\ \frac{399383}{72648576000}\end{bmatrix}f_n+h^4\begin{bmatrix}\frac{3}{20} & \frac{27}{40} & \frac{27}{10}\\ \frac{2}{45} & \frac{-2}{5} & \frac{22}{15}\\ \frac{1}{45} & \frac{-13}{120} & \frac{19}{60}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}$$
$$+h^4u^2\begin{bmatrix}\frac{9}{560} & \frac{9}{1120} & \frac{-9}{140}\\ \frac{1}{126} & \frac{1}{105} & \frac{-3}{70}\\ \frac{1}{252} & \frac{3}{1120} & \frac{-29}{1680}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}+h^4u^4\begin{bmatrix}\frac{9}{5600} & \frac{-9}{11200} & \frac{-9}{2800}\\ \frac{109}{113400} & \frac{-1}{2700} & \frac{-3}{1400}\\ \frac{109}{226800} & \frac{-3}{11200} & \frac{-137}{151200}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}$$
$$+h^4u^6\begin{bmatrix}\frac{79}{492800} & \frac{-19}{140800} & \frac{-13}{61600}\\ \frac{257}{249480} & \frac{-1}{11880} & \frac{-13}{92400}\\ \frac{257}{4989600} & \frac{-19}{422400} & \frac{-859}{13305600}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}+h^4u^8\begin{bmatrix}\frac{7223}{448448000} & \frac{-13763}{896896000} & \frac{-3953}{224224000}\\ \frac{96373}{9081072000} & \frac{-15199}{1513512000} & \frac{-3953}{336336000}\\ \frac{96373}{18162144000} & \frac{-13763}{2690688000} & \frac{-68879}{12108096000}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix} \tag{2.14}$$

Substituting the schemes that made up the block into equations (10)-(13), gives the equation below

$$\begin{bmatrix}y'_{n+3}\\ y'_{n+2}\\ y'_{n+1}\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}y'_n+h^3\begin{bmatrix}\frac{3}{8}\\ \frac{1}{3}\\ \frac{3}{8}\end{bmatrix}f_n+h^3u^2\begin{bmatrix}\frac{3}{160}\\ \frac{1}{90}\\ \frac{3}{160}\end{bmatrix}f_n+h^3u^4\begin{bmatrix}\frac{3}{2240}\\ \frac{1}{2520}\\ \frac{3}{2240}\end{bmatrix}f_n+h^3u^6\begin{bmatrix}\frac{31}{268800}\\ \frac{1}{75600}\\ \frac{31}{268800}\end{bmatrix}f_n+$$
$$h^3u^8\begin{bmatrix}\frac{13}{1182720}\\ \frac{1}{2395008}\\ \frac{13}{1182720}\end{bmatrix}f_n$$
$$+h^3\begin{bmatrix}\frac{3}{8} & 0 & \frac{1}{24}\\ \frac{9}{8} & \frac{1}{3} & \frac{-5}{24}\\ \frac{9}{8} & \frac{4}{3} & \frac{19}{24}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}+h^3u^2\begin{bmatrix}\frac{3}{160} & 0 & \frac{11}{1140}\\ \frac{-3}{160} & \frac{1}{90} & \frac{1}{288}\\ \frac{-3}{160} & \frac{-1}{45} & \frac{-43}{1440}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}+h^3u^4\begin{bmatrix}\frac{3}{2240} & 0 & \frac{19}{20160}\\ \frac{-3}{2240} & \frac{1}{2520} & \frac{-11}{20160}\\ \frac{-3}{2240} & \frac{-1}{1260} & \frac{-1}{576}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}$$
$$+h^3u^6\begin{bmatrix}\frac{31}{268800} & 0 & \frac{247}{2419200}\\ \frac{-31}{268800} & \frac{1}{75600} & \frac{-43}{483840}\\ \frac{-31}{268800} & \frac{-1}{37800} & \frac{-311}{2419200}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix}+h^3u^8\begin{bmatrix}\frac{13}{1182720} & 0 & \frac{1013}{95800320}\\ \frac{-13}{1182720} & \frac{1}{2395008} & \frac{-139}{13685760}\\ \frac{-13}{1182720} & \frac{-1}{1197504} & \frac{1093}{95800320}\end{bmatrix}\begin{bmatrix}f_{n+3}\\ f_{n+2}\\ f_{n+1}\end{bmatrix} \tag{2.15}$$

3. Analysis of the block method

3.1. Order and Error Constant

The technique used by [4] in finding the order of a method is also adopted in establishing the order of this new developed block method. With this, the proposed block method is of order $p = 4$ and error constants given by the vector

$$C_7=\left[\frac{-1}{240},\frac{-1}{80},\frac{-79}{120}\right]^T$$

3.2. Zero Stability of the Block

Definiton: The block method is zero-stable if the roots of the first characteristic polynomial have a modulus less than or equal to one, with any roots of modulus one being

simple.
$\rho(z) = \det[zA^{(0)} - A^{(i)} = 0]$

$$\left[z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$= z^2(z-1) = 0, \; z = 0, 0, 1$$

Therefore, the block is zero-stable.

3.3. Consistency

The trigonometric-fitted block method is consistent because its order is greater than 1.

3.4. Convergence

Theorem 3.1[5]: For a linear multistep method to be convergent, it is both necessary and sufficient for the method to be consistent and zero-stable. Therefore, the proposed block method is convergent, as it is both zero-stable and consistent.

4. Numerical Experiments

This section evaluates the performance of the new method by applying it to some second-order odes problems. The results of these test problems are presented in tabular form.

Test Problem 1 :

$$y'' = y + xe^{(3x)}, \;\; y(0) = \frac{-3}{32}, \; y'(0) = \frac{-5}{32}, \; h = 0.1, \; \varpi = 10$$

Exact Solution : $y(x) = \frac{(4x-3)}{32e^{-3x}}$

Table 1: Numerical results and absolute error for Test Problem 1, compared with the error reported in [6]

x	Exact	Computed	Error	Error in [6]
0.0025	-0.0941409157618488	-0.0941409157618444	4.4E(-15)	1.186113E(-10)
0.005	-0.0945324041423388	-0.0945324041423240	1.48E(-14)	2.999645E(-10)
0.0075	-0.0949244516083878	-0.0949244516083558	3.20E(-14)	4.578383E(-10)
0.001	-0.0953170443907003	-0.0953170443906451	5.52E(-14)	7.570066E(-10)
0.0125	-0.0957101684809806	-0.0957101684808957	8.49E(-14)	1.121167E(-09)
0.015	-0.0961038096291134	-0.0961038096289918	1.216E(-13)	1.461016E(-09)
0.0175	-0.0964979533403162	-0.0964979533401514	1.648E(-13)	1.947207E(-09)
0.02	-0.0968925848722641	-0.0968925848720495	2.146E(-13)	2.500714E(-09)
0.0225	-0.0972876892321841	-0.0972876892319130	2.711E(-13)	3.029054E(-09)
0.025	-0.0976832511739197	-0.0976832511735851	3.346E(-13)	3.708945E(-09)

Test Problem 2:

$$y'' = -1000y - 1001y', \quad y(0) = 1, \ y'(0) = -1, \ h = 0.1 \ \varpi = 10$$

Exact Solution: $y(x) = \exp(-x)$

Table 2: Numerical results and absolute error for Test Problem 2, compared with the error reported in [7]

x	Exact	Computed	Error	Error in [7]
0.1	0.904837418035959520	0.904837414702050700	3.333908816E-(9)	3.332446E(-9)
0.2	0.818730753077981820	0.818730752337564200	7.404176161E(-10)	6.388696E(-9)
0.3	0.740818220681717770	0.740818222267337070	1.585619303E-(9)	9.158701E(-9)
0.4	0.670320046035639330	0.670320045222151160	8.134881657E(-10)	1.163977E(-8)
0.5	0.606530659712633420	0.606530660715495220	1.002861794E(-9)	1.383497 E(-8)
0.6	0.548811636094026390	0.548811638417330610	2.323304216E(-9)	1.575187E(-8)
0.7	0.496585303791409470	0.496585304404218890	6.128094143E(-10)	1.740135E(-8)
0.8	0.449328964117221560	0.449328966001801610	1.884580048E(-9)	1.879678 E(-8)
0.9	0.406569659740599110	0.406569662292013910	2.551414802E(-9)	1.995322E(-8)
1.0	0.367879441171442330	0.367879442516645170	1.345202838E(-9)	2.088685E(-8)

Test Problem 3 : **Resonance Vibration of a Machine**

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger which moves vertically up and down by a fly wheel spinning at constant set speed. The constant rotational speed of the fly wheel makes the impact force on the metal sheet, and therefore the supporting base, intermittent and cyclic. The bearing base on which the metal sheet is situated has a mass, $M = 2000kg$. The force acting on the base follows a function: $F(t) = 2000\sin(10t)$,in which t-time in seconds. The base is supported by an elastic pad with an equivalent spring constant $k = 2x10^5 N/m$. Determine the differential equation for the instantaneous position of the base $y(t)$ if the base is initially depressed down by an amount $0.1m$.

Solution :

The mass-spring system above is modeled as differential equation as: The Bearing base mass = $2000kg$

Spring constant $k = 2x10^5 N/m$

Force (ma) on the metal sheet = $m\frac{d^2y}{dt^2} = my''$

i.e. $ma = my'' = 2000\sin(10t)$; where$a = y''$

Initial conditions on the system are $y(t_0) = y_0$; $\frac{dy}{dt}|_{t=0} = y'(t_0) = y'_0; t_0 = 0, y'_0 = 0.1$

Therefore, the governing equation for the instantaneous position of the base $y(t)$ is given by

$$My'' + ky = F(t); y(t_0) = y_0,\ y^{'}(t_0) = y^{'}(0)$$

$$2000y'' + 2 \times 10^5 y = 2000\sin 10t,\ y_0^{'} = 0,\ y_0 = 0.1\ w = 10$$

Exact Solution : $y(t) = \frac{1}{10}\cos 10t + \frac{1}{200}\sin 10t - \frac{t}{20}\cos 10t$

Table 3: Numerical results and absolute error for Test Problem 1, compared with the error reported in [8]

x	Exact	Computed	Error	Error in [8]
0.01	0.099502081528397705	0.099502081546866334	1.8468629E(-11)	3.849660E(-10)
0.02	0.098019937860258227	0.098019937945210906	8.4952679E(-11)	8.702471E(-10)
0.03	0.095578245212178891	0.095578245436552547	2.24373656E(-10)	2.929087E(-9)
0.04	0.092211069123825990	0.092211069584013060	4.60187070E(-10)	4.479915E(-9)
0.05	0.087961427477332355	0.087961428291232137	8.13899782E(-10)	4.075598E(-9)
0.06	0.082880767013213972	0.082880768317830665	1.304616693E(-9)	7.575420E(-9)
0.07	0.077028359509141588	0.077028361457765235	1.948623647E(-9)	1.060883E(-9)
0.08	0.070470624551825494	0.070470627310838490	2.759012996E(-9)	1.747335E(-9)
0.09	0.063280386517985873	0.063280390263343552	3.745357679E(-9)	1.836375E(-9)
0.1	0.055536073981512756	0.055536078894951775	4.913439019E(-9)	2.762184E(-8)

5. Conclusion

The results from Tables 1-3 indicate that the new method performs favorably compared to existing ones. This novel approach introduces a class of continuous second-derivative block methods for solving ODEs, developed through collocation and interpolation techniques. By integrating polynomial and trigonometric functions and implementing them via MAPLE code, approximate solutions are generated effectively.

These block methods have continuous coefficients and exhibit essential properties like consistency, zero stability, and convergence, providing a solid framework that ensures reliable solutions for ODEs.

References

[1] Awoyemi, D. O.,(2001) "A new sixth order algorithms for General Second order ordinary differential equation", Inter. J. Computer Math., 77(2001)117-124. https://dx.doi.org/10.1080/00207160108805054

[2] Ogunware, B. G., Kuboye, J. O., Abolarin, O. E., & Mmaduakor, C. O. (2021) " Half-step numerical model of order ten for solution of first and second orders ordinary differential equations", Journal of Interdisciplinary Mathematics. https://doi.org/10.1080/09720502.2021.1887620

[3] Adeyefa, E. O., & Ibrahim, A. A.(2016) "A Sixth-order Self-Starting Algorithms for Second Order Initial Value Problems of ODEs", British Journal of Mathematics and Computer Science, pp. 1-8. https://doi.org/10.9734/BJMCS/2016/24322

[4] Enoch, O. O., & Alakofa, C. O. (2024) "A fifth order block methods for solving second order stiff ordinary differential equations using trigonometric functions and polynomial function as the basis function", African Scientific reports 3(2024)156. DOI: 10.46481/asr2024.3.2.156

[5] Fatunla, S. O. (1988) *Numerical methods for initial value problems in ordinary differential equations*, Academic press inc. Harcourt Brace Jovanovich Publishers, New York.

[6] Abhulimen, C. E. & Aigbiremhon, A. (2018) "Three-Step Block Method for Solving Second Order Differential Equations", International Journal Of Mathematical Analysis And Optimization Theory And Applications, PP. 364 - 381, http://ijmao.unilag.edu.ng/article/view/53.

[7] Adeniran, A. O. & Longe, I. O. (2019) "Solving Directly Second Order Initial Value Problems with Lucas Polynomial", Journal of Advances in Mathematics and Computer Science, pp. 1-7. https://doi.org/10.9734/JAMCS/2019/v32i430152

[8] Obarhua, F. & Adegboro, J. (2021) "An Order Four Continuous Numerical Method for Solving General Second Order Ordinary Differential Equations". Journal of the Nigerian Society of Physical Sciences, J. Nig. Soc. Phys. Sci. 3,42–47, 10.46481/jnsps.2021.150