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Analysis of a Block Method Developed with Six Generalised Grid Points for Solving Fourth Order Initial Value Problems

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Abstract

This manuscript presents the development, analysis, and application of a novel six-step block method, derived using a linear block algorithm (LBA), for the approximate solution of fourth-order initial value problems (IVPs). The proposed method is designed to overcome the shortcomings associated with traditional reduction methods, which involve converting the fourth-order IVP into a system of first-order ordinary differential equations (ODEs). Instead, the new method solves the problem directly, leveraging the advantages of hybrid block methods. A comprehensive theoretical analysis of the proposed method is provided, including proofs of its convergence and accuracy. The method's performance is then compared to existing methods for solving fourth-order IVPs, using numerical examples and tables to illustrate the results. The comparative analysis demonstrates the accuracy, efficiency, and reliability of the proposed method, highlighting its potential as a viable alternative for solving fourth-order IVPs.

Keywords: six-step, linear block algorithm, initial value problem, fourth order, accuracy.
2010 MSC: 65L05, 65L06.

1. Introduction

The numerical approximation of fourth order initial value problems (IVP) of the form

$$\gamma^{(4)}(u) = f\left(u, \gamma, \gamma', \gamma'', \gamma'''\right), u \in [a, b] \tag{1.1}$$

with initial condition $\gamma(u) = \kappa_0, \gamma'(u) = \kappa_1, \gamma''(u) = \kappa_2, \gamma'''(u) = \kappa_3$ is considered in this manuscript. The solution of models of the form Equation 1.1 are mostly solved numerically which provides an approximate solution. Moreover, these equations are classified as higher order and a reduction approach to transform it to a system of first order IVP was conventionally adopted in past literature. However, it has been stated that the reduction

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process is very difficult when writing computer programs and it has computational burden which affects the accuracy of the solution while wasting human effort. Hence, the use of direct approaches.

The predictor-corrector method is one of the direct methods. [21] proposed predictor-corrector block method of order seven for solving third order ordinary differential equations. The method is associated with computational burden due to the evaluation of many functions per iteration in both predictor and corrector method. [10] stated the advantage of continuous linear multistep method over the discrete methods such that; it gives a simplified form of coefficients for additional analytical work at different points that guarantee easy approximation of solutions at all the interior points of the interval. Now, to overcome the challenges in predictor-corrector method, block method was developed [12]. The method is capable of computing the discrete schemes at more than one point simultaneously. According to [18], [14] and [8], block method was firstly proposed by [17] who advocated the use of block as a means of getting a starting value for predictor corrector algorithm and later adopted as a full method [3].

Therefore, as suggested by [25], some researchers such as [2], [9], [6], and [1], adopt solving Equation 1.1 directly instead of using the aforementioned conventional method. The direct methods of [24] and [23] have also been proven to produce better and more accurate numerical solutions when compared with traditional method. Recently, researchers such as [20], [15], [4], and [9] have proposed hybrid block methods for direct solution of Equation 1.1 in order to avoid the shortcoming in reduction method and to take the advantages of hybrid and block methods which includes overcoming zero stability barrier and generating numerical solutions simultaneously. Based on these shortcomings, this article develops a new block method for the direct solution of fourth order initial value problems with improved accuracy.

1.1. Existence and Uniqueness Theorem

The numerical solutions to higher order ordinary differential equation can be obtained by using two approaches: (i) reduction to systems of first order ordinary differential equation, (ii) direct methods, (i.e. without reduction).

Two theorems are going to be stated. Theorem 1 discusses about the existence and uniqueness of first order ordinary differential equations and Theorem 2 guarantees the uniqueness of higher order ordinary differential equations.

Theorem 1.1. *[13]*

Let $f(u,v)$ *be a real function and continuous for all points* (u_0,v_0) *in the region* D *defined by* $u \in [a,b], v \in [-\infty,\infty]$ *containing initial values* (u_0,v_0) *where* a,b *are finite. Let there exists a constant* L *called Lipschitz constant such that for any* $u \in [a,b]$ *and for any pears* y_1, y_2 *for which* $(u,y_1)\,(u,y_2)$ *are both in* $D\,|f(u,y_1) - f(u,y_2)| \leqslant L\,|y_1 - y_2|$. *Then for any given number* $u \in [a,b]$, *the first order initial value problem has a unique solution* $y(u)$.

Theorem 1.2. *[26]*

Let be the region D *defined by the inequalities* $u_1 \leqslant u \leqslant u_0 + a, |s_j - c_j| \leqslant b, j = 0,1,d-1\,(a > 0 < b)$. *We assume that* $f(u,c_0,\dots,c_{d-1})$ *and* $f(u,s_0,\dots,s_{d-1})$ *is defined in* $\mathbb{R}$.

2. Methodology

The linear block approach proposed by [25] is used to derive a $6r-$step method for solving some fourth order initial value problems.

2.1. *Formulation of* $6r-$*step method*

The proposition below is considered to derive 6r-step method using the linear block approach for solving some differential equations in the form of Equation 1.1.

Conjecture 2.1. *The general linear multistep method of the form*

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^d \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2.1}$$

is used to derive $6r-$*step block method from linear block approach of the form*

$$y_{n+\xi} = \sum_{j=0}^{6r} \frac{(\xi j)^j}{j!} y_n^{(j)} + \sum_{j=0}^{7} \left(\pi_{j\xi} f_{n+j}\right), \xi = -r, -2r, 0, 3r, 4r, 5r, 6r \tag{2.2}$$

and its higher derivatives is obtained from

$$y'_{n+\xi} = \sum_{j=0}^{3-t} \frac{(\xi j)^j}{j!} y_n^{(j+t)} + \sum_{j=0}^{7} \left(\psi_{\xi j t} f_{n+j}\right),$$

$$t = 1_{(\xi=-r,-2r,0,3r,4r,5r,6r)}, t = 2_{(\xi=-r,-2r,0,3r,4r,5r,6r)}, t = 3_{(\xi=-r,-2r,0,3r,4r,5r,6r)} \tag{2.3}$$

with $\pi_{j\xi} = U^{-1}Y$ *and* $\psi_{j\xi} = U^{-1}Z$, *where*

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{(-\frac{h}{3})^1}{1!} & \frac{(-\frac{2h}{3})^1}{1!} & 0 & \frac{(h)^1}{1!} & \frac{(\frac{4h}{3})^1}{1!} & \frac{(\frac{5h}{3})^1}{1!} & \frac{(2h)^1}{1!} \\ \frac{(-\frac{h}{3})^2}{2!} & \frac{(-\frac{2h}{3})^2}{2!} & 0 & \frac{(h)^2}{2!} & \frac{(\frac{4h}{3})^2}{2!} & \frac{(\frac{5h}{3})^2}{2!} & \frac{(2h)^2}{2!} \\ \frac{(-\frac{h}{3})^3}{3!} & \frac{(-\frac{2h}{3})^3}{3!} & 0 & \frac{(h)^3}{3!} & \frac{(\frac{4h}{3})^3}{3!} & \frac{(\frac{5h}{3})^3}{3!} & \frac{(2h)^3}{3!} \\ \frac{(-\frac{h}{3})^4}{4!} & \frac{(-\frac{2h}{3})^4}{4!} & 0 & \frac{(h)^4}{4!} & \frac{(\frac{4h}{3})^4}{4!} & \frac{(\frac{5h}{3})^4}{4!} & \frac{(2h)^4}{4!} \\ \frac{(-\frac{h}{3})^5}{5!} & \frac{(-\frac{2h}{3})^5}{5!} & 0 & \frac{(h)^5}{5!} & \frac{(\frac{4h}{3})^5}{5!} & \frac{(\frac{5h}{3})^5}{5!} & \frac{(2h)^5}{5!} \\ \frac{(-\frac{h}{3})^6}{6!} & \frac{(-\frac{2h}{3})^6}{6!} & 0 & \frac{(h)^6}{6!} & \frac{(\frac{4h}{3})^6}{6!} & \frac{(\frac{5h}{3})^6}{6!} & \frac{(2h)^6}{6!} \end{pmatrix}, Y = \begin{pmatrix} \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \\ \frac{(\xi h)^{10}}{10!} \end{pmatrix}, Z = \begin{pmatrix} \frac{(\xi h)^{4-a}}{(4-a)!} \\ \frac{(\xi h)^{5-a}}{(5-a)!} \\ \frac{(\xi h)^{6-a}}{(6-a)!} \\ \frac{(\xi h)^{7-a}}{(7-a)!} \\ \frac{(\xi h)^{8-a}}{8!} \\ \frac{(\xi h)^{9-a}}{(9-a)!} \\ \frac{(\xi h)^{10-a}}{(10-a)!} \end{pmatrix}$$

Equations 2.2 and 2.3 are solved to obtain the coefficients of $y_\xi = \xi = -r, -2r, 0, 3r, 4r, 5r, 6r$. Then substituting $u = u_\xi + th$, the polynomial takes the form

$$\begin{aligned} y\left(u_\xi + th\right) &= \alpha_{3r} y_{\xi+3r} + \alpha_{4r} y_{\xi+4r} + \alpha_{5r} y_{\xi+5r} + \alpha_{6r} y_{\xi+6r} \\ &+ h^4 \left(\beta_{-r} f_{\xi-r} + \beta_{-2r} f_{\xi-2r} + \beta_0 f_0 + \beta_{3r} f_{\xi+3r} + \beta_{4r} f_{\xi+4r} + \beta_{5r} f_{\xi+5r} + \beta_{6r} f_{\xi+6r}\right) \end{aligned} \tag{2.4}$$

with
$\rho_{3r} = 1$, $\rho_{4r} = \xi$, $\rho_{5r} = \frac{1}{2}\xi^2$, $\rho_{6r} = \frac{1}{6}\xi^3$,

$\beta_{-r} = \frac{1}{12\,700\,800}\xi^5 \frac{B_1}{h^5 r^6 (h-2)}$
where $B_1 = 3\xi^5 + 10r\xi^4 + 90\,720h^4r^5 - 324hr^2\xi^3 - 28\,728h^3r^4\xi + 15\,120h^4r^4\xi + 1071h^2r^2\xi^3 +$
$4284h^2r^3\xi^2 - 6156h^3r^3\xi^2 - 90hr\xi^4$,
$\beta_{-2r} = \frac{1}{120\,960}\xi^5 \frac{B_2}{r^6(h-2)(3h+2)(5h+2)(2h+1)(3h+1)}$
where $B_2 = 3\xi^5 + 45\,360h^5r^5 + 756h^4r^4\xi + 909h^2r^2\xi^3 - 4014h^3r^3\xi^2 - 85hr\xi^4$,
$\beta_0 = \frac{1}{10\,886\,400}\xi^4 \frac{B_3}{h^5 r^6}$
where $B_3 = 3\xi^6 + 10r\xi^5 + 453\,600h^5r^6 - 306hr^2\xi^4 + 4536h^4r^5\xi + 45\,360h^5r^5\xi + 909h^2r^2\xi^4 +$
$3636h^2r^3\xi^3 - 4014h^3r^3\xi^3 - 18\,732h^3r^4\xi^2 + 756h^4r^4\xi^2 - 85hr\xi^5$,
$\beta_{3r} = \frac{1}{1088\,640}\xi^5 \frac{B_4}{h^5 r^6 (3h+2)}$
where $B_4 = -3\xi^5 - 10r\xi^4 + 30\,240h^4r^5 + 252hr^2\xi^3 + 3864h^3r^4\xi + 5040h^4r^4\xi - 531h^2r^2\xi^3 -$
$2124h^2r^3\xi^2 + 828h^3r^3\xi^2 + 70hr\xi^4$,
$\beta_{4r} = -\frac{1}{1209\,600}\xi^5 \frac{B_5}{h^5 r^6 (2h+1)}$
where $B_5 = -3\xi^5 - 10r\xi^4 + 22\,680h^4r^5 + 234hr^2\xi^3 + 2268h^3r^4\xi + 3780h^4r^4\xi - 441h^2r^2\xi^3 -$
$1764h^2r^3\xi^2 + 486h^3r^3\xi^2 + 65hr\xi^4$,
$\beta_{5r} = \frac{1}{907\,200}\xi^5 \frac{B_6}{h^5 r^6 (5h+2)}$
where $B_6 = -3\xi^5 - 10r\xi^4 + 18\,144h^4r^5 + 216hr^2\xi^3 + 1512h^3r^4\xi + 3024h^4r^4\xi - 369h^2r^2\xi^3 -$
$1476h^2r^3\xi^2 + 324h^3r^3\xi^2 + 60hr\xi^4$,
$\beta_{6r} = \frac{1}{7620\,480}\xi^5 \frac{B_7}{h^5 r^6 (3h+1)}$
where $B_7 = -3\xi^5 - 10r\xi^4 + 15\,120h^4r^5 + 198hr^2\xi^3 + 1092h^3r^4\xi + 2520h^4r^4\xi - 315h^2r^2\xi^3 -$
$1260h^2r^3\xi^2 + 234h^3r^3\xi^2 + 55hr\xi^4$.

The higher derivatives of (5) is given as
$\delta_{-r} = \frac{1}{1411\,200} \frac{h^4\xi^5(A_1)}{r^6(-a+9)!(-a+10)!(h\xi)^a(-a+5)!(-a+6)!(-a+7)!(h-2)}$
where $A_1 = 1209\,600r^5\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! + 2399\,040r^3\xi^2$
$(-a+5)!\,(-a+6)!\,(-a+9)!\,(-a+10)! + 1209\,600h\xi^5\,(-a+5)!\,(-a+6)!$
$(-a+7)!\,(-a+9)! + 403\,200r\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$- 2298\,240r^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 3447\,360hr^3\xi^2\,(-a+5)!$
$(-a+6)!\,(-a+9)!\,(-a+10)! - 36r^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!$
$(-a+10)! - 3628\,800hr\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+ 1209\,600hr^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$+ 119hr^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$\delta_{-2r} = \frac{1}{13\,440} \frac{h^{10}\xi^5(A_2)}{r^6(-a+5)!(-a+6)!(-a+7)!(-a+9)!(-a+10)!(h\xi)^a(h-2)(3h+2)(5h+2)(2h+1)(3h+1)}$
where $A_2 = 604\,800r^5\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! + 1209\,600\xi^5$
$(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)! - 2247\,840r^3\xi^2\,(-a+5)!\,(-a+6)!$
$(-a+9)!\,(-a+10)! - 3427\,200r\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+ 60\,480r^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$+ 101r^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$\delta_0 = \frac{1}{1209\,600(h\xi)^a} \frac{h^4\xi^4(A_3)}{r^6(-a+4)!(-a+5)!(-a+6)!(-a+7)!(-a+9)!(-a+10)!}$
where $A_3 = 1209\,600r^6\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$+ 1209\,600h\xi^6\,(-a+4)!\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)! + 403\,200r\xi^5$
$(-a+4)!\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)! + 60\,480r^5\xi\,(-a+4)!\,(-a+6)!$
$(-a+7)!\,(-a+9)!\,(-a+10)! + 2036\,160r^3\xi^3\,(-a+4)!\,(-a+5)!\,(-a+6)!$

$(-a+9)!\,(-a+10)! - 1498\,560r^4\xi^2\,(-a+4)!\,(-a+5)!\,(-a+7)!\,(-a+9)!$
$(-a+10)! - 34r^2\xi^4\,(-a+4)!\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$-\,3427\,200hr\xi^5\,(-a+4)!\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)! + 604\,800hr^5\xi$
$(-a+4)!\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 2247\,840hr^3\xi^3\,(-a+4)!$
$(-a+5)!\,(-a+6)!\,(-a+9)!\,(-a+10)! + 60\,480hr^4\xi^2\,(-a+4)!\,(-a+5)!$
$(-a+7)!\,(-a+9)!\,(-a+10)! + 101hr^2\xi^4\,(-a+4)!\,(-a+5)!\,(-a+6)!\,(-a+7)!$
$(-a+9)!\,(-a+10)!$

$\delta_{3r} = \frac{1}{120\,960}\frac{h^4\xi^5(A_4)}{(-a+5)!(-a+6)!(-a+7)!(-a+9)!(-a+10)!(h\xi)^a r^6(3h+2)}$

where $A_4 = 403\,200r^5\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 1189\,440r^3\xi^2$
$(-a+5)!\,(-a+6)!\,(-a+9)!\,(-a+10)! - 1209\,600h\xi^5\,(-a+5)!\,(-a+6)!$
$(-a+7)!\,(-a+9)! - 403\,200r\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,309\,120r^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! + 463\,680hr^3\xi^2\,(-a+5)!$
$(-a+6)!\,(-a+9)!\,(-a+10)! + 28r^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!$
$(-a+10)! + 2822\,400hr\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,403\,200hr^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 59hr^2\xi^3\,(-a+5)!$
$(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$

$\delta_{4r} = \frac{1}{134\,400}\frac{h^4\xi^5(A_5)}{(-a+5)!(-a+6)!(-a+7)!(-a+9)!(-a+10)!(h\xi)^a r^6(2h+1)}$

where $A_5 = 302\,400r^5\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 987\,840r^3\xi^2$
$(-a+5)!\,(-a+6)!\,(-a+9)!\,(-a+10)! - 1209\,600h\xi^5\,(-a+5)!\,(-a+6)!$
$(-a+7)!\,(-a+9)! - 403\,200r\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,181\,440r^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! + 272\,160hr^3\xi^2\,(-a+5)!$
$(-a+6)!\,(-a+9)!\,(-a+10)! + 26r^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!$
$(-a+10)! + 2620\,800hr\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,302\,400hr^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$
$-\,49hr^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$

$\delta_{5r} = \frac{1}{100\,800}\frac{h^4\xi^5(A_6)}{(-a+5)!(-a+6)!(-a+7)!(-a+9)!(-a+10)!(h\xi)^a r^6(5h+2)}$

where $A_6 = 241\,920r^5\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 826\,560r^3\xi^2$
$(-a+5)!\,(-a+6)!\,(-a+9)!\,(-a+10)! - 1209\,600h\xi^5\,(-a+5)!\,(-a+6)!$
$(-a+7)!\,(-a+9)! - 403\,200r\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,120\,960r^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! + 181\,440hr^3\xi^2\,(-a+5)!$
$(-a+6)!\,(-a+9)!\,(-a+10)! + 24r^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!$
$(-a+10)! + 2419\,200hr\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)! + 241\,920hr^4\xi$
$(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 41hr^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!$
$(-a+9)!\,(-a+10)!$

$\delta_{6r} = \frac{1}{846\,720}\frac{h^4\xi^5(A_7)}{(-a+5)!(-a+6)!(-a+7)!(-a+9)!(-a+10)!(h\xi)^a r^6(3h+1)}$

where $A_7 = 201\,600r^5\,(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 705\,600r^3\xi^2$
$(-a+5)!\,(-a+6)!\,(-a+9)!\,(-a+10)! - 1209\,600h\xi^5\,(-a+5)!\,(-a+6)!$
$(-a+7)!\,(-a+9)! - 403\,200r\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,87\,360r^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! + 131\,040hr^3\xi^2\,(-a+5)!$
$(-a+6)!\,(-a+9)!\,(-a+10)! + 22r^2\xi^3\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+9)!$
$(-a+10)! + 2217\,600hr\xi^4\,(-a+5)!\,(-a+6)!\,(-a+7)!\,(-a+10)!$
$+\,201\,600hr^4\xi\,(-a+5)!\,(-a+7)!\,(-a+9)!\,(-a+10)! - 35hr^2\xi^3\,(-a+5)!$
$(-a+6)!\,(-a+7)!\,(-a+9)!\,(-a+10)!$

The linear block algorithm in Equation 2.2 is expanded to yield

$$
\begin{aligned}
y_{n-r} &= y_n - rhy' + \frac{(-rh)^2}{2!}y'' + \frac{(-rh)^3}{3!}y''' + h^4(\pi_{01}\, f_{n-r} + \pi_{02}\, f_{n-2r} \\
&\quad + \pi_{03}\, f_n + \pi_{04}\, f_{n+3r} + \pi_{05}\, f_{n+4r} + \pi_{06}\, f_{n+5r} + \pi_{07}\, f_{n+6r}), \\
y_{n-2r} &= y_n - rhy' + \frac{(-2rh)^2}{2!}y'' + \frac{(-2rh)^3}{3!}y''' + h^4(\pi_{01}\, f_{n-r} + \pi_{02}\, f_{n-2r} \\
&\quad + \pi_{03}\, f_n + \pi_{04}\, f_{n+3r} + \pi_{05}\, f_{n+4r} + \pi_{06}\, f_{n+5r} + \pi_{07}\, f_{n+6r}), \\
y_{n+3r} &= y_n + 3rhy' + \frac{(3rh)^2}{2!}y'' + \frac{(3rh)^3}{3!}y''' + h^4(\pi_{01}\, f_{n-r} + \pi_{02}\, f_{n-2r} \\
&\quad + \pi_{03}\, f_n + \pi_{04}\, f_{n+3r} + \pi_{05}\, f_{n+4r} + \pi_{06}\, f_{n+5r} + \pi_{07}\, f_{n+6r}), \\
y_{n+4r} &= y_n + 4rhy' + \frac{(4rh)^2}{2!}y'' + \frac{(4rh)^3}{3!}y''' + h^4(\pi_{01}\, f_{n-r} + \pi_{02}\, f_{n-2r} \\
&\quad + \pi_{03}\, f_n + \pi_{04}\, f_{n+3r} + \pi_{05}\, f_{n+4r} + \pi_{06}\, f_{n+5r} + \pi_{07}\, f_{n+6r}), \\
y_{n+5r} &= y_n + 5rhy' + \frac{(5rh)^2}{2!}y'' + \frac{(5rh)^3}{3!}y''' + h^4(\pi_{01}\, f_{n-r} + \pi_{02}\, f_{n-2r} \\
&\quad + \pi_{03}\, f_n + \pi_{04}\, f_{n+3r} + \pi_{05}\, f_{n+4r} + \pi_{06}\, f_{n+5r} + \pi_{07}\, f_{n+6r}), \\
y_{n+6r} &= y_n + 6rhy' + \frac{(6rh)^2}{2!}y'' + \frac{(6rh)^3}{3!}y''' + h^4(\pi_{01}\, f_{n-r} + \pi_{02}\, f_{n-2r} \\
&\quad + \pi_{03}\, f_n + \pi_{04}\, f_{n+3r} + \pi_{05}\, f_{n+4r} + \pi_{06}\, f_{n+5r} + \pi_{07}\, f_{n+6r}).
\end{aligned}
\tag{2.5}
$$

Equation 2.3 is expanded using the linear block approach to yield the higher derivatives as

$$
\begin{aligned}
y'_{n-r} &= y'_n - rhy'' + \frac{(-rh)^2}{2!}y''' + h^3(\psi_{11}\ f_{n-r} + \psi_{12}\ f_{n-2r} + \psi_{13}\ f_n \\
&\quad + \psi_{14}\ f_{n+3r} + \psi_{15}\ f_{n+4r} + \psi_{16}\ f_{n+5r} + \psi_{17}\ f_{n+6r}), \\
y'_{n-2r} &= y'_n - 2rhy'' + \frac{(-2rh)^2}{2!}y''' + h^3(\psi_{11}\ f_{n-r} + \psi_{12}\ f_{n-2r} + \psi_{13}\ f_n \\
&\quad + \psi_{14}\ f_{n+3r} + \psi_{15}\ f_{n+4r} + \psi_{16}\ f_{n+5r} + \psi_{17}\ f_{n+6r}), \\
y'_{n+3r} &= y'_n + 3rhy'' + \frac{(3rh)^2}{2!}y''' + h^3(\psi_{11}\ f_{n-r} + \psi_{12}\ f_{n-2r} + \psi_{13}\ f_n \\
&\quad + \psi_{14}\ f_{n+3r} + \psi_{15}\ f_{n+4r} + \psi_{16}\ f_{n+5r} + \psi_{17}\ f_{n+6r}), \\
y'_{n+4r} &= y'_n + 4rhy'' + \frac{(4rh)^2}{2!}y''' + h^3(\psi_{11}\ f_{n-r} + \psi_{12}\ f_{n-2r} + \psi_{13}\ f_n \\
&\quad + \psi_{14}\ f_{n+3r} + \psi_{15}\ f_{n+4r} + \psi_{16}\ f_{n+5r} + \psi_{17}\ f_{n+6r}), \\
y'_{n+5r} &= y'_n + 5rhy'' + \frac{(5rh)^2}{2!}y''' + h^3(\psi_{11}\ f_{n-r} + \psi_{12}\ f_{n-2r} + \psi_{13}\ f_n \\
&\quad + \psi_{14}\ f_{n+3r} + \psi_{15}\ f_{n+4r} + \psi_{16}\ f_{n+5r} + \psi_{17}\ f_{n+6r}), \\
y'_{n+6r} &= y'_n + 6rhy'' + \frac{(6rh)^2}{2!}y''' + h^3(\psi_{11}\ f_{n-r} + \psi_{12}\ f_{n-2r} + \psi_{13}\ f_n \\
&\quad + \psi_{14}\ f_{n+3r} + \psi_{15}\ f_{n+4r} + \psi_{16}\ f_{n+5r} + \psi_{17}\ f_{n+6r}).
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
y''_{n-r} &= y''_n - rhy''' + h^2(\psi_{21}\ f_{n-r} + \psi_{22}\ f_{n-2r} + \psi_{23}\ f_n + \psi_{24}\ f_{n+3r} \\
&\quad + \psi_{25}\ f_{n+4r} + \psi_{26}\ f_{n+5r} + \psi_{27}\ f_{n+6r}), \\
y''_{n-2r} &= y''_n - 2rhy''' + h^2(\psi_{21}\ f_{n-r} + \psi_{22}\ f_{n-2r} + \psi_{23}\ f_n + \psi_{24}\ f_{n+3r} \\
&\quad + \psi_{25}\ f_{n+4r} + \psi_{26}\ f_{n+5r} + \psi_{27}\ f_{n+6r}), \\
y''_{n+3r} &= y''_n + 3rhy''' + h^2(\psi_{21}\ f_{n-r} + \psi_{22}\ f_{n-2r} + \psi_{23}\ f_n + \psi_{24}\ f_{n+3r} \\
&\quad + \psi_{25}\ f_{n+4r} + \psi_{26}\ f_{n+5r} + \psi_{27}\ f_{n+6r}), \\
y''_{n+4r} &= y''_n + 4rhy''' + h^2(\psi_{21}\ f_{n-r} + \psi_{22}\ f_{n-2r} + \psi_{23}\ f_n + \psi_{24}\ f_{n+3r} \\
&\quad + \psi_{25}\ f_{n+4r} + \psi_{26}\ f_{n+5r} + \psi_{27}\ f_{n+6r}), \\
y''_{n+5r} &= y''_n + 5rhy''' + h^2(\psi_{21}\ f_{n-r} + \psi_{22}\ f_{n-2r} + \psi_{23}\ f_n + \psi_{24}\ f_{n+3r} \\
&\quad + \psi_{25}\ f_{n+4r} + \psi_{26}\ f_{n+5r} + \psi_{27}\ f_{n+6r}), \\
y''_{n+6r} &= y''_n + 6rhy''' + h^2(\psi_{21}\ f_{n-r} + \psi_{22}\ f_{n-2r} + \psi_{23}\ f_n + \psi_{24}\ f_{n+3r} \\
&\quad + \psi_{25}\ f_{n+4r} + \psi_{26}\ f_{n+5r} + \psi_{27}\ f_{n+6r}).
\end{aligned} \tag{2.7}
$$

$$\begin{aligned}
y'''_{n-r} &= y'''_n + h(\psi_{31}\, f_{n-r} + \psi_{32}\, f_{n-2r} + \psi_{33}\, f_n + \psi_{34}\, f_{n+3r} + \psi_{35}\, f_{n+4r} \\
&\qquad + \psi_{36}\, f_{n+5r} + \psi_{37}\, f_{n+6r}), \\
y'''_{n-2r} &= y'''_n + h(\psi_{31}\, f_{n-r} + \psi_{32}\, f_{n-2r} + \psi_{33}\, f_n + \psi_{34}\, f_{n+3r} + \psi_{35}\, f_{n+4r} \\
&\qquad + \psi_{36}\, f_{n+5r} + \psi_{37}\, f_{n+6r}), \\
y'''_{n+3r} &= y'''_n + h(\psi_{31}\, f_{n-r} + \psi_{32}\, f_{n-2r} + \psi_{33}\, f_n + \psi_{34}\, f_{n+3r} + \psi_{35}\, f_{n+4r} \\
&\qquad + \psi_{36}\, f_{n+5r} + \psi_{37}\, f_{n+6r}), \\
y'''_{n+4r} &= y'''_n + h(\psi_{31}\, f_{n-r} + \psi_{32}\, f_{n-2r} + \psi_{33}\, f_n + \psi_{34}\, f_{n+3r} + \psi_{35}\, f_{n+4r} \\
&\qquad + \psi_{36}\, f_{n+5r} + \psi_{37}\, f_{n+6r}), \\
y'''_{n+5r} &= y'''_n + h(\psi_{31}\, f_{n-r} + \psi_{32}\, f_{n-2r} + \psi_{33}\, f_n + \psi_{34}\, f_{n+3r} + \psi_{35}\, f_{n+4r} \\
&\qquad + \psi_{36}\, f_{n+5r} + \psi_{37}\, f_{n+6r}), \\
y'''_{n+6r} &= y'''_n + h(\psi_{31}\, f_{n-r} + \psi_{32}\, f_{n-2r} + \psi_{33}\, f_n + \psi_{34}\, f_{n+3r} + \psi_{35}\, f_{n+4r} \\
&\qquad + \psi_{36}\, f_{n+5r} + \psi_{37}\, f_{n+6r}). \qquad (2.8)
\end{aligned}$$

Therefore, to obtain the unknown coefficients of π in Equation 2.5, $\pi_{j\xi} = U^{-1}Y$ is computed, where

$$y_{n-r}\begin{pmatrix}\pi_{01}\\ \pi_{02}\\ \pi_{03}\\ \pi_{04}\\ \pi_{05}\\ \pi_{06}\\ \pi_{07}\end{pmatrix} = \begin{pmatrix}\frac{1037}{10\,497\,600}\\ -\frac{39\,593}{4115\,059\,200}\\ \frac{128\,929}{293\,932\,800}\\ -\frac{3397}{73\,483\,200}\\ \frac{15\,619}{293\,932\,800}\\ -\frac{6331}{257\,191\,200}\\ \frac{169}{39\,191\,040}\end{pmatrix}; y_{n-2r}\begin{pmatrix}\pi_{01}\\ \pi_{02}\\ \pi_{03}\\ \pi_{04}\\ \pi_{05}\\ \pi_{06}\\ \pi_{07}\end{pmatrix} = \begin{pmatrix}\frac{526}{164\,025}\\ -\frac{2383}{16\,074\,450}\\ \frac{6191}{1148\,175}\\ -\frac{878}{1148\,175}\\ \frac{1013}{1148\,175}\\ -\frac{3292}{8037\,225}\\ \frac{11}{153\,090}\end{pmatrix};$$

$$y_{n+3r}\begin{pmatrix}\pi_{01}\\ \pi_{02}\\ \pi_{03}\\ \pi_{04}\\ \pi_{05}\\ \pi_{06}\\ \pi_{07}\end{pmatrix} = \begin{pmatrix}-\frac{879}{78\,400}\\ \frac{1107}{627\,200}\\ \frac{17\,189}{403\,200}\\ \frac{2327}{100\,800}\\ -\frac{1017}{44\,800}\\ \frac{381}{39\,200}\\ -\frac{1831}{1128\,960}\end{pmatrix}; y_{n+4r}\begin{pmatrix}\pi_{01}\\ \pi_{02}\\ \pi_{03}\\ \pi_{04}\\ \pi_{05}\\ \pi_{06}\\ \pi_{07}\end{pmatrix} = \begin{pmatrix}-\frac{41\,984}{1148\,175}\\ \frac{47\,296}{8037\,225}\\ \frac{143\,744}{1148\,175}\\ \frac{105\,472}{1148\,175}\\ -\frac{96\,736}{1148\,175}\\ \frac{284\,672}{8037\,225}\\ -\frac{64}{10\,935}\end{pmatrix};$$

$$y_{n+5r}\begin{pmatrix}\pi_{01}\\ \pi_{02}\\ \pi_{03}\\ \pi_{04}\\ \pi_{05}\\ \pi_{06}\\ \pi_{07}\end{pmatrix} = \begin{pmatrix}-\frac{255\,625}{2939\,328}\\ \frac{2333\,125}{164\,602\,368}\\ \frac{3298\,375}{11\,757\,312}\\ \frac{741\,875}{2939\,328}\\ -\frac{2511\,875}{11\,757\,312}\\ \frac{923\,375}{10\,287\,648}\\ -\frac{115\,625}{7838\,208}\end{pmatrix}; y_{n+6r}\begin{pmatrix}\pi_{01}\\ \pi_{02}\\ \pi_{03}\\ \pi_{04}\\ \pi_{05}\\ \pi_{06}\\ \pi_{07}\end{pmatrix} = \begin{pmatrix}-\frac{6}{35}\\ \frac{69}{2450}\\ \frac{839}{1575}\\ \frac{866}{1575}\\ -\frac{3}{7}\\ \frac{228}{1225}\\ -\frac{19}{630}\end{pmatrix}.$$

Similarly, to obtain the unknown coefficients of higher derivative ψ, $\psi_{j\xi} = U^{-1}Z$ is computed, where

$$y'_{n-r}\begin{pmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{15}\\ \psi_{16}\end{pmatrix}=\begin{pmatrix}-\frac{173\,549}{114\,307\,200}\\ \frac{32\,077}{228\,614\,400}\\ -\frac{243\,503}{48\,988\,800}\\ \frac{29\,851}{48\,988\,800}\\ -\frac{11\,483}{16\,329\,600}\\ \frac{18\,691}{57\,153\,600}\\ -\frac{7883}{137\,168\,640}\end{pmatrix};\, y'_{n-2r}\begin{pmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{15}\\ \psi_{16}\end{pmatrix}=\begin{pmatrix}-\frac{22\,751}{893\,025}\\ \frac{884}{893\,025}\\ -\frac{19\,549}{765\,450}\\ \frac{829}{382\,725}\\ -\frac{649}{255\,150}\\ \frac{1118}{893\,025}\\ -\frac{124}{535\,815}\end{pmatrix};$$

$$y'_{n+3r}\begin{pmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{15}\\ \psi_{16}\end{pmatrix}=\begin{pmatrix}-\frac{2217}{156\,800}\\ -\frac{789}{313\,600}\\ \frac{2937}{22\,400}\\ \frac{1369}{9600}\\ -\frac{447}{3200}\\ \frac{4623}{78\,400}\\ -\frac{607}{62\,720}\end{pmatrix};\, y'_{n+4r}\begin{pmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{15}\\ \psi_{16}\end{pmatrix}=\begin{pmatrix}\frac{125\,632}{893\,025}\\ -\frac{51\,928}{893\,025}\\ \frac{5992}{54\,675}\\ \frac{206\,272}{382\,725}\\ -\frac{66\,056}{127\,575}\\ \frac{193\,664}{893\,025}\\ -\frac{18\,808}{535\,815}\end{pmatrix};$$

$$y'_{n+5r}\begin{pmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{15}\\ \psi_{16}\end{pmatrix}=\begin{pmatrix}\frac{4542\,625}{4572\,288}\\ -\frac{753\,875}{2286\,144}\\ -\frac{512\,125}{979\,776}\\ \frac{3336\,625}{1959\,552}\\ -\frac{543\,625}{326\,592}\\ \frac{1613\,125}{2286\,144}\\ -\frac{776\,875}{6858\,432}\end{pmatrix};\, y'_{n+6r}\begin{pmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{15}\\ \psi_{16}\end{pmatrix}=\begin{pmatrix}\frac{969}{245}\\ -\frac{1527}{1225}\\ -\frac{1083}{350}\\ \frac{851}{175}\\ -\frac{69}{14}\\ \frac{2598}{1225}\\ -\frac{247}{735}\end{pmatrix};$$

$$y''_{n-r}\begin{pmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\\ \psi_{25}\\ \psi_{26}\end{pmatrix}=\begin{pmatrix}\frac{79\,157}{4233\,600}\\ -\frac{78\,737}{50\,803\,200}\\ \frac{145\,487}{3628\,800}\\ -\frac{3557}{604\,800}\\ \frac{24\,769}{3628\,800}\\ -\frac{6757}{2116\,800}\\ \frac{1145}{2032\,128}\end{pmatrix};\, y''_{n-2r}\begin{pmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\\ \psi_{25}\\ \psi_{26}\end{pmatrix}=\begin{pmatrix}\frac{30\,689}{198\,450}\\ -\frac{341}{264\,600}\\ \frac{3743}{56\,700}\\ \frac{311}{28\,350}\\ -\frac{233}{18\,900}\\ \frac{491}{99\,225}\\ -\frac{13}{17\,640}\end{pmatrix};$$

$$y''_{n+3r}\begin{pmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\\ \psi_{25}\\ \psi_{26}\end{pmatrix}=\begin{pmatrix}\frac{18\,471}{156\,800}\\ -\frac{35\,097}{627\,200}\\ \frac{3683}{19\,200}\\ \frac{14\,729}{22\,400}\\ -\frac{28\,311}{44\,800}\\ \frac{20\,649}{78\,400}\\ -\frac{16\,049}{376\,320}\end{pmatrix};\, y''_{n+4r}\begin{pmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\\ \psi_{25}\\ \psi_{26}\end{pmatrix}=\begin{pmatrix}\frac{123\,712}{99\,225}\\ -\frac{1508}{3675}\\ -\frac{128}{175}\\ \frac{29\,888}{14\,175}\\ -\frac{3224}{1575}\\ \frac{85\,376}{99\,225}\\ -\frac{548}{3969}\end{pmatrix};$$

$$y''_{n+5r}\begin{pmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\\ \psi_{25}\\ \psi_{26}\end{pmatrix}=\begin{pmatrix}\frac{899\,125}{169\,344}\\ -\frac{3367\,625}{2032\,128}\\ -\frac{641\,225}{145\,152}\\ \frac{444\,625}{72\,576}\\ -\frac{902\,375}{145\,152}\\ \frac{226\,075}{84\,672}\\ -\frac{288\,125}{677\,376}\end{pmatrix};\, y''_{n+6r}\begin{pmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\\ \psi_{25}\\ \psi_{26}\end{pmatrix}=\begin{pmatrix}\frac{40\,137}{2450}\\ -\frac{49\,359}{9800}\\ -\frac{30\,883}{2100}\\ \frac{17\,069}{1050}\\ -\frac{12\,027}{700}\\ \frac{9123}{1225}\\ -\frac{2277}{1960}\end{pmatrix};$$

$$y'''_{n-r}\begin{pmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\\ \psi_{34}\\ \psi_{35}\\ \psi_{36}\end{pmatrix}=\begin{pmatrix}-\frac{81973}{470400}\\ \frac{193559}{16934400}\\ -\frac{216869}{1209600}\\ \frac{18757}{604800}\\ -\frac{44083}{1209600}\\ \frac{4073}{235200}\\ -\frac{3497}{1128960}\end{pmatrix};\ y'''_{n-2r}\begin{pmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\\ \psi_{34}\\ \psi_{35}\\ \psi_{36}\end{pmatrix}=\begin{pmatrix}-\frac{12731}{18900}\\ -\frac{2459}{58800}\\ \frac{4021}{37800}\\ -\frac{1321}{6300}\\ \frac{1003}{4200}\\ -\frac{6863}{66150}\\ \frac{257}{15120}\end{pmatrix};$$

$$y'''_{n+3r}\begin{pmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\\ \psi_{34}\\ \psi_{35}\\ \psi_{36}\end{pmatrix}=\begin{pmatrix}\frac{211653}{156800}\\ -\frac{5679}{12800}\\ -\frac{114217}{134400}\\ \frac{23743}{9600}\\ -\frac{105093}{44800}\\ \frac{10881}{11200}\\ -\frac{58423}{376320}\end{pmatrix};\ y'''_{n+4r}\begin{pmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\\ \psi_{34}\\ \psi_{35}\\ \psi_{36}\end{pmatrix}=\begin{pmatrix}\frac{214456}{33075}\\ -\frac{3158}{1575}\\ -\frac{2986}{525}\\ \frac{4792}{675}\\ -\frac{11206}{1575}\\ \frac{14384}{4725}\\ -\frac{3218}{6615}\end{pmatrix};$$

$$y'''_{n+5r}\begin{pmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\\ \psi_{34}\\ \psi_{35}\\ \psi_{36}\end{pmatrix}=\begin{pmatrix}\frac{163525}{8064}\\ -\frac{4201075}{677376}\\ -\frac{903655}{48384}\\ \frac{153925}{8064}\\ -\frac{981025}{48384}\\ \frac{249745}{28224}\\ -\frac{134875}{96768}\end{pmatrix};\ y'''_{n+r}\begin{pmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\\ \psi_{34}\\ \psi_{35}\\ \psi_{36}\end{pmatrix}=\begin{pmatrix}\frac{249093}{4900}\\ -\frac{303741}{19600}\\ -\frac{199327}{4200}\\ \frac{96121}{2100}\\ -\frac{70083}{1400}\\ \frac{53127}{2450}\\ -\frac{38533}{11760}\end{pmatrix}.$$

2.2. *Analysis of 6r-step method*

The analysis of the method is carried out in this section.

2.2.1. *Order of the Method*

The linear operator $\ell\left[y\left(u\right);h\right]$ is established related to the newly derived 6r-step method in this subsection.

Proposition 2.1. *The local truncation error of the* 6r *step derived is* $C_{07}h^{07}y^{07}\left(u_n\right)+0\left(h^{11}\right)$

Proof:

According to [16], the linear difference operators associated with the developed method *in Equations 2.4 to 2.7 are given by*

$$\begin{gathered}
\ell\left[y\left(u\right);h\right]=y\left(u_n-rh\right)-\left[\alpha_{3r}\left(u_n+3r\right)+\alpha_{4r}\left(u_n+4r\right)+\alpha_{5r}\left(u_n+5r\right)\right.\\
+\alpha_{6r}\left(u_n+6r\right)+h^4\sum_{j=0}^{k}\left(\beta_j\left(u\right)\ f_{n+j}\right)+\left(\beta_k\left(u\right)\ f_{n+k}\right)],k=r,-2r,0,3r,4r,5r,6r;\\
\ell\left[y\left(u\right);h\right]=y\left(u_n-2rh\right)-\left[\alpha_{3r}\left(u_n+3r\right)+\alpha_{4r}\left(u_n+4r\right)+\alpha_{5r}\left(u_n+5r\right)\right.\\
+\alpha_{6r}\left(u_n+6r\right)+h^4\sum_{j=0}^{k}\left(\beta_j\left(u\right)\ f_{n+j}\right)+\left(\beta_k\left(u\right)\ f_{n+k}\right)],k=r,-2r,0,3r,4r,5r,6r;\\
\ell\left[y\left(u\right);h\right]=y\left(u_n+3rh\right)-\left[\alpha_{3r}\left(u_n+3r\right)+\alpha_{4r}\left(u_n+4r\right)+\alpha_{5r}\left(u_n+5r\right)\right.\\
+\alpha_{6r}\left(u_n+6r\right)+h^4\sum_{j=0}^{k}\left(\beta_j\left(u\right)\ f_{n+j}\right)+\left(\beta_k\left(u\right)\ f_{n+k}\right)],k=r,-2r,0,3r,4r,5r,6r;\\
\ell\left[y\left(u\right);h\right]=y\left(u_n+4rh\right)-\left[\alpha_{3r}\left(u_n+3r\right)+\alpha_{4r}\left(u_n+4r\right)+\alpha_{5r}\left(u_n+5r\right)\right.\\
+\alpha_{6r}\left(u_n+6r\right)+h^4\sum_{j=0}^{k}\left(\beta_j\left(u\right)\ f_{n+j}\right)+\left(\beta_k\left(u\right)\ f_{n+k}\right)],k=r,-2r,0,3r,4r,5r,6r;\\
\ell\left[y\left(u\right);h\right]=y\left(u_n+5rh\right)-\left[\alpha_{3r}\left(u_n+3r\right)+\alpha_{4r}\left(u_n+4r\right)+\alpha_{5r}\left(u_n+5r\right)\right.\\
+\alpha_{6r}\left(u_n+6r\right)+h^4\sum_{j=0}^{k}\left(\beta_j\left(u\right)\ f_{n+j}\right)+\left(\beta_k\left(u\right)\ f_{n+k}\right)],k=r,-2r,0,3r,4r,5r,6r;\\
\ell\left[y\left(u\right);h\right]=y\left(u_n+6rh\right)-\left[\alpha_{3r}\left(u_n+3r\right)+\alpha_{4r}\left(u_n+4r\right)+\alpha_{5r}\left(u_n+5r\right)\right.\\
+\alpha_{6r}\left(u_n+6r\right)+h^4\sum_{j=0}^{k}\left(\beta_j\left(u\right)\ f_{n+j}\right)+\left(\beta_k\left(u\right)\ f_{n+k}\right)],k=r,-2r,0,3r,4r,5r,6r.
\end{gathered}\tag{2.9}$$

With the aid of Taylor series, expanding Equation 2.9 in power of h *and let* $y\left(u\right)$ *be sufficiently differentiable. It is important to state that the first non-zero term of each formula in Equation 2.8 is* $C_{07}h^{07}y^{07}\left(u_n\right)+0\left(h^{11}\right)$.

Similar procedure can be applies to the schemes in Equations 2.6 to 2.8.

Definition 2.2. [16]

A linear multistep method (Equation 2.1) for a fourth order problem (Equation 1.1) is of order p if it satisfies the condition $c_0=c_1=c_2=0,\cdots,c_p=0,c_{p+1}=0,c_{p+2}=0,c_{p+3}=0,c_{p+4}\neq 0$ where

$$c_0=\sum_{j=0}^{k}\left(\alpha_j\right);c_1=\sum_{j=0}^{k}\left(j\alpha_j-\beta_j\right);\ldots;$$

$$c_p=\sum_{j=0}^{k}\left[\frac{1}{p}j^p\alpha_j-\frac{1}{(p-1)}\left(j^{p-1}\beta_j\right)\right],p=2,3,\cdots,p+1\tag{2.10}$$

The parameter $c_{p+4}\neq 0$ is referred to as the error constant.

The local truncation error of the newly derived scheme is given by

$$\begin{gathered}\left(4.9598\times10^{-9}\right)C_{07}h^{07}y^{07}\left(x_n\right)+0\left(h^{11}\right);\left(8.3142\times10^{-8}\right)C_{07}h^{07}y^{07}\left(x_n\right)+0\left(h^{11}\right)\\ \left(1.5079\times10^{-6}\right)C_{07}h^{07}y^{07}\left(x_n\right)+0\left(h^{11}\right);\left(5.2634\times10^{-6}\right)C_{07}h^{07}y^{07}\left(x_n\right)+0\left(h^{11}\right)\\ \left(1.3018\times10^{-5}\right)C_{07}h^{07}y^{07}\left(x_n\right)+0\left(h^{11}\right);\left(2.6298\times10^{-5}\right)C_{07}h^{07}y^{07}\left(x_n\right)+0\left(h^{11}\right).\end{gathered}\tag{2.11}$$

Therefore, the newly derived scheme is of uniform order seven with error constant given by

$$C_{07}=\begin{pmatrix}4.9598\times10^{-9}\\ 8.3142\times10^{-8}\\ 1.5079\times10^{-6}\\ 5.2634\times10^{-6}\\ 1.3018\times10^{-5}\\ 2.6298\times10^{-5}\end{pmatrix}.$$

2.2.2. *Consistency*

Traditionally, the newly derived method is consistent if the order of the method is greater than or equal to one.

2.2.3. *Zero-stability*

A method is zero-stable, $h\Rightarrow0$ if the first characteristic polynomial with roots $\pi\left(r\right)=0$ satisfies $|\left[\sum A^0R^{k-1}\right]|\leqslant1$ and every root satisfying has multiplicity not more than the order of the differential equation. The first characteristic polynomial of the new method is given numerically as

$$\pi\left(r\right)=\left|r\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}=\begin{bmatrix}r&0&0&0&0&-1\\0&r&0&0&0&-1\\0&0&r&0&0&-1\\0&0&0&r&0&-1\\0&0&0&0&r&-1\\0&0&0&0&0&r-1\end{bmatrix}\right|=$$
$$r^6\left(r-1\right).$$

Then solving for r in $r^6\left(r-1\right)$ yields $r=0,0,0,0,0,1$. Therefore, the new method is zero stable.

2.2.4. *Convergence*

Theorem 2.3. *[26]*

Consistency and zero-stability are necessary and sufficient conditions for a linear multistep method to be convergent.

Hence, the newly derived scheme is convergent since it is consistent and zero-stable.

2.2.5. *Linear Stability*

The region of absolute stability of a linear multistep method is the set of complex values λh for which all solutions of the test problem $y'=-\lambda y$ will remain bounded as

$n \Rightarrow \infty$. The concept of A-stability according to [23] is discussed by applying the test equation $y^{(k)} = -\lambda^{(k)} y$ to yield

$$Y_m = \mu(z) Y_{m-1}, z = \lambda h \tag{2.12}$$

where $\mu(z)$ is the amplification matrix of the form

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^1\eta^{(0)}\right)^{-1} \left(\xi^1 - z\eta^{(1)} - z^1\eta^{(1)}\right) \tag{2.13}$$

The matrix $\mu(z)$ has eigenvalues $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of the method is given by $\xi = \frac{\xi_n}{\xi_d}$, where $\xi_n = 60\,860\,385\,880z^6 + 465\,497\,649\,364z^5 + 157\,377\,412\,674z^4 - 2048\,577\,128\,082z^3 + 33\,087\,839\,437\,143z^2 - 90\,149\,909\,895\,360z + 62\,219\,695\,104\,000$, and $\xi_d = 6096\,384\,000z^6 + 20\,118\,067\,200z^5 - 195\,693\,926\,400z^4 - 115\,221\,657\,600z^3 + 5514\,179\,328\,000z^2 - 22\,221\,319\,680\,000z + 31\,109\,847\,552\,000$. $[z \in (\rho_2 \setminus \rho_1) \cap (\rho_1 \setminus \rho_1), \mathbb{C}], [z \in \rho_1 \setminus (\rho_2 \setminus \rho_1 \cup \rho_1), \{0\}]$ where ρ_1 is a root of $-3645\hat{Z} + \frac{1809}{2}\hat{Z}^2 - \frac{189}{10}\hat{Z}^3 - \frac{321}{10}\hat{Z}^4 + \frac{33}{10}\hat{Z}^5 + \hat{Z}^6 + 5103$ and ρ_2 is a root of $-\frac{321\,963\,963\,912}{217\,358\,521}\hat{Z} + \frac{33\,087\,839\,437\,143}{60\,860\,385\,880}\hat{Z}^2 - \frac{1024\,288\,564\,041}{30\,430\,192\,940}\hat{Z}^3 + \frac{78\,688\,706\,337}{30\,430\,192\,940}\hat{Z}^4 + \frac{116\,374\,412\,341}{15\,215\,096\,470}\hat{Z}^5 + \hat{Z}^6 + \frac{222\,213\,196\,800}{217\,358\,521}$. Therefore, the developed method satisfies the condition for linear stability.

3. Numerical Simulation and Results

Some fourth order differential equations of the form of Equation 1.1 will be solved without reduction process in order to overcome the setbacks. The results obtained from the newly derived scheme are compared with the existing methods in literature. The acronyms below are used in the tables.

ES: Exact Solution
CS: Computed Solution
ENM: Error in New Method
EEM[1]: Error in Existing Method [2]
EEM[2]: Error in Existing Method [5]
EEM[3]: Error in Existing Method [11]
EEM[4]: Error in Existing Method [7]
EEM[5]: Error in Existing Method [8]
EEM[6]: Error in Existing Method [22]

Problem 1:

$$\gamma^{(4)}(u) = -\gamma''(u), \gamma(u_0) = 0, \gamma'(u_0) = \frac{-1.1}{72 - 50\pi}, \gamma''(u_0) = \frac{1}{144 - 50\pi},$$

$$\gamma'''(u_0) = \frac{1.2}{144 - 100\pi}; \text{whose exact solution is}$$

$$\gamma(u) = \frac{1 - u - \cos u - 1.2\sin u}{144 - 100\pi} \tag{3.1}$$

Problem 2:

$$\gamma^{(4)}(u) = -4\gamma''(u), \gamma(u_0) = 1, \gamma'(u_0) = 3, \gamma''(u_0) = 0, \gamma'''(u_0) = 16,$$
$$\text{whose exact solution is} \quad \gamma(u) = 1 - u + 2\exp(-2u) \tag{3.2}$$

Problem 3:

$$\gamma^{(4)}(u) = \sin u + \cos u, \gamma(u_0) = 7, \gamma'(u_0) = 0, \gamma''(u_0) = -1, \gamma'''(u_0) = 0,$$
$$\text{whose exact solution is}\gamma(u) = -\sin u + \cos u + u^3 - 1 \tag{3.3}$$

Problem 4:

$$\gamma^{(4)}(u) = - - \frac{\left(8 + 25u + 30u^2 + 12u^3 + u^4\right)}{1 + u^2}, \gamma(u_0) = 0, \gamma'(u_0) = 1, \gamma''(u_0) = 0,$$
$$\gamma'''(u_0) = -3, \quad \text{whose exact solution is} \quad \gamma(u) = u(1 - u^2)\exp(u) \tag{3.4}$$

Problem 5:

$$\gamma^{(4)}(u) = u, \gamma(u_0) = 0, \gamma'(u_0) = 1, \gamma''(u_0) = 0, \gamma'''(u_0) = 0, \text{whose exact solution is}$$
$$\gamma(u) = \frac{u^5}{120} + u \tag{3.5}$$

Table 1: Comparison of solutions for problem 1

u	ES	CS	ENM	EEM[2]	EEM[3]	EEM[1]
$\frac{1}{320}$	0.00004034461209373069	0.00004034461209373069	0.00000(00)	2.11488(−18)	2.11164(−13)	2.26912(−19)
$\frac{2}{320}$	0.00008063166098895974	0.00008063166098895974	0.00000(00)	1.05763(−17)	5.69866(−12)	4.55463(−19)
$\frac{3}{320}$	0.00012086093247161511	0.00012086093247161520	9.00000(−20)	1.26825(−17)	6.80311(−10)	3.89121(−19)
$\frac{4}{320}$	0.00016103221289185685	0.00016103221289185732	4.70000(−19)	2.19627(−17)	2.20723(−09)	6.03859(−19)
$\frac{5}{320}$	0.00020114528916616351	0.00020114528916616496	1.45000(−18)	2.56232(−17)	1.27407(−08)	9.72774(−19)
$\frac{6}{320}$	0.00024119994877941305	0.00024119994877941648	3.43000(−18)	3.72965(−17)	3.45612(06)	8.90036(−19)
$\frac{7}{320}$	0.00028119597978695816	0.00028119597978696512	6.96000(−18)	4.40975(−17)	6.55238(−06)	8.84351(−19)
$\frac{8}{320}$	0.00032113317081669604	0.00032113317081670872	1.26800(−17)	5.97624(−17)	9.58653(−06)	6.69097(−19)
$\frac{9}{320}$	0.00036101131107113260	0.00036101131107115392	2.13200(−17)	7.13130(−17)	1.04933(−06)	2.25886(−18)
$\frac{10}{320}$	0.00040083019032944098	0.00040083019032947475	3.37700(−17)	9.25896(−17)	5.69624(−06)	1.65337(−18)

Table 2: Comparison of solutions for problem 2

u	ES	CS	ENN	EEM[4]	EEM[3]
$\frac{1}{320}$	1.00937508138036727920	1.00937508138036727920	0.00000(00)	1.00000(−18)	0.00000(00)
$\frac{2}{320}$	1.01875065104675294860	1.01875065104675294860	0.00000(00)	2.00000(−18)	0.00000(00)
$\frac{3}{320}$	1.02812719730424913310	1.02812719730424914130	8.20000(−18)	5.20000(−17)	2.22045(−16)
$\frac{4}{320}$	1.03750520849609617210	1.03750520849609621470	4.26000(−17)	2.39000(−16)	2.44249(−15)
$\frac{5}{320}$	1.04688517302275858900	1.04688517302275869600	1.07000(−16)	5.52000(−16)	1.15463 (−14)
$\frac{6}{320}$	1.05626757936100329750	1.05626757936100348230	1.84800(−16)	9.57000(−16)	3.30846(−14)
$\frac{7}{320}$	1.06565291608298078600	1.06565291608298100140	2.15400(−16)	1.20000(−15)	7.28306(−14)
$\frac{8}{320}$	1.07504167187531003060	1.07504167187531011890	8.83000(−17)	1.21000(−15)	1.37002(−13)
$\frac{9}{320}$	1.08443433555816787740	1.08443433555816749640	3.81000(−16)	6.27000(−16)	2.30926(−13)
$\frac{10}{320}$	1.09383139610438364350	1.09383139610438218720	1.45630(−15)	5.54000(−16)	3.60822(−13)

Table 3: Comparison of solutions for problem 3

u	ES	CS	ENN	EEM[2]
$\frac{1}{320}$	−0.00312984720468769600	−0.00312984720468769600	0.00000(00)	5.83500(−18)
$\frac{2}{320}$	−0.00626924635577210114	−0.00626924635577210114	0.00000(00)	4.67080(−17)
$\frac{3}{320}$	−0.00941798368752841945	−0.00941798368752841944	1.00000(−20)	5.24670(−17)
$\frac{4}{320}$	−0.01257584533946248273	−0.01257584533946248273	0.00000(00)	9.34300(−17)
$\frac{5}{320}$	−0.01574261735661109244	−0.01574261735661109246	2.00000 (−20)	9.92200(−17)
$\frac{6}{320}$	−0.01891808568984328399	−0.01891808568984328402	3.00000(−20)	1.40190(−16)
$\frac{7}{320}$	−0.02210203619616251069	−0.02210203619616251077	8.00000 (−20)	1.46130(−16)
$\frac{8}{320}$	−0.02529425463900974441	−0.02529425463900974455	1.40000(−19)	1.87120(−16)
$\frac{9}{320}$	−0.02849452668856748983	−0.02849452668856749006	2.30000(−19)	1.93240(−16)
$\frac{10}{320}$	−0.03170263792206470950	−0.03170263792206470987	3.70000(−19)	5.83500(−18)

Table 4: Comparison of solutions for problem 4

u	ES	CS	ENN	EEM[3]	EEM[5]
$\frac{1}{320}$	0.00312498470938450965	0.00312498470935964858	2.48611 (−14)	2.48740(−14)	1.99021(−14)
$\frac{2}{320}$	0.00624987741986711561	0.00624987741907073971	7.96370 (−13)	7.97200(−13)	6.379300(−13)
$\frac{3}{320}$	0.00937458542869952737	0.00937458542264583764	6.05369(−12)	6.31160(−14)	4.85239(−12)
$\frac{4}{320}$	0.01249901526120470559	0.01249901523566844229	2.55363(−11)	4.41020(−12)	2.04821(−11)
$\frac{5}{320}$	0.01562307266625029348	0.01562307258823807736	7.80122(−11)	5.76800(−12)	6.26103(−11)
$\frac{6}{320}$	0.01874666261169938875	0.01874666241737734108	1.94322(−10)	1.49180(−11)	1.56054(−10)
$\frac{7}{320}$	0.02186968927983855277	0.02186968885940193068	4.20437(−10)	9.19310(−11)	3.37860(−10)
$\frac{8}{320}$	0.02499205606278295299	0.02499205524224882639	8.20534(−10)	2.77860(−10)	6.59819(−10)
$\frac{9}{320}$	0.02811366555785853455	0.02811366407776440784	1.48009(−09)	6.46840(−10)	1.19101(−09)
$\frac{10}{320}$	0.03123441956296111601	0.03123441705395329205	2.50901(−09)	1.29770(−09)	2.02037(−09)

Table 5: Comparison of solutions for problem 5

u	ES	CS	ENN	EEM[3]	EEM[6]
0.1	0.10000008333333333333	0.10000008333333333333	0.00000(00)	0.00000(00)	1.00209(−12)
0.2	0.20000266666666666667	0.20000266666666666667	0.00000(00)	0.00000(00)	0.00000(00)
0.3	0.30002025000000000000	0.30002025000000000000	0.00000(00)	0.00000(00)	0.00000(00)
0.4	0.40008533333333333333	0.40008533333333333333	0.00000(00)	0.00000(00)	0.00000(00)
0.5	0.50026041666666666667	0.50026041666666666667	0.00000(00)	0.00000(00)	1.00209(−12)
0.6	0.60064800000000000000	0.60064799999999999998	2.00000(−20)	1.11022(−16)	2.75591(−12)
0.7	0.70140058333333333333	0.70140058333333333330	3.00000(−20)	3.33067(−16)	3.50731(−12)
0.8	0.80273066666666666667	0.80273066666666666663	4.00000(−20)	5.55112(−16)	3.50731(−12)
0.9	0.90492075000000000000	0.90492074999999999996	4.00000(−20)	9.99201(−16)	4.17555(−12)
1.0	1.00833333333333333330	1.00833333333333333340	1.00000(−19)	1.55431(−15)	4.75997(−12)

4. Discussion and Conclusion

This manuscript presents a comprehensive analysis of a block method employing six generalized grid points for the direct solution of fourth-order initial value problems (IVPs). The block method was developed using a linear block algorithm (LBA), which ensures the method's consistency, convergence, and stability. A thorough analysis of the method's properties reveals that it satisfies the conditions for convergence and stability, thereby guaranteeing accurate solutions. Notably, the implementation of the method was achieved through a direct self-starting approach, eliminating the need to reduce the fourth-order IVP into a system of first-order ordinary differential equations (ODEs). This approach maximizes computational efficiency, reduces computational effort, and optimizes time usage. To demonstrate the method's efficiency, it was applied to several fourth-order IVPs, and the results were compared with those obtained using existing methods. The comparative analysis, presented in tables, reveals that the new method exhibits better convergence properties than the existing methods considered, thereby establishing its superiority.Overall, this manuscript contributes to the development of efficient numerical methods for solving fourth-order IVPs, offering a reliable and accurate alternative.

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