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A three-parameter logarithmic generalization of the Hilbert integral inequality

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Abstract

In this paper, we establish a new generalized form of the Hilbert integral inequality involving three adjustable parameters and a logarithmic term. We prove that the resulting constant factor is sharp and best possible. Furthermore, we derive related forms of Hilbert-type integral inequalities, including those based on sharp lower bounds of the logarithmic function.

Keywords: Hilbert integral inequality, Cauchy-Schwarz integral inequality, logarithmic inequality.
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1. Introduction

The classical Hilbert integral inequality is one of the most important results in mathematical analysis. It is mainly used to bound integrals involving products of functions. It has applications in diverse areas, including harmonic analysis and operator theory, where it is used to establish bounds and convergence criteria. The formal statement of the Hilbert integral inequality is as follows: Let $p : [0,+\infty) \mapsto [0,+\infty)$ and $q : [0,+\infty) \mapsto [0,+\infty)$ be two functions such that $0 < \int_0^{+\infty} [p(x)]^2 dx < +\infty$ and $0 < \int_0^{+\infty} [q(y)]^2 dy < +\infty$. Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{p(x)q(y)}{x+y} dx dy < \pi \left\{\int_0^{+\infty} [p(x)]^2 dx\right\}^{1/2} \left\{\int_0^{+\infty} [q(y)]^2 dy\right\}^{1/2}. \tag{1.1}$$

Since its discovery, the Hilbert integral inequality has inspired extensive research. For detailed studies, see [5], [9], [8], [10], [11], and [1]. It has also been generalized in various ways. Among these generalizations, some have introduced adjustable parameters. Focusing on the upper bound with $\left\{\int_0^{+\infty} [p(x)]^2 dx\right\}^{1/2} \left\{\int_0^{+\infty} [q(y)]^2 dy\right\}^{1/2}$, we can highlight the following result from [7]:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{p(x)q(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin[\pi/(2\lambda)]} \left\{\int_0^{+\infty} [p(x)]^2 dx\right\}^{1/2} \left\{\int_0^{+\infty} [q(y)]^2 dy\right\}^{1/2},$$

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*Corresponding author: christophe.chesneau@gmail.com for any $\lambda \in (1/2, 1]$. The classical Hilbert integral inequality is thus obtained by taking $\lambda = 1$.

Another notable one-parameter generalization is proposed in [12, Example 2.1.9]. It is formulated as follows:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\left\{\arctan[\sqrt{x/y}]\right\}^{\alpha}}{x+y} p(x)q(y)dxdy < \frac{2}{\alpha+1}\left(\frac{\pi}{2}\right)^{\alpha+1}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},$$

for any $\alpha > -1$. The classical Hilbert integral inequality is thus recovered by taking $\alpha = 0$.

A comprehensive state of the art can be found in [2] and [12], together with the references therein. The classical and generalized Hilbert integral inequalities are fundamental in various fields such as harmonic analysis, probability theory, partial differential equations and mathematical physics because of their versatility and wide range of applications.

In this paper, we investigate a new generalization of the Hilbert integral inequality that has the feature to introduce three adjustable parameters, and a logarithmic term. More precisely, we discuss the best possible constant κ such that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\beta+\mu x/y)}{\gamma x+y} p(x)q(y)dxdy \leqslant \kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},$$

where β, γ and μ denote the adjustable parameters. Of course, κ logically depends on these parameters. To the best of our knowledge, this is the first attempt to establish such a three-parameter logarithmic type of Hilbert integral inequality. We then investigate the best possible constant ξ such that

$$\int_0^{+\infty}\left[\int_0^{+\infty} \frac{\log(\beta+\mu x/y)}{\gamma x+y} p(x)dx\right]^2 dy \leqslant \xi\int_0^{+\infty}[p(x)]^2dx.$$

Again, ξ logically depends on these parameters, but it must be obtained in the sharpest possible way. In addition, thanks to well known lower bounds of the logarithmic functions, other Hilbert integral inequalities are generated.

The statements of the main results and the detailed proofs, are presented in Section 2. The complementary results are developed in Section 3. A conclusion is given in Section 4.

2. Main results

The theorem below presents the new Hilbert integral inequality of the logarithmic type with three parameters.

Theorem 2.1. *Let* $\beta \geqslant 1$, $\mu \geqslant 0$ *and* $\gamma > 0$. *Let* $p : [0,+\infty) \mapsto [0,+\infty)$ *and* $q : [0,+\infty) \mapsto [0,+\infty)$ *such that* $0 < \int_0^{+\infty}[p(x)]^2dx < +\infty$ *and* $0 < \int_0^{+\infty}[q(y)]^2dy < +\infty$. *Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\beta+\mu x/y)}{\gamma x+y} p(x)q(y)dxdy \leqslant \kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}, \tag{2.1}$$

where

$$\kappa = \frac{2\pi}{\gamma^{1/2}} \log \left[\left(\frac{\mu}{\gamma} \right)^{1/2} + \beta^{1/2} \right]. \tag{2.2}$$

Furthermore, the inequality is a strict one, and the constant κ *is the best possible one.*

Proof. Using a suitable decomposition of the main integrated term (taking into account that $\beta \geqslant 1$ and $\mu \geqslant 0$ make the logarithmic term positive and introducing the basic relation $1 = (x/y)^{1/4}(y/x)^{1/4}$), the Cauchy-Schwarz integral inequality and the Fubini-Tonelli integral theorem, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} p(x) q(y) dx dy \\
&= \int_0^{+\infty} \int_0^{+\infty} \left\{ \left[\frac{\log(\beta + \mu x/y)}{\gamma x + y} \right]^{1/2} \left(\frac{x}{y} \right)^{1/4} p(x) \right\} \\
&\times \left\{ \left[\frac{\log(\beta + \mu x/y)}{\gamma x + y} \right]^{1/2} \left(\frac{y}{x} \right)^{1/4} q(y) \right\} dx dy \\
&\leqslant \left\{ \int_0^{+\infty} \left[\int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} \left(\frac{x}{y} \right)^{1/2} dy \right] [p(x)]^2 dx \right\}^{1/2} \\
&\times \left\{ \int_0^{+\infty} \left[\int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} \left(\frac{y}{x} \right)^{1/2} dx \right] [q(y)]^2 dy \right\}^{1/2} \\
&= \left\{ \int_0^{+\infty} \Xi(x) [p(x)]^2 dx \right\}^{1/2} \left\{ \int_0^{+\infty} \Omega(y) [q(y)]^2 dy \right\}^{1/2},
\end{aligned} \tag{2.3}$$

where

$$\Xi(x) = \int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} \left(\frac{x}{y} \right)^{1/2} dy$$

and

$$\Omega(y) = \int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} \left(\frac{y}{x} \right)^{1/2} dx.$$

The calculation of $\Xi(x)$ and $\Omega(y)$ determines the expression of κ. To do this, we need the lemma below about a referenced integral result.

Lemma 2.2. *For any* $a \geqslant 0$ *and* $b \geqslant 0$ *such that* $a + b > 0$, *and* $c > 0$, *we have*

$$\int_0^{+\infty} \frac{\log(a + bx^2)}{c + x^2} dx = \frac{\pi}{c^{1/2}} \log \left[(bc)^{1/2} + a^{1/2} \right].$$

See [4, Item 4.295.1].

Applying the changes of variables $y = xu$ and $u = v^2$, the property of the logarithmic function, and Lemma 2.2 two times, with two different parameter configurations, i.e.,

$(a = 0, b = 1, c = \gamma)$ and $(a = \mu, b = \beta, c = \gamma)$, we get

$$\begin{aligned}\Xi(x) &= \int_0^{+\infty} \frac{\log(\beta + \mu/u)}{\gamma x + xu} \left(\frac{1}{u}\right)^{1/2} x du = \int_0^{+\infty} \frac{\log(\beta + \mu/u)}{u^{1/2}(\gamma + u)} du \\ &= 2\int_0^{+\infty} \frac{\log(\beta + \mu/v^2)}{\gamma + v^2} du = 2\int_0^{+\infty} \frac{\log\left[(\mu + \beta v^2)/v^2\right]}{\gamma + v^2} du \\ &= -2\int_0^{+\infty} \frac{\log(0 + 1 \times v^2)}{\gamma + v^2} du + 2\int_0^{+\infty} \frac{\log(\mu + \beta v^2)}{\gamma + v^2} du \\ &= -2\frac{\pi}{\gamma^{1/2}} \log\left[(1 \times \gamma)^{1/2} + 0^{1/2}\right] + 2\frac{\pi}{\gamma^{1/2}} \log\left[(\beta\gamma)^{1/2} + \mu^{1/2}\right] \\ &= \frac{2\pi}{\gamma^{1/2}} \log\left[\left(\frac{\mu}{\gamma}\right)^{1/2} + \beta^{1/2}\right] = \kappa.\end{aligned}$$

For $\Omega(y)$, by changing the variables $x = yu$ and using similar techniques as developed above, we obtain

$$\begin{aligned}\Omega(y) &= \int_0^{+\infty} \frac{\log(\beta + \mu u)}{\gamma u y + y} \left(\frac{1}{u}\right)^{1/2} y du = \int_0^{+\infty} \frac{\log(\beta + \mu u)}{u^{1/2}(1 + \gamma u)} du \\ &= \frac{2}{\gamma}\int_0^{+\infty} \frac{\log(\beta + \mu v^2)}{1/\gamma + v^2} dv = \frac{2}{\gamma}\pi\gamma^{1/2} \log\left[\left(\frac{\mu}{\gamma}\right)^{1/2} + \beta^{1/2}\right] \\ &= \frac{2\pi}{\gamma^{1/2}} \log\left[\left(\frac{\mu}{\gamma}\right)^{1/2} + \beta^{1/2}\right] = \kappa. \end{aligned} \tag{2.4}$$

So we have established that $\Xi(x) = \Omega(y) = \kappa$. It follows from Equation (2.3) that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} p(x)q(y)dxdy \\ &\leqslant \left\{\int_0^{+\infty} \kappa[p(x)]^2 dx\right\}^{1/2} \left\{\int_0^{+\infty} \kappa[q(y)]^2 dy\right\}^{1/2} \\ &= \kappa\left\{\int_0^{+\infty} [p(x)]^2 dx\right\}^{1/2} \left\{\int_0^{+\infty} [q(y)]^2 dy\right\}^{1/2}.\end{aligned}$$

Let us now demonstrate that the inequality is a strict one, and the constant κ is the best possible one.

The inequality in Equation (2.1) is an equality if and only if, for almost all $x > 0$ and $y > 0$, there is $\sigma > 0$ and $\omega > 0$ such that

$$\sigma\frac{\log(\beta + \mu x/y)}{\gamma x + y}\left(\frac{x}{y}\right)^{1/2} [p(x)]^2 = \omega\frac{\log(\beta + \mu x/y)}{\gamma x + y}\left(\frac{y}{x}\right)^{1/2} [q(y)]^2,$$

which can be reduced to $\sigma x[p(x)]^2 = \omega y[q(y)]^2$. Since this equality holds for almost all $x > 0$ and $y > 0$, there is a constant $\epsilon > 0$ such that $\sigma x[p(x)]^2 = \omega y[q(y)]^2 = \epsilon$. This implies that

$$\int_0^{+\infty} [p(x)]^2 dx = \frac{\omega}{\sigma}\int_0^{+\infty} x^{-1} dx = +\infty.$$

This is absurd because of the initial assumption $\int_0^{+\infty}[p(x)]^2 dx < +\infty$. The inequality in Equation (2.1) is therefore strict.

Let us now discuss the constant κ. We will show that it is the best possible. Suppose that there exist a better constant, say $\phi \in (0,\kappa]$, such that, for any $p : [0,+\infty) \mapsto [0,+\infty)$ and $q : [0,+\infty) \mapsto [0,+\infty)$ satisfying $0 < \int_0^{+\infty}[p(x)]^2 dx < +\infty$ and $0 < \int_0^{+\infty}[q(y)]^2 dy < +\infty$, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\beta+\mu x/y)}{\gamma x+y} p(x)q(y)dxdy\\ &\leqslant \phi\left\{\int_0^{+\infty}[p(x)]^2 dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2 dy\right\}^{1/2}. \qquad (2.5)\end{aligned}$$

For any positive integer n, we consider $p_n : [0,+\infty) \mapsto [0,+\infty)$ such that $p_n(x) = 0$ for any $x \in (0,1)$, and $p_n(x) = x^{-(1+1/n)/2}$ for $x \in [1,+\infty)$, and $q_n : [0,+\infty) \mapsto [0,+\infty)$ such that $q_n(y) = p_n(y)$ for any $y \in (0,+\infty)$. Then we have

$$\int_0^{+\infty}[p_n(x)]^2 dx = \int_0^{+\infty}[q_n(y)]^2 dy = \int_1^{+\infty} y^{-(1+1/n)} dy = n.$$

It follows from Equation (2.5) that

$$\begin{aligned}\phi &= \frac{1}{n}\phi\left\{\int_0^{+\infty}[p_n(x)]^2 dx\right\}^{1/2}\left\{\int_0^{+\infty}[q_n(y)]^2 dy\right\}^{1/2}\\ &\geqslant \frac{1}{n}\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\beta+\mu x/y)}{\gamma x+y} p_n(x)q_n(y)dxdy. \qquad (2.6)\end{aligned}$$

Let us now examine this last integral term. Using the definition of $p_n(x)$ and $q_n(y)$, the change of variables $x = uy$, the Fubini-Tonelli theorem and the Chasles integral relation, we get

$$\begin{aligned}
&\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p_n(x)q_n(y)dxdy\\
&=\int_{1}^{+\infty}\left[\int_{1}^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}x^{-(1+1/n)/2}dx\right]y^{-(1+1/n)/2}dy\\
&=\int_{1}^{+\infty}\left[\int_{1/y}^{+\infty}\frac{\log(\beta+\mu u)}{y(1+\gamma u)}u^{-(1+1/n)/2}y^{-(1+1/n)/2}y\,du\right]y^{-(1+1/n)/2}dy\\
&=\int_{1}^{+\infty}\left[\int_{1/y}^{+\infty}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\right]y^{-(1+1/n)}dy\\
&=\int_{1}^{+\infty}\left[\int_{1/y}^{1}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\right]y^{-(1+1/n)}dy\\
&+\int_{1}^{+\infty}\left[\int_{1}^{+\infty}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\right]y^{-(1+1/n)}dy\\
&=\int_{0}^{1}\left[\int_{1/u}^{+\infty}y^{-(1+1/n)}dy\right]\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\\
&+\left[\int_{1}^{+\infty}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\right]\left[\int_{1}^{+\infty}y^{-(1+1/n)}dy\right]\\
&=\int_{0}^{1}[nu^{1/n}]\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du+n\left[\int_{1}^{+\infty}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\right]\\
&=n\left\{\int_{0}^{1}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1-1/n)/2}du+\int_{1}^{+\infty}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du\right\}. \qquad (2.7)
\end{aligned}$$

Equations (2.6) and (2.7) give

$$\phi\geqslant\int_{0}^{1}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1-1/n)/2}du+\int_{1}^{+\infty}\frac{\log(\beta+\mu u)}{1+\gamma u}u^{-(1+1/n)/2}du.$$

Applying the Fatou lemma, which is possible because the integrated functions are positive, noting that the considered inferior limits are equal to 1 and using a part of Equation (2.4), we obtain

$$\begin{aligned}
\phi &\geqslant \lim_{n\to+\infty}\inf \int_0^1 \frac{\log(\beta+\mu u)}{1+\gamma u} u^{-(1-1/n)/2} du \\
&+ \lim_{n\to+\infty}\inf \int_1^{+\infty} \frac{\log(\beta+\mu u)}{1+\gamma u} u^{-(1/n+1)/2} du \\
&\geqslant \int_0^1 \frac{\log(\beta+\mu u)}{1+\gamma u} u^{-1/2} \left[\lim_{n\to+\infty}\inf u^{1/(2n)}\right] du \\
&+ \int_1^{+\infty} \frac{\log(\beta+\mu u)}{1+\gamma u} u^{-1/2} \left[\lim_{n\to+\infty}\inf u^{-1/(2n)}\right] du \\
&= \int_0^1 \frac{\log(\beta+\mu u)}{u^{-1/2}(1+\gamma u)} du + \int_1^{+\infty} \frac{\log(\beta+\mu u)}{u^{-1/2}(1+\gamma u)} du \\
&= \int_0^{+\infty} \frac{\log(\beta+\mu u)}{u^{1/2}(1+\gamma u)} du \\
&= \kappa.
\end{aligned}$$

Since $\phi \in (0,\kappa]$, we necessarily have $\phi = \kappa$, proving that it is the best possible constant. The theorem is demonstrated. □

Let us now discuss a reformulation of Theorem 2.1. Since $\log(\beta+\mu x/y) = \log[(x/y)(\mu+\beta y/x)] = \log(x/y) + \log(\mu+\beta y/x)$, under the integral convergence assumptions, Equation (2.1) implies that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{\log(\mu+\beta y/x)}{\gamma x+y} p(x)q(y)dxdy + \int_0^{+\infty}\int_0^{+\infty} \frac{\log(x/y)}{\gamma x+y} p(x)q(y)dxdy \\
&\leqslant \kappa \left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2} \left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},
\end{aligned}$$

where κ is given in Equation (2.2). The interest of this form is that, for any $x > 0$ and $y > 0$, neither $\log(\mu+\beta y/x)$ nor $\log(x/y)$ is of constant sign, so the left term is of some complexity to handle for finding a sharp upper bound.

In addition, from Theorem 2.1, the following inequalities can be derived:

- Taking $\beta = e$, $\gamma = 1$ and $\mu = 0$, we have $\kappa = 2\pi\log(e^{1/2}) = \pi$, and the inequality in Equation (2.1) becomes the classical Hilbert integral inequality.
- Taking $\beta = 1$, $\gamma = 1$ and $\mu = 1$, the inequality in Equation (2.1) becomes

  $$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(1+x/y)}{x+y} p(x)q(y)dxdy \leqslant \kappa \left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2} \left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},$$

  where $\kappa = 2\pi\log(2)$. As far as we know, this is a new Hilbert integral inequality in the literature.

- Taking $\mu = \beta$ and $\gamma = \beta$, since $\log(\beta + \beta x/y) = \log(\beta) + \log(1 + x/y)$, the inequality in Equation (2.1) becomes

$$\log(\beta) \int_0^{+\infty} \int_0^{+\infty} \frac{1}{\beta x + y} p(x)q(y)dxdy + \int_0^{+\infty} \int_0^{+\infty} \frac{\log(1 + x/y)}{\beta x + y} p(x)q(y)dxdy$$
$$\leqslant \kappa \left\{ \int_0^{+\infty} [p(x)]^2 dx \right\}^{1/2} \left\{ \int_0^{+\infty} [q(y)]^2 dy \right\}^{1/2},$$

where

$$\kappa = \frac{2\pi}{\beta^{1/2}} \log\left(1 + \beta^{1/2}\right).$$

Other examples can be derived by adjusting the parameters β, γ and μ.

In a way, Theorem 2.1 is completed with the finding below. The focus is on a bivariate integral that involves a logarithmic term and a single function $p(x)$.

Theorem 2.3. *Let* $\beta \geqslant 1$, $\mu \geqslant 0$ *and* $\gamma > 0$. *Let* $p : [0, +\infty) \mapsto [0, +\infty)$ *such that* $0 < \int_0^{+\infty} [p(x)]^2 dx < +\infty$. *Then we have*

$$\int_0^{+\infty} \left[\int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} p(x)dx \right]^2 dy \leqslant \kappa^2 \int_0^{+\infty} [p(x)]^2 dx,$$

where κ *is given by Equation* (2.2). *In addition, this inequality implies Equation* (2.1).

Proof. We introduce the following intermediary function $r : [0, +\infty) \mapsto [0, +\infty)$ defined by

$$r(y) = \int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} p(x)dx.$$

It follows from Theorem 2.1 applied to $q(y) = r(y)$ that

$$\int_0^{+\infty} [r(y)]^2 dy = \int_0^{+\infty} \left[\int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} p(x)dx \right] r(y)dy$$
$$= \int_0^{+\infty} \int_0^{+\infty} \frac{\log(\beta + \mu x/y)}{\gamma x + y} p(x)r(y)dxdy$$
$$\leqslant \kappa \left\{ \int_0^{+\infty} [p(x)]^2 dx \right\}^{1/2} \left\{ \int_0^{+\infty} [r(y)]^2 dy \right\}^{1/2}.$$

This implies that

$$\left\{ \int_0^{+\infty} [r(y)]^2 dy \right\}^{1/2} \leqslant \kappa \left\{ \int_0^{+\infty} [p(x)]^2 dx \right\}^{1/2},$$

and

$$\int_0^{+\infty} [r(y)]^2 dy \leqslant \kappa^2 \int_0^{+\infty} [p(x)]^2 dx,$$

so that

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)dx\right]^2 dy \leqslant \kappa^2\int_0^{+\infty}[p(x)]^2dx.$$

The claimed result is demonstrated.

Let us now prove that this inequality implies Equation (2.1). Suppose that

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)dx\right]^2 dy \leqslant \kappa^2\int_0^{+\infty}[p(x)]^2dx.$$

Let $q : [0,+\infty) \mapsto [0,+\infty)$ such that $0 < \int_0^{+\infty}[q(y)]^2dy < +\infty$. Applying the Cauchy-Schwarz integral inequality and the above assumption, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)dxdy\\
&=\int_0^{+\infty}\left[\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)dx\right]q(y)dy\\
&\leqslant\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)dx\right]^2dy\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\\
&\leqslant\left\{\kappa^2\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\\
&=\kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}.
\end{aligned}$$

The proof is concluded with the establishment of the inequality in Equation (2.1). □

Specifying values for β, γ and μ, examples of new integral inequalities follow.

3. Complementary results

The proposition below can be seen as an extended version of Theorem 2.1, with the addition of a new parameter.

Proposition 3.1. *Let* $\beta \geqslant 1$, $\mu \geqslant 0$, $\gamma > 0$ *and* $\zeta \in [0,1]$. *Let* $p : [0,+\infty) \mapsto [0,+\infty)$ *and* $q : [0,+\infty) \mapsto [0,+\infty)$ *such that* $0 < \int_0^{+\infty}[p(x)]^2dx < +\infty$ *and* $0 < \int_0^{+\infty}[q(y)]^2dy < +\infty$. *Then we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{[\log(\beta+\mu x/y)]^{\zeta}}{\gamma x+y}p(x)q(y)dxdy\\
&\leqslant \kappa^{\zeta}\left[\frac{\pi}{\gamma^{1/2}}\right]^{1-\zeta}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},
\end{aligned}$$

where κ *is given by Equation* (2.2)*, so that*

$$\kappa^{\zeta}\left[\frac{\pi}{\gamma^{1/2}}\right]^{1-\zeta}=\frac{\pi}{\gamma^{1/2}}\left\{2\log\left[\left(\frac{\mu}{\gamma}\right)^{1/2}+\beta^{1/2}\right]\right\}^{\zeta}.$$

Proof. Let us consider the case $\zeta \in (0,1)$. Using the Hölder integral inequality with the parameter $1/\zeta$, Theorem 2.1 and the following variant of the classical Hilbert integral inequality:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{p(x)q(y)}{\gamma x+y}dxdy \leqslant \frac{\pi}{\gamma^{1/2}}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}, \tag{3.1}$$

we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{[\log(\beta+\mu x/y)]^{\zeta}}{\gamma x+y}p(x)q(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty}\left[\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)\right]^{\zeta}\left[\frac{p(x)q(y)}{\gamma x+y}\right]^{1-\zeta}dxdy \\
&\leqslant \left[\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)dxdy\right]^{\zeta}\left[\int_0^{+\infty}\int_0^{+\infty}\frac{p(x)q(y)}{\gamma x+y}dxdy\right]^{1-\zeta} \\
&\leqslant \left[\kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\right]^{\zeta} \times \\
&\left[\frac{\pi}{\gamma^{1/2}}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\right]^{1-\zeta} \\
&= \kappa^{\zeta}\left[\frac{\pi}{\gamma^{1/2}}\right]^{1-\zeta}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}.
\end{aligned}$$

For the cases $\zeta = 0$ and $\zeta = 1$, the desired inequalities correspond to those in Theorem 2.1 and Equation (3.1). The proposition is demonstrated. □

Taking $\zeta = 1$, Proposition 3.1 becomes Theorem 2.1. The other cases give an original statement. We can also mention that the constant obtained is the best possible.

The result below makes use of the properties of the logarithmic function for innovation in the context of the Hilbert integral inequality.

Proposition 3.2. *Let* $\beta \geqslant 1$, $\delta \geqslant 1$ $\mu \geqslant 0$, $\nu \geqslant 0$ *and* $\gamma > 0$. *Let* $p : [0,+\infty) \mapsto [0,+\infty)$ *and* $q : [0,+\infty) \mapsto [0,+\infty)$ *such that* $0 < \int_0^{+\infty}[p(x)]^2dx < +\infty$ *and* $0 < \int_0^{+\infty}[q(y)]^2dy < +\infty$. *Then we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{\log\left[\beta\delta+(\beta\nu+\delta\mu)x/y+\mu\nu x^2/y^2\right]}{\gamma x+y}p(x)q(y)dxdy \\
&\leqslant \kappa_{\star}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},
\end{aligned}$$

where

$$\kappa_{\star} = \frac{2\pi}{\gamma^{1/2}}\log\left\{\left[\left(\frac{\mu}{\gamma}\right)^{1/2}+\beta^{1/2}\right]\left[\left(\frac{\nu}{\gamma}\right)^{1/2}+\delta^{1/2}\right]\right\}.$$

Furthermore, the inequality is a strict one, and the constant $\kappa_{\star}$ *is the best possible one.*

Proof. Let us notice that

$$\beta\delta+(\beta\nu+\delta\mu)\frac{x}{y}+\mu\nu\frac{x^2}{y^2}=\left(\beta+\mu\frac{x}{y}\right)\left(\delta+\nu\frac{x}{y}\right).$$

Therefore, we have

$$\log\left[\beta\delta+(\beta\nu+\delta\mu)\frac{x}{y}+\mu\nu\frac{x^2}{y^2}\right]=\log\left(\beta+\mu\frac{x}{y}\right)+\log\left(\delta+\nu\frac{x}{y}\right).$$

It follows from Theorem 2.1, applied under two different configurations of parameters, that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{\log\left[\beta\delta+(\beta\nu+\delta\mu)x/y+\mu\nu x^2/y^2\right]}{\gamma x+y}p(x)q(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)dxdy+\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\delta+\nu x/y)}{\gamma x+y}p(x)q(y)dxdy\\&\leqslant(\kappa_1+\kappa_2)\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},\end{aligned}$$

where, thanks to Equation (2.2),

$$\kappa_1=\frac{2\pi}{\gamma^{1/2}}\log\left[\left(\frac{\mu}{\gamma}\right)^{1/2}+\beta^{1/2}\right]$$

and

$$\kappa_2=\frac{2\pi}{\gamma^{1/2}}\log\left[\left(\frac{\nu}{\gamma}\right)^{1/2}+\delta^{1/2}\right].$$

Now, let us notice that

$$\begin{aligned}\kappa_1+\kappa_2&=\frac{2\pi}{\gamma^{1/2}}\log\left[\left(\frac{\mu}{\gamma}\right)^{1/2}+\beta^{1/2}\right]+\frac{2\pi}{\gamma^{1/2}}\log\left[\left(\frac{\nu}{\gamma}\right)^{1/2}+\delta^{1/2}\right]\\&=\frac{2\pi}{\gamma^{1/2}}\log\left\{\left[\left(\frac{\mu}{\gamma}\right)^{1/2}+\beta^{1/2}\right]\left[\left(\frac{\nu}{\gamma}\right)^{1/2}+\delta^{1/2}\right]\right\}=\kappa_\star.\end{aligned}$$

The strict inequality and the "optimality" of $\kappa_\star$ is a direct consequence of Equation (2.2), and the optimality of κ_1 and κ_2 in their respective settings. The result is established. □

The proposition below is about new three-parameter Hilbert integral inequalities derived from lower bounds of the logarithmic function and Theorem 2.1.

Proposition 3.3. *Let* $\beta\geqslant 1$, $\mu\geqslant 0$ *and* $\gamma>0$. *Let* $p:[0,+\infty)\mapsto[0,+\infty)$ *and* $q:[0,+\infty)\mapsto[0,+\infty)$ *such that* $0<\int_0^{+\infty}[p(x)]^2dx<+\infty$ *and* $0<\int_0^{+\infty}[q(y)]^2dy<+\infty$. *Then we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{(\beta-1)y+\mu x}{(\gamma x+y)[\mu^2x^2+2(\beta+5)\mu xy+(\beta^2+10\beta+1)y^2]^{1/2}}p(x)q(y)dxdy\\&\leqslant\frac{\kappa}{2\sqrt{3}}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\end{aligned}$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{(\beta-1)y+\mu x}{(\beta+1)y+\mu x}p(x)q(y)dxdy \leqslant \frac{\kappa}{2}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},$$

where κ *is given by Equation* (2.2).

Proof. The following sharp logarithmic inequalities are well known: For any $z \geqslant 0$, we have

$$\frac{2z}{2+z} \leqslant \frac{z}{(1+z+z^2/12)^{1/2}} \leqslant \log(1+z).$$

See, for instance, [6] and [3]. Using these two inequalities and taking $z = \beta - 1 + \mu x/y \geqslant 0$, we derive the following two results:

$$\log\left(\beta+\mu\frac{x}{y}\right) = \log\left[1+\left(\beta-1+\mu\frac{x}{y}\right)\right] \geqslant \frac{\beta-1+\mu x/y}{[\beta+\mu x/y+(\beta-1+\mu x/y)^2/12]^{1/2}}$$
$$= 2\sqrt{3}\frac{(\beta-1)y+\mu x}{[\mu^2x^2+2(\beta+5)\mu xy+(\beta^2+10\beta+1)y^2]^{1/2}} \tag{3.2}$$

and

$$\log\left(\beta+\mu\frac{x}{y}\right) = \log\left[1+\left(\beta-1+\mu\frac{x}{y}\right)\right] \geqslant \frac{2(\beta-1+\mu x/y)}{\beta+1+\mu x/y} = 2\frac{(\beta-1)y+\mu x}{(\beta+1)y+\mu x}. \tag{3.3}$$

From Equation (3.2), Theorem 2.1 and the positivity of $p(x)$ and $q(y)$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{(\beta-1)y+\mu x}{(\gamma x+y)[\mu^2x^2+2(\beta+5)\mu xy+(\beta^2+10\beta+1)y^2]^{1/2}}p(x)q(y)dxdy\\
&\leqslant \frac{1}{2\sqrt{3}}\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)dxdy\\
&\leqslant \frac{1}{2\sqrt{3}}\left[\kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\right]\\
&= \frac{\kappa}{2\sqrt{3}}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}.
\end{aligned}$$

Similarly, from Equation (3.3), Theorem 2.1 and the positivity of $p(x)$ and $q(y)$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{(\beta-1)y+\mu x}{(\beta+1)y+\mu x}p(x)q(y)dxdy\\
&\leqslant \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)dxdy\\
&\leqslant \frac{1}{2}\left[\kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}\right]\\
&= \frac{\kappa}{2}\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2}.
\end{aligned}$$

The two desired inequalities are proved. ☐

Since the inequalities in Proposition 3.3 are obtained by using lower bounds of the logarithmic function, it is not claimed that they are optimal. However, they illustrate how Theorem 2.1 can be used as an inequality result beyond the main logarithmic integral form. The generality and originality of this theorem opens up some possibilities for research in mathematical analysis, which we leave for future work.

We end this paper by a proposition on another kind of three-parameter logarithmic generalization of the Hilbert integral inequality.

Proposition 3.4. *Technically, in their respective settings, Theorems 2.1 and 2.3 are also valid with*

$$\frac{\log(\beta+\mu xy)}{1+\gamma xy}$$

as the main integrated term, instead of

$$\frac{\log(\beta+\mu x/y)}{\gamma x+y}.$$

Hence,

- *we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu xy)}{1+\gamma xy}p(x)q(y)dxdy \leqslant \kappa\left\{\int_0^{+\infty}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}[q(y)]^2dy\right\}^{1/2},$$

- *we have*

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{\log(\beta+\mu xy)}{1+\gamma xy}p(x)dx\right]^2dy \leqslant \kappa^2\int_0^{+\infty}[p(x)]^2dx,$$

  where κ is given by Equation (2.2).

Proof. The proof follows those of Theorems 2.1 and 2.3, line by line. The more technical point is to notice that, with the change of variables $y = u/x$, based on Equation (2.4), we have

$$\int_0^{+\infty}\frac{\log(\beta+\mu xy)}{1+\gamma xy}\left(\frac{x}{y}\right)^{1/2}dy = \int_0^{+\infty}\frac{\log(\beta+\mu u)}{u^{1/2}(1+\gamma u)}du = \kappa$$

and, with the change of variables $x = u/y$,

$$\int_0^{+\infty}\frac{\log(\beta+\mu xy)}{1+\gamma xy}\left(\frac{y}{x}\right)^{1/2}dx = \int_0^{+\infty}\frac{\log(\beta+\mu u)}{u^{1/2}(1+\gamma u)}du = \kappa.$$

The strict inequality and the optimality of the constant κ can be proved with arguments similar to those used in the proof of Theorem 2.1. This concludes the proof. ☐

With the presented results, we have filled a gap in the topic of Hilbert integral inequalities of logarithmic type. Future work includes exploring the extension of these inequalities to higher dimensions and investigating potential applications in related mathematical fields, mainly in functional analysis.

4. Conclusion

In this paper, we have established a new generalized logarithmic version of the Hilbert integral inequality. It is based on the following integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu x/y)}{\gamma x+y}p(x)q(y)dxdy,$$

where β, μ and γ are three adjustable parameters. The detailed proof is given, as well as the exact values of the constants. The optimality of the obtained inequality is also proved. Thanks to these new results, other new integral inequalities have been established, including one based on the following variant:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\log(\beta+\mu xy)}{1+\gamma xy}p(x)q(y)dxdy.$$

Future directions include using these inequalities in concrete applications and extending them to more dimensions.

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