SABA Publishing

Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2025)6(1) : 35-45
doi:10.48185/jmam.v6i1.1263

Ninth and Twelfth-Order Iterative Methods for Roots of Nonlinear Equations

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• *Received: 24 July 2024* • *Accepted: 03 June 2025*

Abstract

This paper introduces two iterative methods for obtaining numerical solutions to nonlinear equations. The proposed methods achieve convergence orders of nine and twelve, respectively. A detailed convergence analysis confirms their superior efficiency indices compared to several existing techniques. Numerical examples are presented to illustrate the performance and to validate the theoretical convergence orders of the proposed methods.

Keywords: Nonlinear equation, iterative method, convergence order,
2010 MSC: 65H05, 49M15, 41A25.

1. Introduction

Nonlinear equation root finding is a vital element of numerical analysis with many practical uses in science and engineering (see [9], [16]). The need for high precision in computational tasks underscores the significance of higher-order numerical methods, [6].

The development of methods for addressing non-linear equation $f(x) = 0$ holds great importance in numerical analysis, owing to their widespread occurrence in fields like science, technology and engineering, [14].

In the numerical solution of equations, particularly differential and integral equations, addressing nonlinear equations is a common challenge (see [4], [13]). This study introduces two new iterative methods devised for the purpose of determining a simple root λ of a nonlinear equation $f(x) = 0$, where $f : I \subset \mathbf{R} \to \mathbf{R}$: is a scalar function on an open interval I .

The Newton-Raphson method, also known as Newton's method (NM):

$$x_{n+1} = x_n - f(x_n)/f'(x_n), \tag{1.1}$$

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*Corresponding author: hbawazir@yahoo.com a well-known and widely used approach for solving nonlinear equations, iteratively produces a sequence of approximations that converge quadratically to a simple root, λ, of the equation $f(x) = 0$.

Traub, [19], began the classification of iterative methods, suggesting a third-order iterative technique. Jarrat, in [10] and [11], introduced a variety of methods consisting of two points and two steps, demanding one function and two derivative evaluations per iteration, along with one parameter to achieve fourth-order convergence.

Over the last few years, many authors have formulated high-order iterative techniques and explored their convergence analysis for solving nonlinear equations, as referenced by ([1], [3], [15],[18]).

Kou et al. [12] suggested altering the Newton method to achieve cubic convergence,

$$\begin{aligned} y_n &= x_n - f(x_n)/f'(x_n) \\ x_{n+1} &= x_n - \frac{f(x_n)(f'(x_n) - f'(y_n))}{2f'(x_n)f'(y_n)}, \end{aligned}$$

in contrast, Weerakoon and Fernando [20] recommended a modification of Newton's method for third-order convergence as

$$\begin{aligned} y_n &= x_n - f(x_n)/f'(x_n) \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) - f'(y_n)}. \end{aligned}$$

Based on Newton's method, Trachoo et al. [18] proposed the following three-step technique with fifth-order convergence,

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_f(x_n))}{2\alpha(1 - L_f(x_n))}\right),\ L_f(x_n) = \frac{f(x_n)f''(x_n)}{(f'(x_n))^2} \\ \omega_n &= y_n - \frac{f(y_n)}{f'(x_n) + f''(x_n)(y_n - x_n)} \\ x_{n+1} &= y_n - \frac{2f(y_n)}{f'(\omega_n) + f'(x_n) + f''(x_n)(y_n - x_n)} \end{aligned}$$

similarly, Nori et al. [15] introduced a three-step approach with fifth-order convergence,

$$\begin{aligned} \upsilon_n &= x_n - \frac{f(x_n)}{f'(x_n)},\ \mu_n = x_n + \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{(x_n - \upsilon_n)f(x_n)^2(f(\upsilon_n) + f(\mu_n))}{f(x_n)^2(f(\mu_n) - f(\upsilon_n)) - 4f(x_n)f(\upsilon_n)^2 - 6f(\upsilon_n)^3}, \end{aligned}$$

Bawazir [2] developed a three-step technique with fifth-order convergence based on Newton's method, as follows

$$\begin{aligned} y_n &= x_n - f(x_n)/f'(x_n),\ \tilde{y}_n = x_n + f(x_n)/f'(x_n), \\ x_{n+1} &= y_n + \frac{2f^2(x_n)f(y_n)}{f'(x_n)\left[f(x_n)(5f(y_n) - f(\tilde{y}_n)) + 2f^2(y_n)\right]}. \end{aligned} \tag{1.2}$$

Expanding on Newton's method, Nori et al. [15] also suggested a four-step method with tenth-order convergence which can be represented by

$$\begin{aligned} \upsilon_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad \sigma_n = \upsilon_n - \frac{f(\upsilon_n)}{f'(\upsilon_n)}, \ \eta_n = \upsilon_n + \frac{f(\upsilon_n)}{f'(\upsilon_n)}, \\ x_{n+1} &= \upsilon_n - \frac{(\upsilon_n - \sigma_n) f(\upsilon_n)^2 (f(\sigma_n) + f(\eta_n))}{f(\upsilon_n)^2 (f(\eta_n) - f(\sigma_n)) - 4 f(\upsilon_n) f(\sigma_n)^2 - 6 f(\sigma_n)^3}. \end{aligned} \tag{1.3}$$

We recommend and analyze four-step, ninth-order and twelfth-order iterative methods for the solution of nonlinear equations.

Suppose the real sequence $< x_n >$ converges to a simple root , λ , of an equation $f(x) = 0$, the convergence order is a positive real number α if the limit

$$\lim_{n\to\infty} \frac{|x_{n+1} - \lambda|}{|x_n - \lambda|^\alpha}$$

leads to a positive real number, say β, [5]. The sequence is said to have quadratic convergence or cubic convergence if $\alpha = 2$ or $\alpha = 3$, respectively.

The relation

$$\varepsilon_{n+1} = \beta \varepsilon_n^\alpha + O(\varepsilon_n^{\alpha+1})$$

is called error equation, where $\varepsilon_n = |x_n - \lambda|$ is the error in the n-th iteration. If we can demonstrate the error equation for any iterative method, then the value of α is its order of convergence, [17].

Let x_{n+1}, x_n and x_{n-1} be three successive iterations that are closer to the root λ. Then, we can approximate the computational order of convergence ρ, see [20], as follows,

$$\rho = \frac{\ln |(x_{n+1} - \lambda)/(x_n - \lambda)|}{\ln |(x_n - \lambda)/(x_{n-1} - \lambda)|}.$$

The index of efficiency can be computed as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method, [7].

In this paper, we present two new higher order iterative methods based on method (1.2) and determine the convergence rates of these methods. Subsequently, different numerical examples are given to validate the theoretical results and show their performance

2. Convergence Analysis

Firstly, we present new ninth-order iterative method as follows

$$\begin{aligned} & y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \tilde{y}_n = x_n + \frac{f(x_n)}{f'(x_n)}, \quad z_n = y_n + \psi_n, \\ & x_{n+1} = z_n + \frac{\psi_n f(z_n)}{[f(y_n) - f(z_n)] \left[1 - 1.5\left(\frac{f(y_n)}{f(x_n)}\right)^2 + 0.5\left(\frac{f(\tilde{y}_n)}{f(x_n)} - 2\right)^2 - 4\left(\frac{f(y_n)}{f(x_n)}\right)^3\right]}, \\ & \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{2.1}$$

where

$$\psi_n = \frac{2f^2(x_n)f(y_n)}{f'(x_n)\,[f(x_n)(5f(y_n) - f(\tilde{y}_n)) + 2f^2(y_n)]}.$$

The convergence result for the method (2.1) is as follows.

Theorem 2.1. *Let* λ *be a simple zero of sufficiently differentiable function* $f : I \subset \mathbf{R} \to \mathbf{R}$ *for an open interval* I *. If* x_0 *is sufficiently close to* λ*, then the method defined by (2.1) is of ninth-order and satisfies the error equation*
$\varepsilon_{n+1} = 2\,c_2(c_2^3 + c_4 - c_2c_3)(14c_2^4 + 2c_3^2 + 5c_2c_4 - 12c_2^2c_3)\varepsilon_n^9 + O(\varepsilon_n^{10})$,
where $\varepsilon_n = x_n - \lambda,\ c_k = f^{(k)}(\lambda)/(k!f'(\lambda))$.

Proof.

Using Taylor expansion of $f(x_n)$ and $f'(x_n)$ about λ, we get

$$f(x_n) = f'(\lambda)[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + O(\varepsilon_n^5)], \tag{2.2}$$

and

$$f'(x_n) = f'(\lambda)[1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + O(\varepsilon_n^4)]. \tag{2.3}$$

By the long division of (2.2) by (2.3), we obtain

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= \varepsilon_n - c_2\varepsilon_n^2 + 2(c_2^2 - c_3)\varepsilon_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)\varepsilon_n^4 \\ &+(6\,c_3^2 + 10c_2c_4 + 8c_2^4 - 4c_5 - 20c_2^2c_3)\varepsilon_n^5 \ + \ (17c_3c_4 + \\ &13c_2c_5 + 52c_2^3c_3 - 28c_2^2c_4 - 5c_6 - 33c_2c_3^2 - 16c_2^5)\varepsilon_n^6 + O(\varepsilon_n^7),\end{aligned} \tag{2.4}$$

therefore,

$$\begin{aligned}d_n &= y_n - \lambda = \varepsilon_n - \frac{f(x_n)}{f'(x_n)} \\ &= c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (4c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 + (4c_5 + 20c_2^2c_3 - \\ &6c_3^2 - 10c_2c_4 - 8c_2^4)\varepsilon_n^5 \ + \ (28c_2^2c_4 + 5c_6 + 33c_2c_3^2 + 16c_2^5 - 17c_3c_4 - \\ &13c_2c_5 - 52c_2^3c_3\,)\varepsilon_n^6 + O(\varepsilon_n^7).\end{aligned} \tag{2.5}$$

Expanding $f(y_n)$ around λ results in

$$f(y_n) = f'(\lambda)[d_n + c_2d_n^2 + c_3d_n^3 + O(d_n^4)], \tag{2.6}$$

therefore, by (2.5), we have

$$\begin{aligned}f(y_n) &= f'(\lambda)[c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (5c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 \\ &+(24c_2^2c_3 + 4c_5 - 12c_2^4 - 6c_3^2 - 10c_2c_4)\varepsilon_n^5 \ + O(\varepsilon_n^6).\end{aligned} \tag{2.7}$$

Now, dividing (2.7) by (2.2), we get

$$\frac{f(y_n)}{f(x_n)} = c_2\varepsilon_n + (2c_3 - 3c_2^2)\varepsilon_n^2 + (8c_2^3 + 3c_4 - 10c_2c_3)\varepsilon_n^3 + O(\varepsilon_n^4), \tag{2.8}$$

therefore,

$$\left(\frac{f(y_n)}{f(x_n)}\right)^2 = c_2^2\varepsilon_n^2 + 2c_2(2c_3 - 3c_2^2)\varepsilon_n^3 + (25c_2^4 + 6c_2c_4 + 4c_3^2 - 32c_2^2c_3)\varepsilon_n^4 + O(\varepsilon_n^5), \tag{2.9}$$

and

$$\left(\frac{f(y_n)}{f(x_n)}\right)^3 = c_2^3\varepsilon_n^3 + 3c_2^2(2c_3 - 3c_2^2)\varepsilon_n^4 + O(\varepsilon_n^5), \tag{2.10}$$

Now, by (2.1) and (2.4), we obtain

$$\begin{aligned}\tilde{d}_n = \tilde{y}_n - \lambda &= \varepsilon_n + f(x_n)/f'(x_n)\\ &= 2\varepsilon_n - c_2\varepsilon_n^2 + 2(c_2^2 - c_3)\varepsilon_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)\varepsilon_n^4\\ &\quad +(6\,c_3^2 + 10c_2c_4 + 8c_2^4 - 4c_5 - 20c_2^2c_3)\varepsilon_n^5 + (17c_3c_4\\ &\quad +13c_2c_5 + 52c_2^3c_3 - 28c_2^2c_4 - 5c_6 - 33c_2c_3^2 - 16c_2^5\ )\varepsilon_n^6\\ &\quad + (16c_2c_6 - 6c_7 - 36c_2^2c_5 + 22c_3c_5 - 92c_2c_3c_4 + 70c_2^3c_4\\ &\quad +12c_4^2 + 126c_2^2c_3^2 - 18c_3^3 - 128c_2^4c_3 + 32c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8).\end{aligned} \tag{2.11}$$

Taylor expansions of $f(\tilde{y}_n)$ around λ is given as

$$f(\tilde{y}_n) = f'(\lambda)[\tilde{d}_n + c_2\tilde{d}_n^2 + c_3\tilde{d}_n^3 + O(\tilde{d}_n^4)], \tag{2.12}$$

and hence, by (2.11), we attain

$$\begin{aligned}&f(\tilde{y}_n)\\ &= f'(\lambda)[2\varepsilon_n + 3c_2\varepsilon_n^2 + 2(3c_3 - c_2^2)\varepsilon_n^3 + (5c_2^3 + 13c_4 - 13c_2c_3)\varepsilon_n^4\\ &\quad +(42c_2^2c_3 + 28c_5 - 12c_2^4 - 18c_3^2 - 34c_2c_4)\varepsilon_n^5 + (28c_2^5\\ &\quad +106c_2^2c_4 + 59c_6 + 103c_2c_3^2 - 119c_2^3c_3 - 83c_3c_4 - 83c_2c_5)\varepsilon_n^6 + O(\varepsilon_n^7).\end{aligned} \tag{2.13}$$

Now, by dividing (2.13) by (2.2), we obtain

$$\frac{f(\tilde{y}_n)}{f(x_n)} - 2 = c_2\varepsilon_n + (4c_3 - 3c_2^2)\varepsilon_n^2 + (8c_2^3 + 11c_4 - 18c_2c_3)\varepsilon_n^3 + O(\varepsilon_n^4), \tag{2.14}$$

therefore,

$$\begin{aligned}&\left(\frac{f(\tilde{y}_n)}{f(x_n)} - 2\right)^2\\ &= c_2^2\varepsilon_n^2 + 2c_2(4c_3 - 3c_2^2)\varepsilon_n^3 + (25c_2^4 + 22c_2c_4 + 16c_3^2 - 60c_2^2c_3)\varepsilon_n^4 + O(\varepsilon_n^5),\end{aligned} \tag{2.15}$$

so, from (2.9), (2.10) and (2.15), we get

$$\begin{aligned}&1 - 1.5\left(\frac{f(y_n)}{f(x_n)}\right)^2 + 0.5\left(\frac{f(\tilde{y}_n)}{f(x_n)} - 2\right)^2 - 4\left(\frac{f(y_n)}{f(x_n)}\right)^3\\ &= 1 - c_2^2\varepsilon_n^2 + 2c_2(c_2^2 - c_3)\varepsilon_n^3 + (11c_2^4 + 2c_2c_4 + 2c_3^2 - 6c_2^2c_3)\varepsilon_n^4 + O(\varepsilon_n^5).\end{aligned} \tag{2.16}$$

Bawazir, [2], proved that

$$\begin{aligned}\hat{d}_n &= z_n - \lambda \\ &= 2\,c_2(c_2^3 + c_4 - c_2c_3)\varepsilon_n^5 + O(\varepsilon_n^6),\end{aligned} \tag{2.17}$$

therefore, by (2.1) and (2.17), we have

$$\psi_n = -d_n + \hat{d}_n, \tag{2.18}$$

therefore, by (2.5) and (2.17), we get

$$\begin{aligned}\psi_n &= -c_2\varepsilon_n^2 + 2(c_2^2 - c_3)\varepsilon_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3)\varepsilon_n^4 \\ &\quad +2\,(6c_2c_4 + 5c_2^4 + 3c_3^2 - 11c_2^2c_3 - 2c_5)\varepsilon_n^5 + O(\varepsilon_n^6).\end{aligned} \tag{2.19}$$

Now, by using Taylor expansion of $f(z_n)$ around λ , we obtain

$$f(z_n) = f'(\lambda)[\hat{d}_n + c_2\hat{d}_n^2 + c_3\hat{d}_n^3 + O(\hat{d}_n^4)], \tag{2.20}$$

therefore, by (2.6) and (2.20), we achieve

$$\begin{aligned}&f(y_n) - f(z_n) \\ &= f'(\lambda)[d_n + c_2d_n^2 - \hat{d}_n + c_3d_n^3 + c_4d_n^4 + c_5d_n^5 - c_2\hat{d}_n^2 + \cdots],\end{aligned} \tag{2.21}$$

in addition, using (2.5) and (2.17), we obtain

$$\begin{aligned}&f(y_n) - f(z_n) \\ &= f'(\lambda)[c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (5c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 \\ &\quad +(4c_5 + 26c_2^2c_3 - 6\,c_3^2 - 12c_2c_4 - 14c_2^4)\varepsilon_n^5 \ + O(\varepsilon_n^6).\end{aligned} \tag{2.22}$$

From (2.18) and (2.21), we have

$$\begin{aligned}&\psi_n^{-1}\,(f(y_n) - f(z_n)) \\ &= f'(\lambda)[-1 - c_2d_n - c_3d_n^2 - c_4d_n^3 + O(\varepsilon_n^7)],\end{aligned} \tag{2.23}$$

therefore, by (2.5), we get

$$\begin{aligned}&\psi_n^{-1}\,(f(y_n) - f(z_n)) \\ &= f'(\lambda)[-1 - c_2^2\varepsilon_n^2 + 2c_2(c_2^2 - c_3)\varepsilon_n^3 + c_2(6c_2c_3 - 4c_2^3 - 3c_4)\varepsilon_n^4 + O(\varepsilon_n^5)].\end{aligned} \tag{2.24}$$

Using (2.16) and (2.24), we obtain

$$\begin{aligned}&\psi_n^{-1}\,(f(y_n) - f(z_n))\left[1 - 1.5\left(\frac{f(y_n)}{f(x_n)}\right)^2 + 0.5\left(\frac{f(\tilde{y}_n)}{f(x_n)} - 2\right)^2 - 4\left(\frac{f(y_n)}{f(x_n)}\right)^3\right] \\ &= f'(\lambda)[-1 + (12c_2^2c_3 - 14c_2^4 - 2c_3^2 - 5c_2c_4)\varepsilon_n^4 + O(\varepsilon_n^5)].\end{aligned} \tag{2.25}$$

Now, by dividing (2.20) by (2.25), we get

$$\begin{aligned}&\frac{\psi_n f(z_n)}{[f(y_n)-f(z_n)]\left[1-1.5\left(\frac{f(y_n)}{f(x_n)}\right)^2+0.5\left(\frac{f(\tilde{y}_n)}{f(x_n)}-2\right)^2-4\left(\frac{f(y_n)}{f(x_n)}\right)^3\right]}\\&=-\hat{d}_n+(14c_2^4+2c_3^2+5c_2c_4-12c_2^2c_3)\varepsilon_n^4\hat{d}_n+O(\varepsilon_n^{10}).\end{aligned}\tag{2.26}$$

Finally, by using (2.1), (2.26) and (2.17), we obtain the following error relation

$$\begin{aligned}\varepsilon_{n+1}&=x_{n+1}-\lambda\\&=\hat{d}_n+[-\hat{d}_n+(14c_2^4+2c_3^2+5c_2c_4-12c_2^2c_3)\varepsilon_n^4\hat{d}_n+O(\varepsilon_n^{10})]\\&=(14c_2^4+2c_3^2+5c_2c_4-12c_2^2c_3)\varepsilon_n^4\hat{d}_n+O(\varepsilon_n^{10})\\&=2\,c_2(c_2^3+c_4-c_2c_3)(14c_2^4+2c_3^2+5c_2c_4-12c_2^2c_3)\varepsilon_n^9+O(\varepsilon_n^{10}),\end{aligned}$$

this indicates that the method described by (2.1) is of ninth-order.

Finally, we construct new twelfth-order iterative method as follows

$$\begin{aligned}&y_n=x_n-\frac{f(x_n)}{f'(x_n)},\quad \tilde{y}_n=x_n+\frac{f(x_n)}{f'(x_n)},\quad z_n=y_n+\psi_n\\&x_{n+1}=z_n-\frac{f(z_n)}{f'(z_n)}\left(1+\frac{f(z_n)(f(z_n)-f(y_n)-\psi_n f'(z_n))}{f(y_n)\,(f(z_n)-f(y_n))}\right),\\&\qquad n=0,1,2,\ldots\end{aligned}\tag{2.27}$$

where

$$\psi_n=\frac{2f^2(x_n)f(y_n)}{f'(x_n)\,[f(x_n)(5f(y_n)-f(\tilde{y}_n))+2f^2(y_n)]}.$$

Theorem 2.2. *Let* λ *be a simple zero of sufficiently differentiable function* $f: I\subset \mathbf{R}\to\mathbf{R}$ *for an open interval* I. *If* x_0 *is sufficiently close to* λ, *then the method defined by (2.27) is of twelfth-order and satisfies the error equation*
$\varepsilon_{n+1}=4\,c_2^3(2c_2^2-c_3)(c_2^3+c_4-c_2c_3)^2\varepsilon_n^{12}+O(\varepsilon_n^{13})$

Proof.
Using Taylor expansion of $f'(z_n)$ about λ, we have

$$f'(z_n)=f'(\lambda)[1+2c_2\hat{d}_n+3c_3\hat{d}_n^2+O(\hat{d}_n^3)],\tag{2.28}$$

dividing (2.20) by (2.28), we get

$$\frac{f(z_n)}{f'(z_n)}=\hat{d}_n-c_2\hat{d}_n^2+2(c_2^2-c_3)\hat{d}_n^3+O(\hat{d}_n^4).\tag{2.29}$$

From (2.18) and (2.28), we obtain

$$\begin{aligned}&\psi_n f'(z_n)\\&=f'(\lambda)[-d_n+\hat{d}_n-2c_2d_n\hat{d}_n+2c_2\hat{d}_n^2-3c_3d_n\hat{d}_n^2+\cdots],\end{aligned}\tag{2.30}$$

therefore, from (2.21) and (2.30), we achieve

$$\begin{aligned} &f(z_n) - f(y_n) - \psi_n f'(z_n) \\ &= f'(\lambda)[-c_2 d_n^2 - c_3 d_n^3 + 2c_2 d_n \hat{d}_n - c_4 d_n^4 - c_5 d_n^5 - 2c_2 \hat{d}_n^2 + \cdots]. \end{aligned} \tag{2.31}$$

From (2.20) and (2.31), we get

$$\begin{aligned} &f(z_n)\left(f(z_n) - f(y_n) - \psi_n f'(z_n)\right) \\ &= f'^2(\lambda)[-c_2 d_n^2 \hat{d}_n - c_3 d_n^3 \hat{d}_n + 2c_2 d_n \hat{d}_n^2 - \cdots], \end{aligned} \tag{2.32}$$

and from (2.6) and (2.21), we obtain

$$\begin{aligned} &f(y_n)\,(f(z_n) - f(y_n)) \\ &= f'^2(\lambda)[-d_n^2 - 2c_2 d_n^3 + d_n \hat{d}_n - (2c_3 + c_2^2) d_n^4 + c_2 d_n^2 \hat{d}_n - \cdots]. \end{aligned} \tag{2.33}$$

Dividing (2.32) by (2.33), we obtain

$$\begin{aligned} &\frac{f(z_n)\,(f(z_n) - f(y_n) - \psi_n f'(z_n))}{f(y_n)\,(f(z_n) - f(y_n))} \\ &= c_2 \hat{d}_n + (c_3 - 2c_2^2) d_n \hat{d}_n + \cdots \end{aligned} \tag{2.34}$$

Now, by using (2.27), (2.29) and (2.34), we obtain

$$\begin{aligned} \varepsilon_{n+1} &= \hat{d}_n - \frac{f(z_n)}{f'(z_n)}\left(1 + \frac{f(z_n)(f(z_n) - f(y_n) - \psi_n f'(z_n))}{\psi_n f'(z_n) f(y_n)}\right) \\ &= \hat{d}_n - (\hat{d}_n - c_2 \hat{d}_n^2 + \cdots)(1 + c_2 \hat{d}_n + (c_3 - 2c_2^2) d_n \hat{d}_n + \cdots) \\ &= (2c_2^2 - c_3) d_n \hat{d}_n^2 + \cdots \end{aligned} \tag{2.35}$$

therefore, by (2.35), (2.5) and (2.17) we get the error relation:

$$\varepsilon_{n+1} = 4\, c_2^3 (2c_2^2 - c_3)(c_2^3 + c_4 - c_2 c_3)^2 \varepsilon_n^{12} + O(\varepsilon_n^{13}).$$

This signifies that the method specified by (2.27) is of twelfth-order.

Method (2.1) requires five function evaluations per iteration, four of f and one of f', whereas method (2.27) requires six function evaluations, four of f and two of f'. Method (2.1) shows an efficiency index of $9^{1/5} \approx 1.5518$, and method (2.27) exhibits an efficiency index of $12^{1/6} \approx 1.5130$. These efficiency indexes outperform Newton's method (NM) with an efficiency index of $2^{1/2} \approx 1.4142$, as well as the tenth order (TO) method, (1.3), with efficiency index of $10^{1/6} \approx 1.4677$.

3. Numerical Examples

In this section, we apply the recently introduced methods, denoted as method (2.1) and method (2.27) through equations (2.1) and (2.27) respectively. The aim is to address nonlinear equations and conduct a comparative analysis with Newton's method (NM), and the tenth order (TO) method, (1.3). The functions utilized are as follows, [8]:

$$\begin{aligned}
f_1(x) &= x^3 + 4x^2 - 10, \ \lambda = 1.3652300134140968879 1373,\\
f_2(x) &= x^5 + x^4 + 4x^2 - 20, \ \lambda = 1.4662790738647226707 0587,\\
f_3(x) &= e^{x^2+7x-30} - 1, \ \lambda = 3,\\
f_4(x) &= (\sin x)^2 - x^2 + 1, \ \lambda = 1.4044916482153411152 4670,\\
f_5(x) &= e^x \sin x + \ln(x^2 + 1), \ \lambda = 0,\\
f_6(x) &= x^3 - \sin^2 x + 3\cos x + 5, \ \lambda = -1.5826870457520698654 0081,\\
f_7(x) &= x^3 - e^{-x}, \ \lambda = 0.7728829591492101247 49629.
\end{aligned}$$

In Matlab (R2017b), all numerical instances were carried out using 200 digits of floating-point precision and variable precision arithmetic. The solution for each of the 7 functions listed above was computed with two distinct initial guesses x_0 , and the iterative processes were terminated when the error $|x_{n+1} - x_n| + |f(x_n)|$ reached less than 10^{-200} .

In Table 1, we show the iterations (IT) needed to attain $|x_{n+1} - x_n| + |f(x_n)| < 10^{-200}$, and Table 2 outlines the computational order ρ for all examples considered.

The numerical outcomes presented in Table 1 point to the fact that the methods, method (2.1) and method (2.27) achieve faster convergence than Newton's method (NM). Additionally, these methods entail a reduced number of iterations, emphasizing the heightened convergence efficiency of the new methods (2.1) and (2.27).

The computational order of NM, (TO), method (2.1), and method (2.27) is displayed in Table 2. The numerical results in Table 2 substantiate that the proposed methods substantiate the theoretical findings detailed in Section 2.

The proposed iterative methods (2.1) and (2.27) elucidated in this document are capable of competing with other efficient equation-solving approaches, such as Newton's method and the tenth order method (TO). The efficiency indexes 1.5518, 1.5130, 1.4142, and 1.4677 for methods, method (2.1), method (2.27), NM, and (TO) respectively, underscore their effectiveness.

Table 1: Numerical results for different methods with stopping criterium $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$

		The number of iterations (IT)			
$f(x)$	x_0	NM (1.1)	TO (1.3)	Method (2.1)	Method (2.27)
f_1	1.9	9	3	3	3
	1	10	3	3	3
f_2	1.2	10	3	3	3
	2	11	3	3	3
f_3	3.5	17	5	6	5
	4	24	8	8	7
f_4	1.6	10	3	3	3
	2.5	11	3	4	3
f_5	3	11	4	3	3
	4.2	11	4	4	3
f_6	-1	10	3	3	3
	-3	11	3	3	3
f_7	0	11	3	4	3
	1.5	11	3	4	3

Table 2: Numerical results for different methods with stopping criterium $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$

		NM (1.1)	TO (1.3)	Method (2.1)	Method (2.27)
$f(x)$	x_0	(IT, ρ)	(IT, ρ)	(IT, ρ)	(IT, ρ)
f_1	1.9	$(9, 2.0037)$	$(3, 10.2305)$	$(3, 9.2594)$	$(3, 12.3924)$
	1	$(10, 2.0018)$	$(3, 10.3010)$	$(3, 9.3086)$	$(3, 12.3938)$
f_2	1.2	$(10, 1.9999)$	$(3, 9.4298)$	$(3, 8.6253)$	$(3, 11.9239)$
	2	$(11, 2.0000)$	$(3, 9.5343)$	$(3, 8.6267)$	$(3, 118996)$
f_3	3.5	$(17, 1.9957)$	$(5, 7.8713)$	$(6, 8.5874)$	$(5, 10.7706)$
	4	$(24, 1.9947)$	$(8, 9.3547)$	$(8, 7.6455)$	$(7, 10.1966)$
f_4	1.6	$(10, 1.8269)$	$(3, 9.9212)$	$(3, 8.9611)$	$(3, 12.0744)$
	2.5	$(11, 2.0006)$	$(3, 9.7953)$	$(4, 8.9877)$	$(3, 12.2067)$
f_5	3	$(11, 2.0002)$	$(4, 9.6135)$	$(3, 8.7380$	$(3, 11.9779)$
	4.2	$(11, 2.0002)$	$(4, 9.9558)$	$(4, 8.9595)$	$(3, 11.9646)$
f_6	-1	$(10, 2.0029)$	$(3, 10.8454)$	$(3, 9.7113)$	$(3, 12.6674)$
	-3	$(11, 2.0022)$	$(3, 10.7783)$	$(3, 9.6765)$	$(3, 12.8226)$
f_7	0	$(11, 2.0002)$	$(3, 9.6018)$	$(4, 8.9650)$	$(3, 12.0836)$
	1.5	$(11, 2.0002)$	$(3, 9.6159)$	$(4, 8.9743)$	$(3, 12.0898)$

4. Conclusion

In conclusion, two new higher order methods, (2.1) and (2.27), have been constructed. The convergence analysis verified that Methods (2.1) and (2.27) are of order nine and twelve, respectively. Method (2.1) shows an efficiency index of $9^{1/5} \approx 1.5518$, and method (2.27) exhibits an efficiency index of $12^{1/6} \approx 1.5130$. These efficiency indexes outperform Newton's method (NM) with an efficiency index of $2^{1/2} \approx 1.4142$, as well as the tenth order (TO) method, (1.3), with efficiency index of $10^{1/6} \approx 1.4677$. The numerical examples in the previous section highlighted the performance and confirmed their theoretical order of convergence.

We can affirm that the recently proposed iterative methods (2.1) and (2.27) elucidated in this document are capable of competing with other efficient equation-solving approaches, such as Newton's method and the tenth order method (TO).

We suggest upgrading methods (2.1) and (2.27) to attain higher orders without increasing the number of function evaluations, potentially resulting in enhanced efficiency indexes.

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