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Statistical approximation for functions of two variables by Bernstein-Chlodowsky Polynomials on a triangular domain

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Abstract

Statistical approximation of continuous functions of two variables using Bernstein-Chlodowsky polynomials on a triangular domain is studied. Further, the weighted approximation of continuous functions of two variables on a triangular domain is investigated statistically. Finally, we study the approximation results in terms of ideal convergence.

Keywords: Bernstein-Chlodowsky polynomials, Positive linear operators, Statistical convergence, Ideal convergence, Korovkin approximation theorem.

1. Introduction and Background

The notion of statistical convergence was introduced by Fast [6] and Schoenberg [23] independently in the same year 1951. The idea of statistical convergence first appeared, under the name of “almost convergence”, in the first edition of celebrated monograph by Zygmund published in Warsaw in 1935 [25]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory and number theory. Statistical convergence has been investigated in summability theory by Fridy [7], Šalát [22], topological groups (Prullage [21]), topological spaces (Cakalli and Khan [4], Di Maio and Kočinac [17]), locally convex spaces (by Maddox [18]), measure theory (Miller [19]), normed spaces and probabilistic normed space.

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Throughout the paper $\mathbb{N}$ will denote the set of all natural numbers and $\mathbb{R}$ will denote the set of real numbers. If $K \subset \mathbb{N}$ and $K_n = \{k \in K : k \leqslant n\}$ then the natural density or asymptotic density of the subset K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{card(K_n)}{n}, \text{ provided the limit exists.}$$

The Korovkin approximation is a convergence statement which approximate a function by a certain sequences of positive linear operators on $C[a, b]$, the space of all real valued continuous functions defined on $[a, b]$ (see [1] for Korovkin types approximation). The classical Bernstein-Chlodowsky polynomials was introduced by Chlodowsky in 1932 as a generalization of Bernstein polynomials on unbounded set. There are some investigations devoted to the problem of approximating continuous functions by one as well as two dimensional Bernstein-Chlodowsky polynomials ([9, 10, 14]). Very recently, Resat et. al.[2] obtain some approximation results for bivariate Bernstein-Kantorovich type operators on a triangular domain (see [3, 20, 24] for more results on approximation).

The aim of this paper, in the line of E. Ibikli [14], is to investigate some problems of approximation for continuous functions of two variables by means of Bernstein-Chlodowsky polynomials on a triangular domain via statistical convergence. Using ideal convergence, we address the same outcomes as earlier parts in the final portion.

2. Statistical approximation for the functions on triangular domain $\triangle_c$

We first recall the formal definitions of statistical convergence and statistical boundedness of a sequence of real numbers.

[6] A sequence $\{y_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$ the set $K(\varepsilon) = \{n \in \mathbb{N} : |y_n - L| \geqslant \varepsilon\}$ has natural density zero i.e., $\delta(K(\varepsilon)) = 0$ for any $\varepsilon > 0$. In this case, we write $st - \lim_{n\to\infty} y_n = L$.

[8] A sequence $\{y_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically bounded if there exists a real number B such that $\delta(\{n \in \mathbb{N} : |y_n| > B\}) = 0$. In fact, every statistically convergent sequence of reals is statistically bounded. For sequences of real-valued functions, Gökhan et al.[12, 13] and Duman and Orhan [5] independently defined pointwise and uniform statistical convergence. Let for any $c > 0$ we denote by $\triangle_c$, the triangular domain

$$\triangle_c = \{(x, y) : x \geqslant 0, y \geqslant 0, x + y \leqslant c\}$$

and (a_n) is the sequence of positive numbers such that the sequence (a_n) is divergent to ∞ and

$$st - \lim_{n\to\infty} \frac{a_n}{n} = 0. \tag{2.1}$$

Then $\triangle_{a_n}$ is the corresponding triangle with $c = a_n$. We recall the Bernstein-Chlodowsky polynomials $(BC)_n(f)$ for a function f of two variables in following form: for $(x, y) \in \triangle_{a_n}$

$$(BC)_n(f, x, y) = \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{x + y}{a_n}\right)^{n-k} \sum_{j=0}^{k} f\left(\frac{k - j}{n} a_n, \frac{j}{n} a_n\right) \binom{k}{j} \left(\frac{x}{a_n}\right)^{k-j} \left(\frac{y}{a_n}\right)^{j} \tag{2.2}$$

By the properties (1), the triangular domain $\triangle_{a_n}$ extends to the infinite quadrant $x \geqslant 0$, $y \geqslant 0$ as $n \to \infty$ and we established a theorem on the approximation of continuous functions by polynomials (2) on an unbounded set. Let $\mathbb{R}_2^{++} = \{(x, y) : x \geqslant 0, y \geqslant 0\}$. We define $B_\rho(\mathbb{R}_2^{++})$ by the set of all functions defined in $\mathbb{R}_2^{++}$ and satisfying

$$| f(x, y) | \leqslant M_f \rho(x, y) \tag{2.3}$$

where $\rho(x, y) = (1 + x^2 + y^2)$ and M_f is a constant depending on the function f only. By $C_\rho(\mathbb{R}_2^{++})$ we denote the space of all continuous functions belonging to $B_\rho(\mathbb{R}_2^{++})$. If we introduce the norm

$$\|f\|_\rho = \sup_{x \geqslant 0, y \geqslant 0} \frac{|f(x, y)|}{\rho(x, y)} \tag{2.4}$$

then $B_\rho(\mathbb{R}_2^{++})$ and $C_\rho(\mathbb{R}_2^{++})$ are normed linear spaces.

In particular, for $(x, y) \in \triangle_c$, we define $\|f\|_{\rho, \triangle_c} = \sup_{(x,y) \in \triangle_c} \frac{|f(x, y)|}{\rho(x, y)}$ but for time being we use $\|f\|_\rho$. If L be a positive linear operator then $L(f) \geqslant 0$ for any positive function f. Also we denote the value of $L(f)$ at a point (x, y) by $L(f; x, y)$.

We prove the following Korovkin type approximation theorem for a function $f \in C_\rho(\mathbb{R}_2^{++})$ on a bounded triangular domain $\triangle_c$. If a sequence of positive linear operator $L_n : C_\rho(\mathbb{R}_2^{++}) \to B_\rho(\mathbb{R}_2^{++})$ fulfils five conditions

$$st - \lim_{n \to \infty} \|L_n(f_{k,m}) - f_{k,m}\|_\rho = 0 \tag{2.5}$$

where $f_{0,0} = 1$; $f_{1,0} = \xi$; $f_{0,1} = \eta$; $f_{2,0} = \xi^2$; $f_{0,2} = \eta^2$, then for a fixed positive number c and for any function $f \in C_\rho(\mathbb{R}_2^{++})$ we have,

$$st - \lim_{n \to \infty} \|L_n(f) - f\|_\rho = 0$$

where the norm $\|f\|_\rho$ is restricted on $\triangle_c$.

Proof. For all $(x, y) \in \triangle_a$ and $f \in C_\rho$ we have

$$| f(x, y) | \leqslant \|f\|_\rho (1 + x^2 + y^2)$$

Hence for all $(x, y) \in \triangle_a$

$$-2\|f\|_\rho (1 + x^2 + y^2) < f(\xi, \eta) - f(x, y) < 2\|f\|_\rho (1 + x^2 + y^2)$$

Further, for any $\varepsilon > 0$ there exists $\sigma > 0$ such that

$$-\varepsilon < f(\xi, \eta) - f(x, y) < \varepsilon$$

only if $(\xi - x)^2 + (\eta - y)^2 < \sigma^2$. Combining these two we can write,

$$-\varepsilon - \frac{2\|f\|_\rho (1 + x^2 + y^2)}{\sigma^2} \{(\xi - x)^2 + (\eta - y)^2\} \quad < \quad f(\xi, \eta) - f(x, y)$$

$$< \quad \varepsilon + \frac{2\|f\|_\rho(1+x^2+y^2)}{\sigma^2}\{(\xi-x)^2+(\eta-y)^2\}.$$

For $(x,y)\in\triangle_a$, $(1+x^2+y^2)$ is bounded and say bounded by M. Hence, from the monotonicity of positive linear operators it follows that,

$$\begin{aligned} -\varepsilon L_n(1)-\frac{2\|f\|_\rho M}{\sigma^2}L_n\{(\xi-x)^2+(\eta-y)^2\} &< L_nf(\xi,\eta;x,y)-f(x,y)L_n(1) \\ &< \varepsilon L_n(1)+\frac{2\|f\|_\rho M}{\sigma^2}L_n\{(\xi-x)^2+(\eta-y)^2\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mid L_nf(\xi,\eta;x,y)-f(x,y)L_n(1;x,y)\mid &< \varepsilon + A\mid L_n(1;x,y)-1(x,y)\mid \\ &+B\mid L_n\,(\xi;x,y)-\xi(x,y)\mid \\ &+C\mid L_n\,(\eta;x,y)-\eta(x,y)\mid \\ &+D\mid L_n\left(\xi^2;x,y\right)-\xi^2(x,y)\mid \\ &+E\mid L_n\left(\eta^2;x,y\right)-\eta^2(x,y)\mid \end{aligned}$$

where, $A=\varepsilon+\frac{4\|f\|_\rho M a^2}{\sigma^2}$, $B=C=\frac{4\|f\|_\rho M a}{\sigma^2}$, $D=E=\frac{2\|f\|_\rho M}{\sigma^2}$. As $(1+x^2+y^2)$ is positive for $(x,y)\in\triangle_c$, dividing both sides by $(1+x^2+y^2)$ we get,

$$\begin{aligned} \frac{\mid L_nf(x,y)-f(x,y)L_n(1;x,y)\mid}{(1+x^2+y^2)} &< \frac{\varepsilon}{(1+x^2+y^2)}+A\frac{\mid L_n(1;x,y)-1(x,y)\mid}{(1+x^2+y^2)} \\ &+B\frac{\mid L_n\,(\xi;x,y)-\xi(x,y)\mid}{(1+x^2+y^2)} \\ &+C\frac{\mid L_n\,(\eta;x,y)-\eta(x,y)\mid}{(1+x^2+y^2)} \\ &+D\frac{\mid L_n\left(\xi^2;x,y\right)-\xi^2(x,y)\mid}{(1+x^2+y^2)} \\ &+E\frac{\mid L_n\left(\eta^2;x,y\right)-\eta^2(x,y)\mid}{(1+x^2+y^2)} \\ &< \varepsilon \sup_{(x,y)\in\triangle_c}\frac{1}{(1+x^2+y^2)} \\ &+A\sup_{(x,y)\in\triangle_c}\frac{\mid L_n(f_{0,0};x,y)-f_{0,0}(x,y)\mid}{(1+x^2+y^2)} \\ &+B\sup_{(x,y)\in\triangle_c}\frac{\mid L_n\,(f_{1,0};x,y)-f_{1,0}(x,y)\mid}{(1+x^2+y^2)} \\ &+C\sup_{(x,y)\in\triangle_c}\frac{\mid L_n\,(f_{0,1};x,y)-f_{0,1}(x,y)\mid}{(1+x^2+y^2)} \\ &+D\sup_{(x,y)\in\triangle_c}\frac{\mid L_n\,(f_{2,0};x,y)-f_{2,0}(x,y)\mid}{(1+x^2+y^2)} \\ &+E\sup_{(x,y)\in\triangle_c}\frac{\mid L_n\,(f_{0,2};x,y)-f_{0,2}(x,y)\mid}{(1+x^2+y^2)}. \end{aligned}$$

Hence

$$\|L_n f - f(x,y)L_n(1)\|_\rho < \varepsilon + A\|L_n(1)-1\|_\rho + B\|L_n(\xi)-\xi\|_\rho + C\|L_n(\eta)-\eta\|_\rho + D\|L_n(\xi^2)-\xi^2\|_\rho + E\|L_n(\eta^2)-\eta^2\|_\rho.$$

Now

$|L_n f(x,y) - f(x,y)| \leqslant | L_n f(x,y) - f(x,y)L_n(1;x,y)| + |f(x,y)|| L_n f_{0,0}(x,y) - f_{0,0}(x,y)|$.

Since f is continuous on $\triangle_c$ and from (5) it follows

$$st - \lim_{n\to\infty} \|L_n(f;x,y) - f\|_\rho = 0.$$

□

Let f be a continuous function in $\mathbb{R}_2^{++}$ and satisfy the condition $| f(x,y) | \leqslant M_f(1+x^2+y^2)$,where M_f is a constant depending on the function f only. Then for any fixed positive number c, the relation

$$st - \lim_{n\to\infty} \max_{(x,y)\in\triangle_c} | (BC)_n(f;x,y) - f(x,y) | = 0$$

Proof. Let $f_{k,m}(u,v) = u^k v^m$. Then by applying Bernstein-Chlodowsky polynomials (2) for $f_{k,m}$ we get,

$$(BC)_n(f_{0,0};x,y) = 1;$$

$$(BC)_n(f_{1,0};x,y) = x;$$

$$(BC)_n(f_{0,1};x,y) = y;$$

$$(BC)_n(f_{2,0};x,y) = x^2 + \frac{x(a_n - x)}{n};$$

$$(BC)_n(f_{0,2};x,y) = y^2 + \frac{y(a_n - y)}{n}.$$

Hence

$$\|(BC)_n(f_{0,0};x,y) - f_{0,0}\|_\rho = 0;$$

$$\|(BC)_n(f_{1,0};x,y) - f_{1,0}\|_\rho = 0;$$

$$\|(BC)_n(f_{0,1};x,y) - f_{0,1}\|_\rho = 0;$$

$$\|(BC)_n(f_{2,0};x,y) - f_{2,0}\|_\rho = \frac{a_n}{n};$$

$$\|(BC)_n(f_{0,2};x,y) - f_{0,2}\|_\rho = \frac{a_n}{n}.$$

By previous theorem as $f(x,y)$ satisfies the conditions (3) and then (1) follows that $(BC)_n(f;x,y)$ uniformly statistical convergent to f. Now for a triangular region $\triangle_c$, no matter however large, for some n, $\triangle_{a_n}$ will contain $\triangle_c$. Hence we get a solution for the approximate problem on a closed subset $\triangle_c$. □

The polynomials (2) cannot approximate the infinitely differentiable function $f^*(x,y) =$

$x^2 + y^2$ on the entire triangular region $\triangle_{a_n}$ because, according to Theorem 2,

$$\sup_{(x,y)\in\triangle_{a_n}} |\, (BC)_n(f^*; x, y) - f^*(x, y)\, | = \frac{a_n^2}{2n}.$$

Moreover, the right hand side will not be statistically convergent by (1) in general.

Let $\{a_n\}$ be a sequence of positive numbers defined as

$$a_n = \begin{cases} n & n = i^2, i = 1, 2, 3, ... \\ n^{\frac{3}{4}} & otherwise. \end{cases}$$

Hence $\{a_n\}$ is a divergent to ∞ and $st - \lim\limits_{n\to\infty} \frac{a_n}{n} = 0$ but $\frac{a_n^2}{n}$ does not converge statistically.

3. Weighted approximation for the functions on triangular domain $\triangle_{a_n}$

The relation

$$st - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_{a_n}} \frac{|\, (BC)_n(f; x, y) - f(x, y)\, |}{(1 + x^2 + y^2)^{1+\gamma}} = 0$$

holds for any continuous function f satisfying (3) and for any positive γ.

Proof. For a given $\varepsilon > 0$ there exists a large $c > 0$ such that,

$$\tfrac{1}{1+x^2+y^2} < \varepsilon \text{ for } x + y > c$$

Since, by definition the sequence $(\frac{a_n}{n})$ is statistically convergent and hence statistically bounded. Then there exists a $C > 0$ such that $\delta(K_1) = 0$ where $K_1 = \{n \in \mathbb{N} :|\, \frac{a_n}{n}\, |> C\}$. Now

$$\begin{aligned} |\, f(x, y)\, | &\leqslant M_f(1 + x^2 + y^2) \\ \Rightarrow |\, (BC)_n(f; x, y)\, | &\leqslant M_f(BC)_n\left(1 + x^2 + y^2\right) \\ \Rightarrow |\, (BC)_n(f; x, y)\, | &\leqslant M_f\left(1 + x^2 + y^2 + \frac{x(a_n - x)}{n} + \frac{y(a_n - y)}{n}\right) \\ &= M_f(1 + x^2 + y^2)\left(1 + \frac{x(a_n - x)}{n(1 + x^2 + y^2)} + \frac{y(a_n - y)}{n(1 + x^2 + y^2)}\right) \\ &\leqslant M_f(1 + x^2 + y^2)\left(1 + \frac{x(a_n - x)}{n(1 + x^2)} + \frac{y(a_n - y)}{n(1 + y^2)}\right) \\ &\leqslant M_f(1 + x^2 + y^2)\left(1 + \frac{(a_n - x)}{2n} + \frac{(a_n - y)}{2n}\right) \\ &\leqslant M_f(1 + x^2 + y^2)\left(1 + \frac{a_n}{n}\right) \\ &\leqslant M_f(1 + x^2 + y^2)(1 + C) \text{ for all } n \in \mathbb{N} \setminus K_1. \end{aligned}$$

Hence for all $x, y \geqslant 0$ and for any continuous function f satisfying (3), we get,

$$|\, (BC)_n(f; x, y)\, |\leqslant M_f'(1 + x^2 + y^2) \text{ for all } n \in \mathbb{N} \setminus K_1.$$

where, $M'_f = M_f(1+C)$ is independent of n. Now, for any continuous function f, satisfying (3) and any $\gamma > 0$

$$\begin{aligned}\sup_{(x,y)\in\triangle_{a_n}} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}} &\leqslant \sup_{(x,y)\in\triangle_c} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}} \\ &\quad + \sup_{(x,y)\in\triangle_{a_n}\setminus\triangle_c} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}} \\ &\leqslant \sup_{(x,y)\in\triangle_c} |(BC)_n(f;x,y)-f(x,y)| \\ &\quad + \sup_{(x,y)\in\triangle_{a_n}\setminus\triangle_c} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}}\end{aligned}$$

By Theorem 2

$$st - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_c} |(BC)_n(f;x,y)-f(x,y)| = 0$$

Again, for all $n \in \mathbb{N} \setminus K_1$

$$\begin{aligned}\sup_{(x,y)\in\triangle_{a_n}\setminus\triangle_c} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}} &\leqslant \left(M'_f + M_f\right) \sup_{(x,y)\in\triangle_{a_n}\setminus\triangle_c} \frac{1}{(1+x^2+y^2)^{\gamma}} \\ &\leqslant \left(M'_f + M_f\right) \varepsilon^{\gamma}.\end{aligned}$$

Since this is true for arbitrary $\varepsilon > 0$,

$$st - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_{a_n}\setminus\triangle_c} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}} = 0.$$

Therefore

$$st - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_{a_n}} \frac{|(BC)_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\gamma}} = 0.$$

□

Here we are interested to construct a sequence of positive linear operators $L_n : C_\rho(\mathbb{R}_2{}^{++}) \to B_\rho(\mathbb{R}_2{}^{++})$ satisfying the conditions:-

$$st - \lim_{n\to\infty} \|L_n(f_{k,m}) - f_{k,m}\|_\rho = 0 \tag{3.1}$$

where $f_{0,0} = 1$; $f_{1,0} = \xi$; $f_{0,1} = \eta$; $f_{2,0} = \xi^2$; $f_{0,2} = \eta^2$, and give a counterexample of a function $f^* \in C_\rho(\mathbb{R}_2{}^{++})$ such that

$$st - \lim_{n\to\infty} \sup \|L_n(f) - f\|_\rho \geqslant 1.$$

For this, consider a sequence of positive linear operators $L_n : C_\rho(\mathbb{R}_2{}^{++}) \to B_\rho(\mathbb{R}_2{}^{++})$

defined by

$$L_n(f;x,y)=\begin{cases} f(x,y)+n(1+x^2+y^2)[f(x+1,y+1)-f(x,y)] & \text{when } (x,y)\in\Delta_n,\ n=k^2\\ f(x,y) & \text{when } (x,y)\in\mathbb{R}^{++}\setminus\Delta_n,\ n=k^2\\ f(x,y) & \text{otherwise}\end{cases}$$

where $k=1,2,3....$ So

$$L_n(f_{0,0};x,y)-f_{0,0}(x,y)=0 \text{ for all } (x,y)\in\mathbb{R}^{++};$$

$$\begin{aligned} L_n(f_{1,0};x,y)-f_{1,0}(x,y) &= n(1+x^2+y^2) \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0 \text{ when } (x,y)\in\mathbb{R}^{++}\setminus\Delta_n,\ n=k^2\\ &= 0 \text{ otherwise},\quad (k=1,2,3...);\end{aligned}$$

$$\begin{aligned} L_n(f_{0,1};x,y)-f_{0,1}(x,y) &= n(1+x^2+y^2) \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0 \text{ when } (x,y)\in\mathbb{R}^{++}\setminus\Delta_n,\ n=k^2\\ &= 0 \text{ otherwise},\quad (k=1,2,3...);\end{aligned}$$

$$\begin{aligned} L_n(f_{2,0};x,y)-f_{2,0}(x,y) &= n(1+x^2+y^2)(2x+1) \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0 \text{ when } (x,y)\in\mathbb{R}^{++}\setminus\Delta_n,\ n=k^2\\ &= 0 \text{ otherwise},\quad (k=1,2,3...);\end{aligned}$$

$$\begin{aligned} L_n(f_{0,2};x,y)-f_{0,2}(x,y) &= n(1+x^2+y^2)(2y+1) \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0 \text{ when } (x,y)\in\mathbb{R}^{++}\setminus\Delta_n,\ n=k^2\\ &= 0 \text{ otherwise},\quad (k=1,2,3...);\end{aligned}$$

Thus

$$st-\lim_{n\to\infty}\|L_n(f_{k,m})-f_{k,m}\|_\rho=0 \text{ for } (k,m)\in\{(0,0),(1,0),(0,1),(2,0),(0,2)\}.$$

Consider the function $f^*(x,y)=(x^2+y^2)\cos\pi(x+y)$. Then

$$\begin{aligned} L_n(f^*;x,y) &= f^*(x,y)+n(1+x^2+y^2)[\{(x+1)^2+(y+1)^2\}\cos\pi(x+y+2)\\ &\quad -(x^2+y^2)\cos\pi(x+y)] \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= f^*(x,y),\ \text{otherwise}\end{aligned}$$

i.e.,

$$\begin{aligned} \frac{L_n(f^*;x,y)-f^*(x,y)}{1+x^2+y^2} &= n[\{(x+1)^2+(y+1)^2\}\cos\pi(x+y+2)\\ &\quad -(x^2+y^2)\cos\pi(x+y)] \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0,\ \text{otherwise}\end{aligned}$$

which implies

$$\begin{aligned}\left|\frac{L_n(f^*;x,y)-f^*(x,y)}{1+x^2+y^2}\right| &= n|\cos\pi(x+y)|[\{(x+1)^2+(y+1)^2\}\\ &\quad -(x^2+y^2)] \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0,\ \text{otherwise.}\end{aligned}$$

Hence

$$\begin{aligned}\|L_n(f^*)-f^*\| &\geqslant n|\cos\pi n|[\left\{\left(\frac{n}{2}+1\right)^2+\left(\frac{n}{2}+1\right)^2\right\}\\ &\quad -\left\{\left(\frac{n}{2}\right)^2+\left(\frac{n}{2}\right)^2\right\}] \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0,\ \text{otherwise}\end{aligned}$$

which implies

$$\begin{aligned}\|L_n(f^*)-f^*\| &\geqslant n(2n+2) \text{ when } (x,y)\in\Delta_n,\ n=k^2\\ &= 0,\ \text{otherwise.}\end{aligned}$$

Hence

$$st-\limsup_{n\to\infty}\|L_n(f)-f\|_\rho \geqslant 1.$$

4. Ideal approximation

Kostyrko et al. [15] (2001) extended the concept of statistical convergence to the concept of ideal convergence in terms of a class of subsets of $P(\mathbb{N})$, namely ideal. If (i) $A,B\in\mathcal{I}$ implies $A\cup B\in\mathcal{I}$ and (ii) $A\in\mathcal{I}, B\subset A$ implies $B\in\mathcal{I}$, then a family $\mathcal{I}\subset P(\mathbb{N})$ is considered an ideal in $\mathbb{N}$. If $\{x\}\in\mathcal{I}$ for each $x\in\mathbb{N}$, then an ideal $\mathcal{I}$ of $\mathbb{N}$ is considered admissible. If $\mathbb{N}\notin\mathcal{I}, \mathcal{I}\neq\{\emptyset\}$, then $\mathcal{I}$ is recognized as a non-trivial proper ideal in $\mathbb{N}$.

Here, we revisit the familiar concepts of $\mathcal{I}$-boundedness and $\mathcal{I}$-convergence.

[16] A real numbers sequence $\{y_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-bounded if there is a number $K>0$ such that $\{k\in\mathbb{N}:|y_k|>K\}\in\mathcal{I}$. [15] The real numbers sequence $\{y_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-convergent to L provided that for every $\varepsilon>0$, the set $\{k\in\mathbb{N}:|y_k-L|\geqslant\varepsilon\}\in\mathcal{I}$. In this case, we write $\mathcal{I}-\lim_{k\to\infty} y_k=L$. It's interesting to note that any real number sequences that are $\mathcal{I}$-convergent are $\mathcal{I}$-bounded. In [11], Gezer et. al. introduced the notion of $\mathcal{I}$-uniform convergence. In this section, we try to investigate the results of previous sections in terms of ideal convergence. Let for any $a>0$ we denote by $\triangle_a$, the triangular domain

$$\triangle_a=\{(x,y):x\geqslant 0,y\geqslant 0,x+y\leqslant a\}$$

and (b_n) is the sequence of positive numbers such that the sequence (b_n) is divergent to ∞ and

$$\mathcal{I}-\lim_{n\to\infty}\frac{b_n}{n}=0. \tag{4.1}$$

Throughout the section $\mathcal{I}$ will denote the non-trivial admissible ideal on $\mathbb{N}$. The following Korovkin type approximation theorem is the ideal version of the Theorem 2. The proof of this theorem is obvious from the proof of the Theorem 2. If a sequence of positive linear operator $L_n : C_\rho(\mathbb{R}_2{}^{++}) \to B_\rho(\mathbb{R}_2{}^{++})$ fulfils five conditions

$$\mathcal{I} - \lim_{n\to\infty} \|L_n(f_{k,m}) - f_{k,m}\|_\rho = 0 \tag{4.2}$$

where $f_{0,0} = 1;\ f_{1,0} = \xi;\ f_{0,1} = \eta;\ f_{2,0} = \xi^2;\ f_{0,2} = \eta^2$, then for a fixed positive number a and for any function $f \in C_\rho(\mathbb{R}_2{}^{++})$ we have,

$$\mathcal{I} - \lim_{n\to\infty} \|L_n(f) - f\|_\rho = 0$$

where the norm $\|f\|_\rho$ is restricted on $\triangle_a$.

For any fixed positive number a, the relation

$$\mathcal{I} - \lim_{n\to\infty} \max_{(x,y)\in\triangle_a} |\,(BC)_n(f;x,y) - f(x,y)\,| = 0$$

holds for all functions f which are continuous in $x \geqslant 0, y \geqslant 0$ and satisfy the condition

$$|\,f(x,y)\,| \leqslant M_f(1 + x^2 + y^2)$$

where M_f is a constant depending on the function f only.

Proof. The proof of this theorem readily follows from the Theorem 4. As $f(x,y)$ satisfies the conditions (3) and then (7) follows that $(BC)_n(f;x,y)$ $\mathcal{I}$-uniformly convergent to $f(x,y)$. □

The polynomials (2) are not able to approximate the infinitely differentiable function $f^*(x,y) = x^2 + y^2$ on the entire triangular region $\triangle_{b_n}$ since,

$$\sup_{(x,y)\in\triangle_{b_n}} |\,(BC)_n(f^*;x,y) - f^*(x,y)\,| = \frac{b_n^2}{2n}$$

and the right hand side, in general, will not be $\mathcal{I}$-convergent by (1).

Let $\mathcal{I}$ be a non-trivial admissible ideal and $A \in \mathcal{I}$. Let (b_n) be a sequence of positive real numbers defined as

$$b_n = \begin{cases} n & n \in A \\ n^{\frac{3}{4}} & n \notin A. \end{cases}$$

Hence (b_n) is a divergent to ∞ and $\mathcal{I} - \lim_{n\to\infty} \frac{b_n}{n} = 0$ but $\frac{b_n^2}{n}$ is not $\mathcal{I}$-convergent. Next, we prove the ideal version of the Theorem 3. The relation

$$\mathcal{I} - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_{b_n}} \frac{|\,(BC)_n(f;x,y) - f(x,y)\,|}{(1 + x^2 + y^2)^{1+\gamma}} = 0$$

holds for any continuous function f satisfying (3) and for any positive γ.

Proof. For a given $\varepsilon > 0$ there exists a large $a > 0$ such that $\frac{1}{1+x^2+y^2} < \varepsilon$ for $x + y > a$.

Since, by definition the sequence $(\frac{b_n}{n})$ is $\mathcal{I}$-convergent and hence $\mathcal{I}$-bounded. Then there exists a $C > 0$ such that $\{n \in \mathbb{N} : \mid \frac{b_n}{n} \mid > C\} \in \mathcal{I}$. Now

$$\begin{aligned}
\mid f(x,y) \mid &\leqslant M_f(1+x^2+y^2) \\
\Rightarrow \mid (BC)_n(f;x,y) \mid &\leqslant M_f (BC)_n \left(1+x^2+y^2\right) \\
\Rightarrow \mid (BC)_n(f;x,y) \mid &\leqslant M_f \left(1+x^2+y^2+\frac{x(b_n-x)}{n}+\frac{y(b_n-y)}{n}\right) \\
&= M_f(1+x^2+y^2)\left(1+\frac{x(b_n-x)}{n(1+x^2+y^2)}+\frac{y(b_n-y)}{n(1+x^2+y^2)}\right) \\
&\leqslant M_f(1+x^2+y^2)\left(1+\frac{x(b_n-x)}{n(1+x^2)}+\frac{y(b_n-y)}{n(1+y^2)}\right) \\
&\leqslant M_f(1+x^2+y^2)\left(1+\frac{(b_n-x)}{2n}+\frac{(b_n-y)}{2n}\right) \\
&\leqslant M_f(1+x^2+y^2)\left(1+\frac{b_n}{n}\right)
\end{aligned}$$

Hence $\{n \in \mathbb{N} : \mid (BC)_n(f;x,y) \mid > M_f(1+x^2+y^2)(1+C)\} \subseteq \{n \in \mathbb{N} : 1+\frac{b_n}{n} > (1+C)\} \in \mathcal{I}$. Hence for all $x, y \geqslant 0$ and for any continuous function f satisfying (3), we get,

$$\{n \in \mathbb{N} : \mid (BC)_n(f;x,y) \mid > M'_f(1+x^2+y^2)\} = K(\text{say}) \in \mathcal{I}$$

where, $M'_f = M_f(1+C)$ is independent of n. Now, for any continuous function f, satisfying (3) and any $\gamma > 0$

$$\begin{aligned}
\sup_{(x,y)\in\triangle_{b_n}} \frac{\mid (BC)_n(f;x,y)-f(x,y) \mid}{(1+x^2+y^2)^{1+\gamma}} &\leqslant \sup_{(x,y)\in\triangle_a} \frac{\mid (BC)_n(f;x,y)-f(x,y) \mid}{(1+x^2+y^2)^{1+\gamma}} \\
&\quad + \sup_{(x,y)\in\triangle_{b_n}\setminus\triangle_a} \frac{\mid (BC)_n(f;x,y)-f(x,y) \mid}{(1+x^2+y^2)^{1+\gamma}} \\
&\leqslant \sup_{(x,y)\in\triangle_a} \mid (BC)_n(f;x,y)-f(x,y) \mid \\
&\quad + \sup_{(x,y)\in\triangle_{b_n}\setminus\triangle_a} \frac{\mid (BC)_n(f;x,y)-f(x,y) \mid}{(1+x^2+y^2)^{1+\gamma}}
\end{aligned}$$

Hence by Theorem 4

$$\mathcal{I}-\lim_{n\to\infty} \sup_{(x,y)\in\triangle_a} \mid (BC)_n(f;x,y)-f(x,y) \mid = 0$$

Again

$$\begin{aligned}
\sup_{(x,y)\in\triangle_{b_n}\setminus\triangle_a} \frac{\mid (BC)_n(f;x,y)-f(x,y) \mid}{(1+x^2+y^2)^{1+\gamma}} &\leqslant \left(M'_f+M_f\right) \sup_{(x,y)\in\triangle_{b_n}\setminus\triangle_a} \frac{1}{(1+x^2+y^2)^{\gamma}} \\
&\leqslant \left(M'_f+M_f\right)\varepsilon^{\gamma} \text{ for all } n \in \mathbb{N}\setminus K \in \mathcal{I}.
\end{aligned}$$

Since this is true for arbitrary $\varepsilon > 0$,

$$\mathcal{I} - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_{b_n}\setminus\triangle_a} \frac{|\,(BC)_n(f;x,y) - f(x,y)\,|}{(1+x^2+y^2)^{1+\gamma}} = 0.$$

Therefore

$$\mathcal{I} - \lim_{n\to\infty} \sup_{(x,y)\in\triangle_{b_n}} \frac{|\,(BC)_n(f;x,y) - f(x,y)\,|}{(1+x^2+y^2)^{1+\gamma}} = 0.$$

□

Open Problem: We leave an open problem whether there exists a function f^* in $C_\rho(\mathbb{R}_2{}^{++})$ such that

$$\mathcal{I} - \limsup_{n\to\infty} \|L_n(f) - f\|_\rho \geqslant 1$$

for any sequence of positive linear operators $L_n : C_\rho(\mathbb{R}_2{}^{++}) \to B_\rho(\mathbb{R}_2{}^{++})$ satisfying the conditions:-

$$\mathcal{I} - \lim_{n\to\infty} \|L_n(f_{k,m}) - f_{k,m}\|_\rho = 0 \tag{4.3}$$

where $f_{k,m} = u^k v^m$.

5. Conclusion

This paper contributes on statistical approximation for continuous functions, satisfying (3), of two variables by means of Bernstein-Chlodowsky polynomials on a triangular domain. Additionally, in section 3, the weighted approximation theorem for continuous functions of two variables, satisfying (3), on a triangular domain (Theorem 3) is investigated in the sense of statistical convergence. The Remark 3 witnesses the existence of a positive linear operator, which justify that Theorem 3 does not hold in general. The ideal analogous results of sections 2 and 3 are witnessed in section 4. Remarkably, these outcomes extend the conclusions of Ibikli's study [14] in both statistical and ideal contexts.

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References

[1] Altomare F. and Campiti M., *Korovkin-type approximation theory and its applications*, Walter de Gruyter, Berlin-New York, 1994.

[2] Aslan R. and Izgi A., *Some approximation results by bivariate Bernstein-Kantrovich type operators on a triangular domain*, Kyungpook Mathematical Journal, 62 (3) (2022), 467-484, https://doi.org/10.5666/KMJ.2022.62.3.467

[3] Cai Q. B., Khan A., Mansoori M. S., Iliyas M., Khan K., *Approximation by λ-Bernstein type operators on triangular domain*, Filomat, 37(6) (2023), 1941-1958, https://doi.org/10.2298/FIL2306941C

[4] Cakalli H. and Khan M. K., *Summability in topological spaces*. Appl. Math. Lett., 24 (3) (2011), 348-352.

[5] Duman O., Orhan C., *μ-statistically convergent function sequences*, Czechoslov. Math. J., 54 (129)(2) (2004), 413-422, https://dml.cz/handle/10338.dmlcz/127899

[6] Fast H., *Sur la convergence Statistique*, Colloq. Math., 2(1951), 241-244.

[7] Fridy J.A., *On statistical convergence*, Analysis 5(4)(1985), 301-313, https://doi.org/10.1524/anly.1985.5.4.301
[8] Fridy J. A., Orhan C. *Statistical limit superior and limit inferior*, Proc. Am. Math. Soc., 125 (12) (1997), 3625-3631.
[9] Gadjiev A. D., Efendiev R.O. and Ibikli E., *Generalized Bernstein-Chlodowsky polynomials*, Rocky Mountain J. Math., 28 (4)(1998), 10.1216/rmjm/1181071716.
[10] Gadjieva E. A. and Ibikli E., *On generalization of Bernstein-Chlodowsky polynomials*, Hacettepe Bull. Natur. Sci. Engrg., 24 (1995), 31-40.
[11] Gezer F., Karakuş S., I *and* I*-convergent function sequences*, Mathematical Communications, 10 (2005), 71-80, https://hrcak.srce.hr/file/1336
[12] Gökhan A., Güngör, M., *On pointwise statistical convergence*, Indian J. Pure Appl. Math., 33(9)(2002), 1379-1384.
[13] Güngör M., Gökhan A., *On uniform statistical convergence*, Int. J. Pure Appl. Math., 19(1)(2005), 17-24.
[14] Ibikli E., *On approximation for functions of two variables on a triangular domain*, Rocky Mountain J. Math., 35(5)(2005) 10.1216/rmjm/1181069649
[15] Kostyrko P., Šalát T. and Wilczyński W., *$\mathcal{I}$-convergence*, Real Anal. Exchange, **26** (2) (2000/2001) 669-685.
[16] Kostyrko P., Mačaj M., Šalát T. and Sleziak M., *$\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points*, Math. Slovaca, 55 (4) (2005) 443-464.
[17] Maio G. Di., Kočinac Lj. D. R., *Statistical convergence in topology*, Topology Appl., 156, (2008), 28-45, 10.1016/j.topol.2008.01.015
[18] Maddox I. J., *Statistical convergence in a locally convex spaces*, Math. Proc. Cambridge Philos. Soc., 104(1)(1988), 141-145, https://doi.org/10.1017/S0305004100065312
[19] Miller H. I., *A measure theoretical subsequence characterization of statistical convergence*. Trans. Amer. Math. Soc., 347 (5) (1995), 1811-1819, 10.1090/S0002-9947-1995-1260176-6
[20] Mursaleen M., Al-Abied A. A. H. and Salman M. A., *Approximation by Stancu-Chlodowsky type -Bernstein operators, Journal of Applied Analysis*, Azerbaijan Journal of Mathematics, 26 (1) (2021) 97-110, https://www.azjm.org/volumes/1001/pdf/1001-5.pdf
[21] Prullage D. L., *Summability in topological groups*, Math. Z., 96(1967), 259-279, 10.1007/bf01123653
[22] Šalát T. , *On statistical convergence of real numbers*, Math. Slovaca, 30(1980), 139-150.
[23] Schoenberg I. J., *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66 (1959) 361-375.
[24] Zhou G., Cai Q. B., *Bivariate -Bernstein operators on triangular domain*, AIMS Mathematics, 9(6) (2024), 14405-14424, 10.3934/math.2024700
[25] Zigmund A., *cyr Trigonometricheskie ryady. Tomy I, II. (Russian) [Trigonometric series] Izdat*. “Mir”, Moscow 1965 Vol. I: 615 pp.; Vol. II: 537 pp. MR0178296.