Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2025)6(1) : 107-116
doi:10.48185/jmam.v6i1.1261

# Summation formulas for Gauss's and Clausen's hypergeometric functions

AHMED ALI ATASH [a], AHMED ALI AL-GONAH[b,*] ,

[a] Department of Mathematics, Faculty of Education-Ataq, University of Shabwah, Shabwah, Yemen
[b] Department of Mathematics, Faculty of Science, University of Aden, Aden, Yemen
[b] Department of Information Technology, Faculty of Computer and Information, Socotra Archipelago University, Socotra, Yemen

• *Received: 21 July 2024* • *Accepted: 01 April 2025* • *Published Online: 15 June 2025*

## Abstract

In this paper, we utilize the general hypergeometric identity established by Masjed-Jamei and Koepf [5] to derive several summation formulas for Gauss's and Clausen's hypergeometric functions. As an application of our main results, we also evaluate specific values of Clausen's hypergeometric function for various arguments.

Keywords: Gauss hypergeometric function, Clausen hypergeometric function, Summation formulas.
*2010 MSC:* 33B15, 33C05, 33C20.

## 1. Introduction

The generalized hypergeometric function ${}_rF_s$ with $r$ numerator parameters and $s$ denominator parameters ($r$ and $s$ are positive integers or zero and $x$ is complex variable) is defined by (see [13])

$$ {}_rF_s\left[\begin{matrix} a_1, a_2, \dots, a_r; \\ \\ b_1, b_2, \dots, b_s; \end{matrix}\ x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(b_1)_n (b_2)_n \dots (b_s)_n} \frac{x^n}{n!}, \qquad (1.1) $$

where $(\lambda)_n$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ denotes the Pochhammer's symbol defined by

$$ (\lambda)_n = \begin{cases} 1 & n = 0, \\ \lambda(\lambda+1)\dots(\lambda+n-1) & n = 1, 2, \dots, \end{cases} $$

---

*Corresponding author: gonah1977@yahoo.com

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \lambda \neq 0, -1, -2, \dots \tag{1.2}$$

and $\Gamma(\lambda)$ is the Gamma function defined by

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t}\, dt, \qquad Re(\lambda) > 0. \tag{1.3}$$

Two important cases of the series (1.1) are appearance in many mathematical problems. The first case is the Gauss's hypergeometric function ${}_2F_1$:

$${}_2F_1\left[\begin{array}{ll} a, b; & \\ & x \\ c; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \qquad c \neq 0, -1, -2, \dots \quad . \tag{1.4}$$

The second case is the Clausen's hypergeometric function ${}_3F_2$:

$${}_3F_2\left[\begin{array}{ll} a, b, c; & \\ & x \\ d, e; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{x^n}{n!}, \qquad d, e \neq 0, -1, -2, \dots \quad . \tag{1.5}$$

The evaluation of the Gauss's and Clausen's hypergeometric functions with different arguments is of ongoing interest, since it appears in many mathematical science problems. Many summation formulas for Gauss's and Clausen's hypergeometric functions with different arguments have been obtained, see [1,3,4,7,8,9,10 ]. Recently Masjed-Jamei and Koepf [5] obtained general hypergeometric identity for generalized hypergeometric function and discussed certain cases of this identity. In this paper, we employ the result of Masjed-Jamei and Koepf [5] to obtain some summation formulas for Gauss's and Clausen's hypergeometric functions. For the application purpose, we evaluate certain new special values for the Clausen's hypergeometric function using our main results.

## 2. Summation theorems for Gauss's hypergeometric function

**Theorem 2.1.** *The following summation formulas for Gauss's hypergeometric function hold ture:*

$${}_2F_1\left[\begin{array}{ll} a, 2; & \\ & x \\ 3; & \end{array}\right] = \frac{2\left[(ax - x - 1)(1-x)^{1-a} + 1\right]}{(a-1)(a-2)x^2} \tag{2.1}$$

and

$${}_2F_1\left[\begin{array}{ll} a, 3; & \\ & x \\ 4; & \end{array}\right] = \frac{3\left[(2(1 - ax + x) + x^2(a^2 - 3a + 2))(1-x)^{1-a} - 2\right]}{(a-1)(a-2)(a-3)x^3}. \tag{2.2}$$

**Proof.** For prove (2.1) and (2.2),we consider the following result [5]:

$${}_pF_q\left[\begin{array}{ll} a_1, \dots, a_{p-1}, n; & \\ & x \\ b_1, \dots, b_{q-1}, m; & \end{array}\right] = \frac{\Gamma(m)}{\Gamma(n)} \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_{p-1})_k (k+1)(k+2)...(k+n-1)\; x^k}{(b_1)_k \dots (b_{q-1})_k\; (k+m-1)!}$$

$$= \frac{\Gamma(m)}{\Gamma(n)} \sum_{j=m-1}^{\infty} \frac{(a_1)_{j-m+1} \cdots (a_{p-1})_{j-m+1}}{(b_1)_{j-m+1} \cdots (b_{q-1})_{j-m+1}}$$

$$\times \frac{(j+2-m)(j+3-m)...(j-m+n)\ x^{j-m+1}}{j!}, \tag{2.3}$$

where $m, n \in N, m > n$.

Setting $p = q + 1 = 2$, $n = 2, m = 3$ in (2.3), we get

$${}_2F_1\left[\begin{matrix} a, 2; \\ \\ 3; \end{matrix}\ x\right] = \frac{2\Gamma(a-2)}{x^2\Gamma(a)}\left[(a-2)x\ {}_1F_0\left[\begin{matrix} a-1; \\ \\ -; \end{matrix}\ x\right] - {}_1F_0\left[\begin{matrix} a-2; \\ \\ -; \end{matrix}\ x\right] + 1\right]. \tag{2.4}$$

Using the following relation [13]:

$${}_1F_0\left[\begin{matrix} a; \\ \\ -; \end{matrix}\ x\right] = \sum_{n=0}^{\infty} \frac{(a)_n\ x^n}{n!} = (1-x)^{-a}, \tag{2.5}$$

we get the required result(2.1). For prove (2.2), setting $p = q + 1 = 2$, $n = 3, m = 4$ in (2.3) and using the same technique as in the proof of (2.1), we obtain the required result(2.2). This completes the proof of Theorem 2.1.

**Remark 2.1.** Note that, the case $p = q + 1 = 2$, $n = 1, m = 2$ in (2.3) is found in [8] as:

$${}_2F_1\left[\begin{matrix} a, 1; \\ \\ 2; \end{matrix}\ x\right] = \frac{(1-x)^{1-a} - 1}{(a-1)x}. \tag{2.6}$$

**Corollary 2.1.** Setting $a = -\frac{3}{2}, -\frac{5}{2}$ in (2.1), we get

$${}_2F_1\left[\begin{matrix} -\frac{3}{2}, 2; \\ \\ 3; \end{matrix}\ x\right] = \frac{4}{35x^2}\left[2 - (5x+2)(1-x)^{\frac{5}{2}}\right], \tag{2.7}$$

$${}_2F_1\left[\begin{matrix} -\frac{5}{2}, 2; \\ \\ 3; \end{matrix}\ x\right] = \frac{4}{63x^2}\left[2 - (7x+2)(1-x)^{\frac{7}{2}}\right]. \tag{2.8}$$

**Remark 2.2.** The special cases of (2.1) when $a = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ yield a known results in [8].

**Corollary 2.2.** Setting $a = -\frac{3}{2}, -\frac{5}{2}$ in (2.2), we get

$${}_2F_1\left[\begin{matrix} -\frac{3}{2}, 3; \\ \\ 4; \end{matrix}\ x\right] = \frac{2}{105x^3}\left[8 - (35x^2 + 20x + 8)(1-x)^{\frac{5}{2}}\right], \tag{2.9}$$

$$ {}_2F_1\left[\begin{array}{c} -\frac{5}{2},3; \\ \\ 4; \end{array} x\right] = \frac{2}{231x^3}\left[8-(63x^2+28x+8)(1-x)^{\frac{7}{2}}\right]. \tag{2.10} $$

**Remark 2.3.** The special cases of (2.2) when $a=-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2}$ yield a known results in [8].

As an application of our results (2.1) and (2.2), we present the following Theorem:

**Theorem 2.2.** *The following summation formula for Gauss's hypergeometric function holds ture:*

$$ {}_2F_1\left[\begin{array}{c} a,2; \\ \\ 4; \end{array} x\right] = \frac{6\left[(ax-3x+2)-(ax^2-x^2-ax-x+2)(1-x)^{1-a}\right]}{(a-1)(a-2)(a-3)x^3}. \tag{2.11} $$

**Proof.** For prove (2.11), we start with the following result (see[8],[12]):

$$ {}_3F_2\left[\begin{array}{c} a,b,c; \\ \\ b+1,c+1; \end{array} x\right] = \frac{1}{c-b}\left(c\,{}_2F_1\left[\begin{array}{c} a,b; \\ \\ b+1; \end{array} x\right] - b\,{}_2F_1\left[\begin{array}{c} a,c; \\ \\ c+1; \end{array} x\right]\right). \tag{2.12} $$

Taking $b=2, c=3$ in (2.12), we get

$$ {}_2F_1\left[\begin{array}{c} a,2; \\ \\ 4; \end{array} x\right] = \left(3\,{}_2F_1\left[\begin{array}{c} a,2; \\ \\ 3; \end{array} x\right] - 2\,{}_2F_1\left[\begin{array}{c} a,3; \\ \\ 4; \end{array} x\right]\right). \tag{2.13} $$

Using (2.1) and (2.2) to the two ${}_2F_1$ in the right hand side of (2.13), we easily obtain the required result (2.11).

**Corollary 2.3.** Setting $a=-\frac{3}{2},-\frac{5}{2}$ in (2.2), we get

$$ {}_2F_1\left[\begin{array}{c} -\frac{3}{2},2; \\ \\ 4; \end{array} x\right] = \frac{8}{105x^3}\left[9x-4+(4+x-5x^2)(1-x)^{\frac{5}{2}}\right], \tag{2.14} $$

$$ {}_2F_1\left[\begin{array}{c} -\frac{5}{2},2; \\ \\ 4; \end{array} x\right] = \frac{8}{231x^3}\left[11x-4+(4+3x-7x^2)(1-x)^{\frac{7}{2}}\right]. \tag{2.15} $$

**Remark 2.4.** The special cases of (2.11) when $a=-\frac{1}{2},\frac{1}{2},\frac{3}{2}$ yield a known results in [8].

For more applications of our results, we present new summation formulas for the following two other special cases of Gauss hypergeometric function:

$$ {}_2F_1\left[\begin{array}{c} a,a+\frac{1}{2}; \\ \\ \frac{3}{2}; \end{array} \frac{x^2}{(1-x)^2}\right] \text{ and } {}_2F_1\left[\begin{array}{c} a,a+\frac{1}{2}; \\ \\ \frac{5}{2}; \end{array} \frac{x^2}{(1-x)^2}\right]. $$

**Theorem 2.3.** *The following summation formula for Gauss's hypergeometric function hold true:*

$$
{}_2F_1\left[\begin{matrix} a, a+\frac{1}{2}; & \\ & \frac{x^2}{(1-x)^2} \\ \frac{5}{2}; & \end{matrix}\right]
$$

$$
= \frac{3(1-x)^{2a}[(2ax-3x+1)-(4ax^2-2x^2-2ax-x+1)(1-2x)^{1-2a}]}{2(2a-1)(2a-2)(2a-3)x^3}. \tag{2.16}
$$

**Proof.** For prove (2.16), we consider the following quadratic transformation (see[2],[7]):

$$
{}_2F_1\left[\begin{matrix} a, b; & \\ & 2x \\ 2b; & \end{matrix}\right] = (1-x)^{-a}{}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2}; & \\ & \frac{x^2}{(1-x)^2} \\ b+\frac{1}{2}; & \end{matrix}\right]. \tag{2.17}
$$

Replacing $a$ by $2a$ and taking $b=2$ in (2.17), we get

$$
{}_2F_1\left[\begin{matrix} a, a+\frac{1}{2}; & \\ & \frac{x^2}{(1-x)^2} \\ \frac{5}{2}; & \end{matrix}\right] = (1-x)^{2a}{}_2F_1\left[\begin{matrix} 2a, 2; & \\ & 2x \\ 4; & \end{matrix}\right]. \tag{2.18}
$$

Now, by apply (2.11) to ${}_2F_1$ in the right-hand side of (2.18), we easily obtain the required result (2.16).

**Corollary 2.4.** Setting $x=\frac{1}{3}$ in (2.16), we get

$$
{}_2F_1\left[\begin{matrix} a, a+\frac{1}{2}; & \\ & \frac{1}{4} \\ \frac{5}{2}; & \end{matrix}\right] = \frac{3\times 2^{2a}[a(3^{2-2a}+1)-2]}{(2a-1)(2a-2)(2a-3)}. \tag{2.19}
$$

**Remark 2.5.** If we replace $a$ by $2a$ and taking $b=1$ in (2.17) and apply (2.6), we easily obtain the following summation formula:

**Theorem 2.4.** *The following summation formula for Gauss's hypergeometric function hold true:*

$$
{}_2F_1\left[\begin{matrix} a, a+\frac{1}{2}; & \\ & \frac{x^2}{(1-x)^2} \\ \frac{3}{2}; & \end{matrix}\right] = \frac{(1-x)^{2a}[(1-2x)^{1-2a}-1]}{2x(2a-1)}. \tag{2.20}
$$

For instance, if we set $x=\frac{1}{3}$ in (2.20), we get

$$
{}_2F_1\left[\begin{matrix} a, a+\frac{1}{2}; & \\ & \frac{1}{4} \\ \frac{3}{2}; & \end{matrix}\right] = \frac{2^{2a-1}(1-3^{1-2a})}{2a-1}, \tag{2.21}
$$

which is a known result [6].

## 3. Summation theorems for Clausen's hypergeometric function

In this section, we will use the results (2.1),(2.2) and (2.6) to obtain four Summation formulas for Clausen's hypergeometric function.

**Theorem 3.1.** *The following summation formula for Clausen's hypergeometric function* ${}_3F_2(x)$ *holds ture:*

$$ {}_3F_2\left[\begin{matrix} a,1,3; \\ & x \\ 2,4; \end{matrix}\right] = \frac{3}{2(a-1)(a-2)(a-3)\,x^3} \times \left[2(2x^2-ax^2+ax-x-1)(1-x)^{1-a}-a^2x^2+5ax^2-6x^2+2\right]. \tag{3.1} $$

**Proof.** Setting $b=1, c=3$ in (2.12), we get

$$ {}_3F_2\left[\begin{matrix} a,1,3; \\ & x \\ 2,4; \end{matrix}\right] = \frac{1}{2}\left(3\,{}_2F_1\left[\begin{matrix} a,1; \\ & x \\ 2; \end{matrix}\right] - {}_2F_1\left[\begin{matrix} a,3; \\ & x \\ 4; \end{matrix}\right]\right). \tag{3.2} $$

Using (2.6) and (2.2) to the two ${}_2F_1$ in the right hand side of (3.2) respectively, we easily obtain the required result (3.1).

**Corollary 3.1.** Setting $a=\frac{1}{2},-\frac{1}{2},-\frac{3}{2},-\frac{5}{2}$ in (3.1), we get

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2},1,3; \\ & x \\ 2,4; \end{matrix}\right] = \frac{1}{5x^3}\left[15x^2-8-4(3x^2-x-2)(1-x)^{\frac{1}{2}}\right], \tag{3.3} $$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2},1,3; \\ & x \\ 2,4; \end{matrix}\right] = \frac{1}{35x^3}\left[35x^2-8-4(5x^2-3x-2)(1-x)^{\frac{3}{2}}\right], \tag{3.4} $$

$$ {}_3F_2\left[\begin{matrix} -\frac{3}{2},1,3; \\ & x \\ 2,4; \end{matrix}\right] = \frac{1}{105x^3}\left[63x^2-8-4(7x^2-5x-2)(1-x)^{\frac{5}{2}}\right], \tag{3.5} $$

$$ {}_3F_2\left[\begin{matrix} -\frac{5}{2},1,3; \\ & x \\ 2,4; \end{matrix}\right] = \frac{1}{231x^3}\left[99x^2-8-4(9x^2-7x-2)(1-x)^{\frac{7}{2}}\right]. \tag{3.6} $$

**Theorem 3.2.** *The following summation formulas for Clausen's hypergeometric function* ${}_3F_2(x)$ *hold ture:*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}(a+2),\frac{1}{2}(a+1),1; \\ & 4x(1-x) \\ a+1,2; \end{matrix}\right] = \frac{(1-x)^{1-a}-1}{(a-1)(1-x)x}, \tag{3.7} $$

$$ {}_3F_2\left[\begin{array}{ll} \frac{1}{2}(a+3), \frac{1}{2}(a+2), 2; & \\ & 4x(1-x) \\ a+2, 3; & \end{array}\right] = \frac{2\left[(ax-x-1)(1-x)^{1-a}+1\right]}{(a-1)(a-2)(1-x)^2x^2}, \tag{3.8} $$

$$ {}_3F_2\left[\begin{array}{ll} \frac{1}{2}(a+4), \frac{1}{2}(a+3), 3; & \\ & 4x(1-x) \\ a+3, 4; & \end{array}\right] = \frac{3\left[(2(1-ax+x)+x^2(a^2-3a+2))(1-x)^{1-a}-2\right]}{(a-1)(a-2)(a-3)(1-x)^3x^3}. \tag{3.9} $$

**Proof.** For prove (3.7), we consider the following result [11]:

$$ {}_4F_3\left[\begin{array}{ll} a, b, \frac{1}{2}(a+b), \frac{1}{2}(a+b+1); & \\ & 4x(1-x) \\ c, 1+a+b-c, a+b; & \end{array}\right] = {}_2F_1\left[\begin{array}{ll} a, b; & \\ & x \\ c; & \end{array}\right] {}_2F_1\left[\begin{array}{ll} a, b; & \\ & x \\ a+b+1-c; & \end{array}\right]. \tag{3.10} $$

Setting $c = a$ in (3.10) and using (2.5), we have

$$ {}_3F_2\left[\begin{array}{ll} \frac{1}{2}(a+b+1), \frac{1}{2}(a+b), b; & \\ & 4x(1-x) \\ a+b, b+1; & \end{array}\right] = (1-x)^{-b}\ {}_2F_1\left[\begin{array}{ll} a, b; & \\ & x \\ b+1; & \end{array}\right]. \tag{3.11} $$

Now, setting $b = 1$ and using (2.6) to the ${}_2F_1$ in the right hand side of (3.11), we easily obtain the required result(3.7). The results (3.8) and(3.9) can be obtained similarly. This completes the proof of Theorem 3.2.

## 4. Special cases

In this section, we apply the results of section three to evaluate some special values for the Clausen's hypergeometric function as follows:

**1.** Taking $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively in (3.3), we get

$$ {}_3F_2\left[\begin{array}{ll} \frac{1}{2}, 1, 3; & \\ & \frac{1}{2} \\ 2, 4; & \end{array}\right] = \frac{2}{5}(14\sqrt{2}-17), \tag{4.1} $$

$$ {}_3F_2\left[\begin{array}{ll} \frac{1}{2}, 1, 3; & \\ & \frac{1}{3} \\ 2, 4; & \end{array}\right] = \frac{1}{5}(72\sqrt{6}-17), \tag{4.2} $$

$$ {}_3F_2\left[\begin{array}{ll} \frac{1}{2}, 1, 3; & \\ & \frac{1}{4} \\ 2, 4; & \end{array}\right] = \frac{4}{5}(66\sqrt{3}-113). \tag{4.3} $$

**2.** Taking $x = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ respectively in (3.4), we get

$$ {}_3F_2\left[\begin{array}{ll} -\frac{1}{2}, 1, 3; & \\ & \frac{1}{3} \\ 2, 4; & \end{array}\right] = \frac{176\sqrt{6}}{105} - \frac{111}{35}, \tag{4.4} $$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, 1, 3; & \\ & \frac{1}{4} \\ 2, 4; & \end{matrix}\right] = \frac{6}{35}(39\sqrt{3}-62), \tag{4.5}$$

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, 1, 3; & \\ & \frac{1}{5} \\ 2, 4; & \end{matrix}\right] = \frac{384\sqrt{5}}{35} - \frac{165}{7}. \tag{4.6}$$

**3.** Taking $x = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ respectively in (3.5), we get

$$ {}_3F_2\left[\begin{matrix} -\frac{3}{2}, 1, 3; & \\ & \frac{1}{4} \\ 2, 4; & \end{matrix}\right] = \frac{27\sqrt{3}}{14} - \frac{52}{21}, \tag{4.7}$$

$$ {}_3F_2\left[\begin{matrix} -\frac{3}{2}, 1, 3; & \\ & \frac{1}{5} \\ 2, 4; & \end{matrix}\right] = \frac{8704\sqrt{5}}{2625} - \frac{137}{21}, \tag{4.8}$$

$$ {}_3F_2\left[\begin{matrix} -\frac{3}{2}, 1, 3; & \\ & \frac{1}{6} \\ 2, 4; & \end{matrix}\right] = \frac{475\sqrt{30}}{189} - \frac{90}{7}. \tag{4.9}$$

**4.** Taking $x = \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ respectively in (3.6), we get

$$ {}_3F_2\left[\begin{matrix} -\frac{5}{2}, 1, 3; & \\ & \frac{1}{5} \\ 2, 4; & \end{matrix}\right] = \frac{38912\sqrt{5}}{28875} - \frac{505}{231}, \tag{4.10}$$

$$ {}_3F_2\left[\begin{matrix} -\frac{5}{2}, 1, 3; & \\ & \frac{1}{6} \\ 2, 4; & \end{matrix}\right] = \frac{4375\sqrt{30}}{4158} - \frac{378}{77}, \tag{4.11}$$

$$ {}_3F_2\left[\begin{matrix} -\frac{5}{2}, 1, 3; & \\ & \frac{1}{7} \\ 2, 4; & \end{matrix}\right] = \frac{39744\sqrt{42}}{26411} - \frac{2051}{231}. \tag{4.12}$$

**5.** Taking $a = \frac{1}{2}, x = \frac{1}{3}$ in (3.6), (3.7) and (3.9) respectively, we obtain

$$ {}_3F_2\left[\begin{matrix} \frac{5}{4}, \frac{3}{4}, 1; & \\ & \frac{8}{9} \\ \frac{3}{2}, 2; & \end{matrix}\right] = 9 - 3\sqrt{6}, \tag{4.13}$$

$$ {}_3F_2\left[\begin{matrix} \frac{7}{4}, \frac{5}{4}, 2; & \\ & \frac{8}{9} \\ \frac{5}{2}, 3; & \end{matrix}\right] = 54 - 21\sqrt{6}, \tag{4.14}$$

$$ {}_3F_2\left[\begin{matrix} \frac{9}{4}, \frac{7}{4}, 3; & \\ & \frac{8}{9} \\ \frac{7}{2}, 4; & \end{matrix}\right] = \frac{1458}{5} - \frac{2349\sqrt{6}}{20}. \tag{4.15}$$

**6.** Taking $a = -\frac{1}{2}, x = \frac{1}{3}$ in (3.6), (3.7) and (3.9) respectively, we obtain

$$ {}_3F_2\left[\begin{matrix} \frac{3}{4}, \frac{1}{4}, 1; \\ \\ \frac{1}{2}, 2; \end{matrix} \; \frac{8}{9}\right] = 3 - \frac{2\sqrt{6}}{3}, \tag{4.16} $$

$$ {}_3F_2\left[\begin{matrix} \frac{5}{4}, \frac{3}{4}, 2; \\ \\ \frac{3}{2}, 3; \end{matrix} \; \frac{8}{9}\right] = \frac{18}{5}(3 - \sqrt{6}), \tag{4.17} $$

$$ {}_3F_2\left[\begin{matrix} \frac{7}{4}, \frac{5}{4}, 3; \\ \\ \frac{5}{2}, 4; \end{matrix} \; \frac{8}{9}\right] = \frac{1458}{35} - \frac{1107\sqrt{6}}{70}. \tag{4.18} $$

**7.** Taking $a = -\frac{3}{2}, x = \frac{1}{3}$ in (3.6), (3.7) and (3.9) respectively, we obtain

$$ {}_3F_2\left[\begin{matrix} \frac{1}{4}, -\frac{1}{4}, 1; \\ \\ -\frac{1}{2}, 2; \end{matrix} \; \frac{8}{9}\right] = \frac{9}{5} - \frac{4\sqrt{6}}{15}, \tag{4.19} $$

$$ {}_3F_2\left[\begin{matrix} \frac{3}{4}, \frac{1}{4}, 2; \\ \\ \frac{1}{2}, 3; \end{matrix} \; \frac{8}{9}\right] = \frac{2}{35}(81 - 22\sqrt{6}), \tag{4.20} $$

$$ {}_3F_2\left[\begin{matrix} \frac{5}{4}, \frac{3}{4}, 3; \\ \\ \frac{3}{2}, 4; \end{matrix} \; \frac{8}{9}\right] = \frac{1}{35}(486 - 167\sqrt{6}). \tag{4.21} $$

**8.** Taking $a = -\frac{5}{2}, x = \frac{1}{3}$ in (3.6), (3.7) and (3.9) respectively, we obtain

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{4}, -\frac{3}{4}, 1; \\ \\ -\frac{3}{2}, 2; \end{matrix} \; \frac{8}{9}\right] = \frac{9}{7} - \frac{8\sqrt{6}}{63}, \tag{4.22} $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{4}, -\frac{1}{4}, 2; \\ \\ -\frac{1}{2}, 3; \end{matrix} \; \frac{8}{9}\right] = \frac{18}{7} - \frac{104\sqrt{6}}{189}, \tag{4.23} $$

$$ {}_3F_2\left[\begin{matrix} \frac{3}{4}, \frac{1}{4}, 3; \\ \\ \frac{1}{2}, 4; \end{matrix} \; \frac{8}{9}\right] = \frac{2}{77}(243 - 73\sqrt{6}). \tag{4.24} $$

**Remark 4.1.** Similarly many other new special values for the Clausen's hypergeometric function may be obtained from our main results (3.1),(3.7),(3.8) and (3.9).

## 5. conclusion

In this paper, we established some summation formulas for Gauss's and Clausen's hypergeometric functions with the help of general hypergeometric identity of Masjed-Jamei and Koepf [5]. As an applications of our main formulas, we evaluated many special values for the Clausen's hypergeometric function with the different arguments . Our main results of this paper can be applied to derive further summation formulas and special values for Gauss's and Clausen's hypergeometric functions and other generalized hypergeometric function given in the literature.

## References

[1] Chen K.W. (2021), Clausen's series ${}_3F_2(1)$ with integral parameter differences, *symmetry*. **13**: pp.1783. https://doi.org/10.3390/sym13101783.

[2] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. (1953), Higher Transcendental Functions Volume I.; McGraw-Hill Book Company: New York, NY, USA; Toronto, ON, Canada; London, UK.

[3] Kim Y.S., Rathie A.K. and Wang X. (2015), $\pi$ and other formulae implied by hypergeometric summation theorems, *Commun. Korean Math. Soc.* **30**: 227–237.http://dx.doi.org/10.4134/CKMS.2015.30.3.227.

[4] Lavoie J.L. (1987), Some summation formulae for the series ${}_3F_2$, *Math. Comput.* **49**: 269–274.

[5] Masjed-Jamei M. and Koepf W. (2018), Some summation theorems for generalized hypergeometric functions, *Axioms*, **7** 38: 1–20. https://doi.org/10.3390/axioms7020038.

[6] Masjed-Jamei M. and Koepf W. (2011), Computing two special cases of Gauss hypergeometric function, *Creat. Math. Inform.*, **2**: 137–146.

[7] Olver F.W.J., Lozier D.W., Boisvert R.F and Clark C.W.(Eds.) (2010), NIST Handbook of Mathematical Functions, Cambridge University Press.

[8] Prudnikov A.P., Brychkov Y.A. and Marichev O.I. (1990), Integrals and Series, Vol. 3: More Special Functions, Gordon and Breach Science Publishers, New York.

[9] Qureshi M.I., Jabee S. and Shadab M. (2018), Truncated Gauss hypergeometric series and its application in digamma function. https://arxiv.org/abs/1805.12548.

[10] Qureshi M.I. and Shadab M. (2020), Some Summation Theorems for Clausen's Hypergeometric functions with Unit Argument, *Appl. Appl. Math.* **15** (1): 469–478. https://digitalcommons.pvamu.edu/aam/vol15/iss1/27.

[11] Rainville E.D. (1960), Special Functions, The Macmillan, New York, Reprintedby Chelsea Publishing Company, Bronx, New York, (1971).

[12] Shpot M.A. and Srivastava H.M. (2015), The Clausenian hypergeometric function ${}_3F_2$ with unit argument and negative integral parameter differences, *Appl. Math. Comput.* **259**: 819–827. http://dx.doi.org/10.1016/j.amc.2015.03.031.

[13] Srivastava H.M. and Manocha H.L. (1984), A Treatise on generating functions, Halsted Press, New York.