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# New proof and variants of a referenced logarithmic-power integral

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**Abstract**

This article contributes to mathematical analysis by (i) presenting an elegant proof of a specific integral, (ii) demonstrating its connection with an existing result, and (iii) introducing previously unexplored variants.

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## 1. Introduction

Integral results in mathematics are essential tools used in all areas of applied science. They play a crucial role in modelling and solving real-world problems. Concrete applications can be found in physics, engineering, finance, economics and environmental sciences. The following books cover the basics of the subject: [1], [12], [13] and [7]. In addition, an overview of the most important integral formulas can be found in the influential book [2].

The pursuit of new integral results is driven by their ability to solve complex mathematical challenges in a variety of fields. Moreover, they often lead to practical applications that improve our understanding of physical phenomena. In addition, advances in integral calculus contribute to the development of sophisticated theories. Recent contributions to this topic include [8], [9], [10] and [11]. They illustrate the continuing relevance and development of integral methods in contemporary research.

In particular, integrals of the logarithmic type derived from those presented in [2] have attracted some attention. We can refer to [4], [5], [6], and again [8] and [10]. In this article, we contribute to the topic by revisiting, in a sense, the following referenced integral:

$$" \int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2} dx,"$$

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where $\alpha > 0$. It is given in [2, Entry 4.291.19, page 557], with a precise expression depending on the arctangent and logarithmic functions, and $\alpha$. In the first part, we propose an elegant proof of this result, which seems to be missing in the current literature. We also show the direct connection between this integral and [2, Entry 4.535.1, page 603]. In the second part, we innovate by determining new variants of this integral, of the following form:

$$\text{``}\int_0^1 \frac{f(x)\log(1+\alpha x)}{(1+\alpha x^2)^2}dx,\text{''} \tag{1.1}$$

where $\alpha > 0$ and $f(x)$ is a particular function, which we have assumed in this article to be polynomial. To the best of our knowledge, these variants are not given in [2] or any other source. We thus extend the available repertoire of integral results, especially those of the logarithmic type. Furthermore, our results open the door to further exploration of related integrals involving more complex functions $f(x)$, as discussed later.

The rest of the article is as follows: Section 2 revisits the proof of the main integral. Section 3 is devoted to its new variants. A conclusion is given in Section 4.

## 2. Revisit of some known integrals

We begin by revisiting the proof of the main integral. We then highlight a connection between it and another well-known integral result.

### *2.1. A logarithmic-power integral*

The proposition below corresponds to [2, Entry 4.291.19, page 557]. The main contribution remains on the proof which uses a judicious change of variables. As far as we know, this proof has not yet been mentioned in the literature.

**Proposition 2.1.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx = \frac{1}{2\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha).$$

**Proof of Proposition 2.1.** For this proof, let us set

$$I = \int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx.$$

With the change of variables $x = (1-y)/(1+\alpha y)$, noticing that

$$\frac{dx}{dy} = \frac{-(1+\alpha y)-(1-y)\alpha}{(1+\alpha y)^2} = -\frac{1+\alpha}{(1+\alpha y)^2},$$

$$1+\alpha x = 1+\alpha\frac{1-y}{1+\alpha y} = \frac{1+\alpha y+\alpha(1-y)}{1+\alpha y} = \frac{1+\alpha}{1+\alpha y},$$

$$\log\left(\frac{1+\alpha}{1+\alpha y}\right) = \log(1+\alpha) - \log(1+\alpha y)$$

and

$$(1+\alpha y)^2+\alpha(1-y)^2 = 1+2\alpha y+\alpha^2y^2+\alpha-2\alpha y+\alpha y^2$$
$$= 1+\alpha^2y^2+\alpha+\alpha y^2 = (1+\alpha)(1+\alpha y^2),$$

we get

$$\begin{aligned}
I &= \int_1^0 \frac{\log\left[1+\alpha(1-y)/(1+\alpha y)\right]}{1+\alpha(1-y)^2/(1+\alpha y)^2}\left[-\frac{1+\alpha}{(1+\alpha y)^2}\right]dy \\
&= (1+\alpha)\int_0^1 \frac{\log\left[(1+\alpha)/(1+\alpha y)\right]}{(1+\alpha y)^2+\alpha(1-y)^2}dy \\
&= \int_0^1 \frac{\log\left[(1+\alpha)/(1+\alpha y)\right]}{1+\alpha y^2}dy \\
&= \log(1+\alpha)\int_0^1 \frac{1}{1+\alpha y^2}dy - \int_0^1 \frac{\log(1+\alpha y)}{1+\alpha y^2}dy \\
&= \log(1+\alpha)\left[\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}y]\right]_{y=0}^{y=1} - I \\
&= \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - I.
\end{aligned}$$

We thus deduce that

$$2I = \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha),$$

and the desired result is obtained by multiplying by 2 on both sides. This completes the proof. □

Thanks to the technique of the proof of Proposition 2.1, we see how the arctangent, square and logarithmic functions naturally appear in the result.

As a special case, if we take $\alpha = 1$, since $\arctan(1) = \pi/4$, we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{1}{2\sqrt{1}}\arctan[\sqrt{1}]\log(1+1) = \frac{\pi}{8}\log(2), \tag{2.1}$$

which also appears in [2, Entry 4.291.8, page 556].

In [2], several variants of this integral are given, but not the form in Equation (1.1), with simple functions $f(x)$. Section 3 of this article fills this gap.

Before that, in the next subsection, we make the direct connection between the result in Proposition 2.1 and another entry in [2].

### *2.2. An arctangent-power integral*

The proposition below corresponds to [2, Entry 4.535.1, page 603] with $p = \sqrt{\alpha}$. The main novelty here is in the proof: we show that it is directly related to Proposition 2.1.

**Proposition 2.2.** *For any* $\alpha > 0$*, we have*

$$\int_0^1 \frac{\arctan[\sqrt{\alpha}x]}{1+\alpha x}dx = \frac{1}{2\alpha}\arctan[\sqrt{\alpha}]\log(1+\alpha).$$

**Proof of Proposition 2.2.** We start with the integral term in Proposition 2.1, whose expression we know. Applying an integration by parts using the arctangent primitive of $1/(1+\alpha x^2)$, we get

$$\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx = \left[\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}x]\log(1+\alpha x)\right]_{x=0}^{x=1} - \int_0^1 \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}x]\frac{\alpha}{1+\alpha x}dx$$
$$= \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \sqrt{\alpha}\int_0^1 \frac{\arctan[\sqrt{\alpha}x]}{1+\alpha x}dx.$$

It follows from Proposition 2.1 that

$$\frac{1}{2\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) = \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \sqrt{\alpha}\int_0^1 \frac{\arctan[\sqrt{\alpha}x]}{1+\alpha x}dx,$$

from which we immediately derive

$$\int_0^1 \frac{\arctan[\sqrt{\alpha}x]}{1+\alpha x}dx = \frac{1}{2\alpha}\arctan[\sqrt{\alpha}]\log(1+\alpha).$$

This concludes the proof. □

So we see how Proposition 2.1 can be used to establish other kinds of integral results, such as the one in Proposition 2.2. We recall that it corresponds to [2, Entry 4.535.1, page 603] with $p = \sqrt{\alpha}$.

As a special case, if we take $\alpha = 1$, we get

$$\int_0^1 \frac{\arctan(x)}{1+x}dx = \frac{1}{2\times 1}\arctan[\sqrt{1}]\log(1+1) = \frac{\pi}{8}\log(2),$$

value also obtained in Equation (2.1).

New integrals derived from our main logarithmic-power integral are set up in the next section.

## 3. Some new integrals

We can see that the integral in Proposition 2.1 has the form in Equation (1.1) with $f(x) = 1+\alpha x^2$. We will now consider other variants using the following functions:

$$f(x) = x(1+\alpha x), \quad f(x) = 1-x, \quad f(x) = x^2, \quad f(x) = 1, \quad f(x) = x,$$

$$f(x) = (1+\alpha x)^2, \quad f(x) = (1-\alpha x)^2, \quad f(x) = 1-\alpha x, \quad f(x) = 1+\alpha x, \quad f(x) = 1-\alpha x^2,$$

in order.

### *3.1. First new variant*

The first variant is the integral in Equation (1.1) with $f(x) = x(1+\alpha x)$. The proof is based on an appropriate integration by parts and Proposition 2.1.

**Proposition 3.1.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2}\log(1+\alpha x)dx$$
$$= \frac{1}{2\sqrt{\alpha}}\left\{\frac{1}{2}\arctan[\sqrt{\alpha}]\log(1+\alpha) + \arctan[\sqrt{\alpha}] - \frac{1}{\sqrt{\alpha}}\log(1+\alpha)\right\}.$$

**Proof of Proposition 3.1.** We start with the integral term in Proposition 2.1, whose expression we know. Applying an integration by parts using the polynomial-logarithmic primitive of $\log(1+\alpha x)$, we get

$$\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx = \left\{\left[\frac{1}{\alpha}(1+\alpha x)\log(1+\alpha x) - x\right]\frac{1}{1+\alpha x^2}\right\}_{x=0}^{x=1}$$
$$-\int_0^1 \left[\frac{1}{\alpha}(1+\alpha x)\log(1+\alpha x) - x\right]\left[-\frac{2\alpha x}{(1+\alpha x^2)^2}\right]dx$$
$$= \left[\frac{1}{\alpha}(1+\alpha)\log(1+\alpha) - 1\right]\frac{1}{1+\alpha}$$
$$+2\int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2}\log(1+\alpha x)dx - \int_0^1 x\frac{2\alpha x}{(1+\alpha x^2)^2}dx.$$

For the last integral term, a new integration by parts gives

$$\int_0^1 x\frac{2\alpha x}{(1+\alpha x^2)^2}dx = \left[x\left(-\frac{1}{1+\alpha x^2}\right)\right]_{x=0}^{x=1} + \int_0^1 \frac{1}{1+\alpha x^2}dx$$
$$= -\frac{1}{1+\alpha} + \left[\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}x]\right]_{x=0}^{x=1} = -\frac{1}{1+\alpha} + \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}].$$

Combining the equalities above and using Proposition 2.1, we obtain

$$\frac{1}{2\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) = \frac{1}{\alpha}\log(1+\alpha) - \frac{1}{1+\alpha}$$
$$+2\int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2}\log(1+\alpha x)dx + \frac{1}{1+\alpha} - \frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}].$$

We immediately derive

$$\int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2}\log(1+\alpha x)dx$$
$$= \frac{1}{2\sqrt{\alpha}}\left\{\frac{1}{2}\arctan[\sqrt{\alpha}]\log(1+\alpha) + \arctan[\sqrt{\alpha}] - \frac{1}{\sqrt{\alpha}}\log(1+\alpha)\right\}.$$

The desired formula is obtained. □

This result will be used to prove other variants presented below.

### 3.2. *Second new variant*

The second variant is the integral in Equation (1.1) with $f(x) = 1 - x$. The proof is based on a suitable decomposition of the main integral, combined with Propositions 2.1 and 3.1.

**Proposition 3.2.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{1-x}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \frac{1}{2\sqrt{\alpha}} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \arctan[\sqrt{\alpha}] + \frac{1}{\sqrt{\alpha}} \log(1+\alpha) \right\}.$$

**Proof of Proposition 3.2.** Noticing that $1 - x = (1 + \alpha x^2) - x(1 + \alpha x)$, we can write

$$\int_0^1 \frac{1-x}{(1+\alpha x^2)^2} \log(1+\alpha x)dx = \int_0^1 \frac{(1+\alpha x^2) - x(1+\alpha x)}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \int_0^1 \frac{1+\alpha x^2}{(1+\alpha x^2)^2} \log(1+\alpha x)dx - \int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2} dx - \int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2} \log(1+\alpha x)dx.$$

Applying Propositions 2.1 and 3.1, we get

$$\int_0^1 \frac{1-x}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \frac{1}{2\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha)$$
$$- \frac{1}{2\sqrt{\alpha}} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] \log(1+\alpha) + \arctan[\sqrt{\alpha}] - \frac{1}{\sqrt{\alpha}} \log(1+\alpha) \right\}$$
$$= \frac{1}{2\sqrt{\alpha}} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \arctan[\sqrt{\alpha}] + \frac{1}{\sqrt{\alpha}} \log(1+\alpha) \right\}.$$

This concludes the proof. □

### 3.3. *Third new variant*

The third variant is the integral in Equation (1.1) with $f(x) = x^2$. The proof is based on Proposition 2.1 and the differentiation of the corresponding integral with respect to $\alpha$.

**Proposition 3.3.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{x^2 \log(1+\alpha x)}{(1+\alpha x^2)^2} dx$$
$$= \frac{1}{(1+\alpha)\sqrt{\alpha}} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] - \frac{3}{4\sqrt{\alpha}} \log(1+\alpha) \right\} + \frac{1}{4\alpha\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha).$$

**Proof of Proposition 3.3.** It follows from Proposition 2.1 that

$$\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx = \frac{1}{2\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha).$$

Differentiating with respect to $\alpha$, we get

$$\frac{\partial}{\partial\alpha}\left[\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx\right] = \frac{\partial}{\partial\alpha}\left[\frac{1}{2\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha)\right]. \tag{3.1}$$

For the left term, applying the Leibniz integral rule which allows to interchange the derivative and integral symbols, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\left[\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2}dx\right] = \int_0^1 \frac{\partial}{\partial\alpha}\left[\frac{\log(1+\alpha x)}{1+\alpha x^2}\right]dx\\
&= \int_0^1 \left[\frac{x}{(1+\alpha x)(1+\alpha x^2)} - \frac{x^2\log(1+\alpha x)}{(1+\alpha x^2)^2}\right]dx\\
&= \int_0^1 \frac{x}{(1+\alpha x)(1+\alpha x^2)}dx - \int_0^1 \frac{x^2\log(1+\alpha x)}{(1+\alpha x^2)^2}dx.
\end{aligned}$$

The second integral term is the integral of interest. For the first integral term, a ratio decomposition work is required. We have

$$\begin{aligned}
&\int_0^1 \frac{x}{(1+\alpha x)(1+\alpha x^2)}dx = \frac{1}{1+\alpha}\int_0^1\left[\frac{1}{1+\alpha x^2}+\frac{x}{1+\alpha x^2}-\frac{1}{1+\alpha x}\right]dx\\
&= \frac{1}{1+\alpha}\left\{\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}x]+\frac{1}{2\alpha}\log(1+\alpha x^2)-\frac{1}{\alpha}\log(1+\alpha x)\right\}_{x=0}^{x=1}\\
&= \frac{1}{1+\alpha}\left\{\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]+\frac{1}{2\alpha}\log(1+\alpha)-\frac{1}{\alpha}\log(1+\alpha)\right\}\\
&= \frac{1}{(1+\alpha)\sqrt{\alpha}}\left\{\arctan[\sqrt{\alpha}]-\frac{1}{2\sqrt{\alpha}}\log(1+\alpha)\right\}.
\end{aligned}$$

On the other hand, for the right term in Equation (3.1), we have

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\left[\frac{1}{2\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha)\right]\\
&= \frac{1}{2\sqrt{\alpha}}\left\{\frac{\log(1+\alpha)}{2\sqrt{\alpha}(1+\alpha)} - \frac{1}{2\alpha}\arctan[\sqrt{\alpha}]\log(1+\alpha) + \frac{1}{1+\alpha}\arctan[\sqrt{\alpha}]\right\}.
\end{aligned}$$

We therefore obtain

$$
\begin{aligned}
&\int_0^1 \frac{x^2 \log(1+\alpha x)}{(1+\alpha x^2)^2} dx \\
&= \int_0^1 \frac{x}{(1+\alpha x)(1+\alpha x^2)} dx - \frac{\partial}{\partial \alpha}\left[\frac{1}{2\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha)\right] \\
&= \frac{1}{(1+\alpha)\sqrt{\alpha}}\left\{\arctan[\sqrt{\alpha}] - \frac{1}{2\sqrt{\alpha}} \log(1+\alpha)\right\} \\
&- \frac{1}{2\sqrt{\alpha}}\left\{\frac{\log(1+\alpha)}{2\sqrt{\alpha}(1+\alpha)} - \frac{1}{2\alpha} \arctan[\sqrt{\alpha}] \log(1+\alpha) + \frac{1}{1+\alpha} \arctan[\sqrt{\alpha}]\right\} \\
&= \frac{1}{(1+\alpha)\sqrt{\alpha}}\left\{\frac{1}{2} \arctan[\sqrt{\alpha}] - \frac{3}{4\sqrt{\alpha}} \log(1+\alpha)\right\} + \frac{1}{4\alpha\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha).
\end{aligned}
$$

The desired result is established. □

### 3.4. *Fourth new variant*

The fourth variant is the integral in Equation (1.1) with $f(x) = 1$. The proof is based on a suitable decomposition of the main integral, and Propositions 2.1 and 3.3.

**Proposition 3.4.** *For any $\alpha > 0$, we have*

$$
\begin{aligned}
&\int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2} dx \\
&= \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha} \arctan[\sqrt{\alpha}] - \frac{3}{4} \log(1+\alpha)\right\}.
\end{aligned}
$$

**Proof of Proposition 3.4.** Using the simple decomposition $1 = 1 + \alpha x^2 - \alpha x^2$, we can write

$$
\begin{aligned}
\int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2} dx &= \int_0^1 \frac{1+\alpha x^2 - \alpha x^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx \\
&= \int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2} dx - \alpha \int_0^1 \frac{x^2 \log(1+\alpha x)}{(1+\alpha x^2)^2} dx.
\end{aligned}
$$

It follows from Propositions 2.1 and 3.3 that

$$
\begin{aligned}
&\int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2} dx = \frac{1}{2\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) \\
&- \alpha\left[\frac{1}{(1+\alpha)\sqrt{\alpha}}\left\{\frac{1}{2} \arctan[\sqrt{\alpha}] - \frac{3}{4\sqrt{\alpha}} \log(1+\alpha)\right\} + \frac{1}{4\alpha\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha)\right] \\
&= \frac{1}{2\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) \\
&- \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha} \arctan[\sqrt{\alpha}] - \frac{3}{4} \log(1+\alpha)\right\} - \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) \\
&= \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha} \arctan[\sqrt{\alpha}] - \frac{3}{4} \log(1+\alpha)\right\}.
\end{aligned}
$$

The desired result is established. □

### 3.5. *Fifth new variant*

The fifth variant is the integral in Equation (1.1) with $f(x) = x$. The proof is based on a suitable decomposition of the main integral, and Propositions 3.2 and 3.4.

**Proposition 3.5.** *For any* $\alpha > 0$, *we have*

$$\int_0^1 \frac{x\log(1+\alpha x)}{(1+\alpha x^2)^2}dx = \frac{1}{4\alpha(1+\alpha)}\left\{2\sqrt{\alpha}\arctan[\sqrt{\alpha}] + (\alpha-2)\log(1+\alpha)\right\}.$$

**Proof of Proposition 3.5.** Using the basic decomposition $x = 1-(1-x)$, we get

$$\begin{aligned}\int_0^1 \frac{x\log(1+\alpha x)}{(1+\alpha x^2)^2}dx &= \int_0^1 \frac{1-(1-x)}{(1+\alpha x^2)^2}\log(1+\alpha x)dx \\ &= \int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2}dx - \int_0^1 \frac{1-x}{(1+\alpha x^2)^2}\log(1+\alpha x)dx.\end{aligned}$$

Propositions 3.2 and 3.4 give

$$\begin{aligned}&\int_0^1 \frac{x\log(1+\alpha x)}{(1+\alpha x^2)^2}dx \\ &= \frac{1}{4\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan[\sqrt{\alpha}] - \frac{3}{4}\log(1+\alpha)\right\} \\ &- \frac{1}{2\sqrt{\alpha}}\left\{\frac{1}{2}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \arctan[\sqrt{\alpha}] + \frac{1}{\sqrt{\alpha}}\log(1+\alpha)\right\} \\ &= -\frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan[\sqrt{\alpha}] - \frac{3}{4}\log(1+\alpha)\right\} \\ &- \frac{1}{2\sqrt{\alpha}}\left\{-\arctan[\sqrt{\alpha}] + \frac{1}{\sqrt{\alpha}}\log(1+\alpha)\right\} \\ &= \frac{1}{4\alpha(1+\alpha)}\left\{2\sqrt{\alpha}\arctan[\sqrt{\alpha}] + (\alpha-2)\log(1+\alpha)\right\}.\end{aligned}$$

The desired formula is obtained. □

### 3.6. *Sixth new variant*

The sixth variant is the integral in Equation (1.1) with $f(x) = (1+\alpha x)^2$. The proof is based on a suitable decomposition of the main integral, and Propositions 3.1, 3.4 and 3.5.

**Proposition 3.6.** *For any* $\alpha > 0$, *we have*

$$\begin{aligned}&\int_0^1 \frac{(1+\alpha x)^2}{(1+\alpha x^2)^2}\log(1+\alpha x)dx \\ &= \frac{1}{4}\left\{\left[\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right]\arctan[\sqrt{\alpha}]\log(1+\alpha) - \log(1+\alpha) + 2\sqrt{\alpha}\arctan[\sqrt{\alpha}]\right\}.\end{aligned}$$

**Proof of Proposition 3.6.** Applying the decomposition $(1+\alpha x)^2 = (1+\alpha x) + \alpha x(1+\alpha x)$, we have

$$\int_0^1 \frac{(1+\alpha x)^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx = \int_0^1 \frac{(1+\alpha x) + \alpha x(1+\alpha x)}{(1+\alpha x^2)^2} \log(1+\alpha x) dx$$
$$= \int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2} dx + \alpha \int_0^1 \frac{x \log(1+\alpha x)}{(1+\alpha x^2)^2} dx + \alpha \int_0^1 \frac{x(1+\alpha x)}{(1+\alpha x^2)^2} \log(1+\alpha x) dx.$$

It follows from Propositions 3.1, 3.4 and 3.5 that

$$\int_0^1 \frac{(1+\alpha x)^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx$$
$$= \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \frac{1}{1+\alpha} \left\{ \frac{1}{2}\sqrt{\alpha} \arctan[\sqrt{\alpha}] - \frac{3}{4} \log(1+\alpha) \right\}$$
$$+ \alpha \left[ \frac{1}{4\alpha(1+\alpha)} \left\{ 2\sqrt{\alpha} \arctan[\sqrt{\alpha}] + (\alpha-2) \log(1+\alpha) \right\} \right]$$
$$+ \alpha \left[ \frac{1}{2\sqrt{\alpha}} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] \log(1+\alpha) + \arctan[\sqrt{\alpha}] - \frac{1}{\sqrt{\alpha}} \log(1+\alpha) \right\} \right]$$
$$= \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \frac{1}{1+\alpha} \left\{ \frac{1}{2}\sqrt{\alpha} \arctan[\sqrt{\alpha}] - \frac{3}{4} \log(1+\alpha) \right\}$$
$$+ \frac{1}{4(1+\alpha)} \left\{ 2\sqrt{\alpha} \arctan[\sqrt{\alpha}] + (\alpha-2) \log(1+\alpha) \right\}$$
$$+ \frac{\sqrt{\alpha}}{2} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] \log(1+\alpha) + \arctan[\sqrt{\alpha}] - \frac{1}{\sqrt{\alpha}} \log(1+\alpha) \right\}$$
$$= \frac{1}{4} \left\{ \left[ \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right] \arctan[\sqrt{\alpha}] \log(1+\alpha) - \log(1+\alpha) + 2\sqrt{\alpha} \arctan[\sqrt{\alpha}] \right\}.$$

The proof ends. □

### 3.7. *Seventh new variant*

The seventh variant is the integral in Equation (1.1) with $f(x) = (1-\alpha x)^2$. The proof is based on a suitable decomposition of the main integral, and Propositions 3.5 and 3.6.

**Proposition 3.7.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{(1-\alpha x)^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx$$
$$= \frac{1}{4} \left[ \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right] \arctan[\sqrt{\alpha}] \log(1+\alpha)$$
$$+ \frac{1}{2(1+\alpha)} \left\{ \sqrt{\alpha}(\alpha-3) \arctan[\sqrt{\alpha}] + \frac{1}{2}(7-5\alpha) \log(1+\alpha) \right\}.$$

**Proof of Proposition 3.7.** The decomposition $(1+\alpha x^2)^2 = (1+\alpha x)^2 - 4\alpha x$ gives

$$\int_0^1 \frac{(1-\alpha x)^2}{(1+\alpha x^2)^2} \log(1+\alpha x)dx = \int_0^1 \frac{(1+\alpha x)^2 - 4\alpha x}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \int_0^1 \frac{(1+\alpha x)^2}{(1+\alpha x^2)^2} \log(1+\alpha x)dx - 4\alpha \int_0^1 \frac{x\log(1+\alpha x)}{(1+\alpha x^2)^2} dx.$$

Applying Propositions 3.5 and 3.6, we find that

$$\int_0^1 \frac{(1-\alpha x)^2}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \frac{1}{4}\left\{\left[\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right] \arctan[\sqrt{\alpha}]\log(1+\alpha) - \log(1+\alpha) + 2\sqrt{\alpha}\arctan[\sqrt{\alpha}]\right\}$$
$$-4\alpha\left[\frac{1}{4\alpha(1+\alpha)}\left\{2\sqrt{\alpha}\arctan[\sqrt{\alpha}] + (\alpha-2)\log(1+\alpha)\right\}\right]$$
$$= \frac{1}{4}\left[\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right] \arctan[\sqrt{\alpha}]\log(1+\alpha)$$
$$+ \frac{1}{2(1+\alpha)}\left\{\sqrt{\alpha}(\alpha-3)\arctan[\sqrt{\alpha}] + \frac{1}{2}(7-5\alpha)\log(1+\alpha)\right\}.$$

The desired result is obtained. □

### 3.8. *Eighth new variant*

The eighth variant is the integral in Equation (1.1) with $f(x) = 1 - \alpha x$. The proof is based on a suitable decomposition of the main integral, and Propositions 3.4 and 3.5.

**Proposition 3.8.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{1-\alpha x}{(1+\alpha x^2)^2} \log(1+\alpha x)dx$$
$$= \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}]\log(1+\alpha) - \frac{1}{1+\alpha}\left\{\sqrt{\alpha}\arctan[\sqrt{\alpha}] + \frac{1}{4}(\alpha-5)\log(1+\alpha)\right\}.$$

**Proof of Proposition 3.8.** We have immediately

$$\int_0^1 \frac{1-\alpha x}{(1+\alpha x^2)^2} \log(1+\alpha x)dx = \int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2} dx - \alpha\int_0^1 \frac{x\log(1+\alpha x)}{(1+\alpha x^2)^2} dx.$$

It follows from Propositions 3.4 and 3.5 that

$$\begin{aligned}
&\int_0^1 \frac{1-\alpha x}{(1+\alpha x^2)^2}\log(1+\alpha x)dx \\
&= \frac{1}{4\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan[\sqrt{\alpha}] - \frac{3}{4}\log(1+\alpha)\right\} \\
&- \alpha\left[\frac{1}{4\alpha(1+\alpha)}\left\{2\sqrt{\alpha}\arctan[\sqrt{\alpha}] + (\alpha-2)\log(1+\alpha)\right\}\right] \\
&= \frac{1}{4\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan[\sqrt{\alpha}] - \frac{3}{4}\log(1+\alpha)\right\} \\
&- \alpha\left[\frac{1}{4\alpha(1+\alpha)}\left\{2\sqrt{\alpha}\arctan[\sqrt{\alpha}] + (\alpha-2)\log(1+\alpha)\right\}\right] \\
&= \frac{1}{4\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \frac{1}{1+\alpha}\left\{\sqrt{\alpha}\arctan[\sqrt{\alpha}] + \frac{1}{4}(\alpha-5)\log(1+\alpha)\right\}.
\end{aligned}$$

This concludes the proof. □

### 3.9. *Ninth new variant*

The ninth variant is the integral in Equation (1.1) with $f(x) = 1+\alpha x$. The proof is based on a suitable decomposition of the main integral, and Propositions 3.4 and 3.5.

**Proposition 3.9.** *For any* $\alpha > 0$*, we have*

$$\int_0^1 \frac{1+\alpha x}{(1+\alpha x^2)^2}\log(1+\alpha x)dx = \frac{1}{4}\log(1+\alpha)\left\{\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]+1\right\}.$$

**Proof of Proposition 3.9.** The following decomposition holds:

$$\int_0^1 \frac{1+\alpha x}{(1+\alpha x^2)^2}\log(1+\alpha x)dx = \int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2}dx + \alpha\int_0^1 \frac{x\log(1+\alpha x)}{(1+\alpha x^2)^2}dx.$$

Propositions 3.4 and 3.5 give

$$\begin{aligned}
&\int_0^1 \frac{1+\alpha x}{(1+\alpha x^2)^2}\log(1+\alpha x)dx \\
&= \frac{1}{4\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan[\sqrt{\alpha}] - \frac{3}{4}\log(1+\alpha)\right\} \\
&+ \alpha\left[\frac{1}{4\alpha(1+\alpha)}\left\{2\sqrt{\alpha}\arctan[\sqrt{\alpha}] + (\alpha-2)\log(1+\alpha)\right\}\right] \\
&= \frac{1}{4\sqrt{\alpha}}\arctan[\sqrt{\alpha}]\log(1+\alpha) + \frac{1}{4}\log(1+\alpha) \\
&= \frac{1}{4}\log(1+\alpha)\left\{\frac{1}{\sqrt{\alpha}}\arctan[\sqrt{\alpha}]+1\right\}.
\end{aligned}$$

This ends the proof. □

### 3.10. Tenth new variant

The tenth variant is the integral in Equation (1.1) with $f(x) = 1 - \alpha x^2$. The proof is based on a suitable decomposition of the main integral, and Propositions 3.3 and 3.4.

**Proposition 3.10.** *For any $\alpha > 0$, we have*

$$\int_0^1 \frac{1-\alpha x^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx = \frac{1}{1+\alpha} \left\{ \frac{3}{2} \log(1+\alpha) - \sqrt{\alpha} \arctan[\sqrt{\alpha}] \right\}.$$

**Proof of Proposition 3.10.** We have

$$\int_0^1 \frac{1-\alpha x^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx = \int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^2} dx - \alpha \int_0^1 \frac{x^2 \log(1+\alpha x)}{(1+\alpha x^2)^2} dx.$$

It follows from Propositions 3.3 and 3.4 that

$$\begin{aligned}
&\int_0^1 \frac{1-\alpha x^2}{(1+\alpha x^2)^2} \log(1+\alpha x) dx \\
&= \frac{1}{4\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) - \frac{1}{1+\alpha} \left\{ \frac{1}{2} \sqrt{\alpha} \arctan[\sqrt{\alpha}] - \frac{3}{4} \log(1+\alpha) \right\} \\
&- \alpha \left[ \frac{1}{(1+\alpha)\sqrt{\alpha}} \left\{ \frac{1}{2} \arctan[\sqrt{\alpha}] - \frac{3}{4\sqrt{\alpha}} \log(1+\alpha) \right\} + \frac{1}{4\alpha\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha) \right] \\
&= \frac{1}{1+\alpha} \left\{ \frac{3}{2} \log(1+\alpha) - \sqrt{\alpha} \arctan[\sqrt{\alpha}] \right\}.
\end{aligned}$$

This concludes the proof. □

Taking $\alpha = 1$, we get

$$\begin{aligned}
&\int_0^1 \frac{1-x^2}{(1+x^2)^2} \log(1+x) dx = \frac{1}{1+1} \left\{ \frac{3}{2} \log(1+1) - \sqrt{1} \arctan[\sqrt{1}] \right\} \\
&= \frac{3}{4} \log(2) - \frac{\pi}{8},
\end{aligned}$$

which also corresponds to [2, Entry 4.291.27, page 557] with $a = 1$ (only for this special case the results are identical).

## 4. Conclusion

Finally, this article contributes to analysis by proposing an elegant proof for the following integral:

$$\int_0^1 \frac{\log(1+\alpha x)}{1+\alpha x^2} dx = \frac{1}{2\sqrt{\alpha}} \arctan[\sqrt{\alpha}] \log(1+\alpha),$$

with $\alpha > 0$, showing its connection with another existing result and introducing variants that have not received attention in the literature. Future research includes the exploration of new relationships between these integrals and their applications to broader scientific problems. We note that some physics problems, such as those derived from Feynman diagrams (see [3]), involve the evaluation of complicated integrals of the logarithmic type. Our results therefore provide valuable evaluation tools in these areas.

Other direct variants may also be considered, such as the following, for which there is no established formula to our knowledge:

$$\int_0^1 \frac{[\log(1+\alpha x)]^2}{1+\alpha x^2} dx,$$

with $\alpha > 0$, and

$$\int_0^1 \frac{\log[\log(1+\alpha x)]}{1+\alpha x^2} dx,$$

with $\alpha > 0$. We can also think about finding formulas for some parameter generalizations, such as

$$\int_0^1 \frac{\log(1+\alpha x)\log(1+\beta x)}{1+\gamma x^2} dx,$$

with $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, or

$$\int_0^1 \frac{\log(1+\alpha x)}{(1+\alpha x^2)^{\beta}} dx,$$

with $\alpha > 0$ and $\beta > 0$. These perspectives require further investigation, which we have left for the future.


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