


# Insight to Multi-derivative Hybrid Linear Multistep Formula for directly Solving Third-order Initial Value Problems of Ordinary Differential Equations

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## Abstract

This article focuses on the multi-derivative hybrid linear multistep formula (MHLMF) for the numerical solution of third-order ordinary differential equations. Power series was used as the basis function in the derivation of the formula. An approximate solution from the basis function was interpolated at some selected off-grid points. In contrast, the third derivative of the approximate solution was located at all points in the grid and outside the grid to generate a system of linear equations to determine the unknown parameters. The derived method was examined to be consistent, convergent, and zero stable. The method was implemented to solve third-order ordinary differential equations, including the Genesio equation, demonstrating that the derived methods efficiently handle non-linear problems. Absolute errors obtained in the numerical experiments established the good performance of the proposed method when compared with other cited methods in the literature.

Keywords: Third-order, , hybrid, interpolation and collocation, linear multistep, Genesio equation.

## 1. Introduction

The solution to differentiation has evolved over the years with different authors bringing their approach to see which method will perform better than the others. Recently, attention has been focused on the numerical solutions to third-order ordinary differential equations. Hence, the concern of this article is to discuss the approximate solution to general third-order ordinary differential equations using the MHLMF. The third-order ODEs is of the form:

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(a) = y_0, y'(a) = \delta_0, y'(b) = y_N \quad (1.1)$$

where  $a, b, y_0, \delta_0, y_N, y_M \in \mathfrak{R}$  and  $f$  is a continuous function and satisfies a Lipschitz condition as given in [1].

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The solution of (1) have been considered by several authors such as [2], [3], [4], [9], [10], and a host of others. In particular, On the solution of singular initial value problems in ordinary differential equations using a new third order inverse Runge-Kutta method [11], a block multistep method for the direct solution of third-order ODEs was suggested in [3], while a sixth-order linear multistep method for third-order ordinary differential equations was considered in [1], [4] considered a Fully implicit Block-Predictor Corrector method for the Numerical Integration of  $y''' = f(x, y, y', y'')$   $y(a) = \psi_1, y'(a) = \psi_2, y''(a) = \psi_3$ . In particular, [5] in their work considered Existence and Solution of Third-Order Integro-Differential Equations (IDE) via Haar Wavelet Method (HWC) where they established some necessary conditions for the existence and uniqueness of a solution to the Third-Order ODEs. Upon the use of the HWC method, computational results about the numerical solution to linear IDEs have been provided. Meanwhile [6] have used Mathematical Analysis of Coupled Systems with Fractional Order Boundary Conditions to successfully established sufficient conditions for the existence and uniqueness of the problem (1) with the help of the mentioned standard fixed point theorems to derived necessary conditions for Ulam stability and its various kinds to the solutions of (1). It is worthy of note that the work by [7] and [8], On Qualitative Analysis of Boundary Value Problem of Variable Order Fractional Delay Differential Equations and the Study of chronic myeloid leukemia with T-cell under fractal-fractional order model have provided more insight to the solution of (1).

In recent times, tremendous attention has been shifted to developing methods for the solution of (1) without reduction to equivalent first order by several authors (see [2], [12], [13], [14], [15], [16]). Many other approaches have been considered by [[17], [18], [19], [20], [21], [23], [24], [25], [26], [27], [28], [29], [29]] with each coming out with varying degree of successes in their approach.

The search for more accurate numerical methods has led to the successful introduction of hybrid linear multistep methods. This work proposes a multi-derivative hybrid linear multistep formula, implemented in block mode, for directly solving (1). The paper is arranged as follows: Section 1 is an introduction, Section 2 discusses how the methods were derived, and Section 3 analyses the method's basic properties. Section 4 gives numerical results to show the advantages of speed and accuracy. Section 5 provides a summary and conclusion.

## 2. Derivation of the Block Method

The solution of (1.1) is considered in the interval [0,3] by allowing  $y(x)$  to be approximated by partial sum of power series polynomial of the form

$$y(x) \approx p(x) = \sum_{j=0}^{14} a_j x^j \quad (2.1)$$

where  $x$  is continuously differentiable and  $a_j$ 's are parameters to be determine. In order to apply the procedure of collocation and interpolation to (2.1), the third, fourth and fifth

derivatives of equation (2.1) are obtained to be

$$y'''(x) \approx p'''(x) = \sum_{j=3}^{14} j(j-1)(j-2)\alpha_j x^{j-3}, \quad (2.2)$$

$$y^{(4)}(x) = \sum_{j=4}^{14} j(j-1)(j-2)(j-3)\alpha_j x^{j-4}, \quad (2.3)$$

and

$$y^{(5)}(x) = \sum_{j=5}^{14} j(j-1)(j-2)(j-3)(j-4)\alpha_j x^{j-5}. \quad (2.4)$$

respectively. Note that the following are equivalent;

$$y'''(x, y(x), y'(x), y''(x)) \approx f(x, y(x), y'(x), y''(x)),$$

$$y^{(4)}(x, y(x), y'(x), y''(x)) \approx g(x, y(x), y'(x), y''(x)),$$

and

$$y^{(5)}(x, y(x), y'(x), y''(x)) \approx \gamma(x, y(x), y'(x), y''(x)).$$

Interpolating (2.1) at  $x = x_{n+j}, j = 0, 1, \frac{5}{2}$  and collocating (2.2) to (2.4) at  $x = x_{n+j}, j = 0, 1, 2,$  and  $3$  yield the system of equations that can be written in matrix form

$$XA = B \quad (2.5)$$

where,

$$X = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & \dots & x_n^{14} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & \dots & x_{n+1}^{14} \\ 1 & x_{n+\frac{5}{2}} & x_{n+\frac{5}{2}}^2 & x_{n+\frac{5}{2}}^3 & x_{n+\frac{5}{2}}^4 & x_{n+\frac{5}{2}}^5 & x_{n+\frac{5}{2}}^6 & x_{n+\frac{5}{2}}^7 & \dots & x_{n+\frac{5}{2}}^{14} \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & \dots & 2184x_n^{11} \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & \dots & 2184x_{n+1}^{11} \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & \dots & 2184x_{n+2}^{11} \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & \dots & 2184x_{n+3}^{11} \\ 0 & 0 & 0 & 0 & 24 & 60x_n & 120x_n^2 & 360x_n^3 & \dots & 24024x_n^{10} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+1} & 120x_{n+1}^2 & 360x_{n+1}^3 & \dots & 24024x_{n+1}^{10} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+2} & 120x_{n+2}^2 & 360x_{n+2}^3 & \dots & 24024x_{n+2}^{10} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+3} & 120x_{n+3}^2 & 360x_{n+3}^3 & \dots & 24024x_{n+3}^{10} \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_n & 720x_n^2 & \dots & 240240x_n^9 \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+1} & 720x_{n+1}^2 & \dots & 240240x_{n+1}^9 \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+2} & 720x_{n+2}^2 & \dots & 240240x_{n+2}^9 \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+3} & 720x_{n+3}^2 & \dots & 240240x_{n+3}^9 \end{pmatrix},$$

$$B = \left( y_n, y_{n+1}, y_{n+\frac{5}{2}}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, g_n, g_{n+1}, g_{n+2}, g_{n+3}, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \gamma_{n+3} \right)$$

$A = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots, a_{14})$ . Solving equation (2.5) for  $a_j$ 's using Gaussian elimination methods and substitute into equation (2.1), yields after simple simplification the continuous scheme

$$y(t) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_{\frac{5}{2}} y_{n+\frac{5}{2}} + h^3 \sum_{j=0}^3 \beta_j f_{n+j} + h^4 \sum_{j=0}^3 \mu_j g_{n+j} + h^5 \sum_{j=0}^3 \nu_j \gamma_{n+j} \quad (2.6)$$

where  $\alpha$ ,  $\beta$ ,  $\mu$ , and  $\nu$  are the coefficients that define the scheme. The coefficients are presented in the appendix. Evaluating (2.6) and its first and second derivatives at  $t = 1, 2, 3$  yields the proposed multi-derivative hybrid Linear multistep method whose formulas are as presented in equations (2.7), (2.8), and (2.9).

$$\left. \begin{aligned} & y_3 - \frac{2y_0}{5} - \frac{8y_{\frac{5}{2}}}{5} + y_1 = \\ & + h^5 \left( \frac{217289\gamma_0}{2724986880} + \frac{434839\gamma_1}{100925440} + \frac{529079\gamma_2}{60555264} + \frac{472159\gamma_3}{2724986880} \right) \\ & + h^4 \left( \frac{3393371g_0}{2724986880} + \frac{4508983g_1}{302776320} - \frac{1395451g_2}{100925440} - \frac{854869g_3}{302776320} \right) \\ & + h^3 \left( \frac{9610663f_0}{1634992128} + \frac{11445965f_1}{60555264} + \frac{17640923f_2}{60555264} + \frac{22539425f_3}{1634992128} \right) \end{aligned} \right\}, \quad (2.7)$$

and additional schemes

$$\left. \begin{aligned} & hy'_0 - \frac{5}{3} y_1 + \frac{7}{5} y_0 + \frac{4}{15} y_{\frac{5}{2}} = \\ & + h^5 \left( \frac{3881737}{42509795328} \gamma_3 + \frac{4098617}{6072827904} \gamma_0 + \frac{3703591}{224919552} \gamma_1 - \frac{2242973}{674758656} \gamma_2 \right) \\ & + h^4 \left( -\frac{15164521}{14169931776} g_3 + \frac{498637229}{42509795328} g_0 - \frac{38405519}{4723310592} g_1 + \frac{2364085}{224919552} g_2 \right) \\ & + h^3 \left( -\frac{32097311}{4723310592} f_2 + \frac{448912339}{127529385984} f_3 + \frac{8336270981}{127529385984} f_0 + \frac{1674766231}{4723310592} f_1 \right) \\ \\ & hy'_1 - \frac{3}{5} y_0 + \frac{1}{3} y_1 + \frac{4}{15} y_{\frac{5}{2}} = \\ & + h^5 \left( -\frac{2873987}{7872184320} \gamma_2 - \frac{28051}{3373793280} \gamma_3 - \frac{224633}{2146959360} \gamma_0 - \frac{8685073}{1574436864} \gamma_1 \right) \\ & + h^4 \left( -\frac{6802337}{1124597760} g_1 + \frac{932939}{10121379840} g_3 + \frac{3823459}{715653120} g_2 - \frac{121305803}{70849658880} g_0 \right) \\ & + h^3 \left( -\frac{118610333}{14169931776} f_0 - \frac{581485}{2024275968} f_3 - \frac{57813323}{1574436864} f_2 - \frac{322164701}{1574436864} f_1 \right) \\ \\ & hy'_2 - \frac{1}{5} y_0 + y_1 - \frac{4}{5} y_{\frac{5}{2}} = \\ & + h^5 \left( \frac{158935}{674758656} \gamma_2 - \frac{1217927}{212548976640} \gamma_3 + \frac{6000287}{212548976640} \gamma_0 + \frac{746639}{306708480} \gamma_1 \right) \\ & + h^4 \left( \frac{167031649}{23616552960} g_1 + \frac{5603879}{70849658880} g_3 - \frac{44526653}{7872184320} g_2 + \frac{103890653}{212548976640} g_0 \right) \\ & + h^3 \left( \frac{319769713}{127529385984} f_0 - \frac{5542015}{18218483712} f_3 - \frac{43856803}{4723310592} f_2 + \frac{427059515}{4723310592} f_1 \right) \\ \\ & hy'_3 - y_0 + \frac{7}{3} y_1 - \frac{4}{3} y_{\frac{5}{2}} = \\ & + h^5 \left( \frac{726667859}{23616552960} \gamma_{+2} + \frac{38652115}{42509795328} \gamma_3 + \frac{1017535}{3864526848} \gamma_0 + \frac{63187031}{7872184320} \gamma_1 \right) \\ & + h^4 \left( \frac{614611703}{23616552960} g_1 - \frac{1103626687}{70849658880} g_3 - \frac{6606427}{238551040} g_2 + \frac{821538331}{212548976640} g_n \right) \\ & + h^3 \left( \frac{2190417767}{127529385984} f_0 + \frac{10750072033}{127529385984} f_3 + \frac{4120540315}{4723310592} f_2 + \frac{2091538957}{4723310592} f_1 \right) \end{aligned} \right\} \quad (2.8)$$

$$\left. \begin{aligned}
 &h^2 y_0'' + \frac{4}{3} y_1 - 4/5 y_0 - \frac{8}{15} y_{\frac{5}{2}} = \\
 &+ h^5 \left( -\frac{33730349}{106274488320} \gamma_3 - \frac{281826763}{106274488320} \gamma_0 - \frac{190232149}{3936092160} \gamma_1 + \frac{2343369}{187432960} \gamma_2 \right) \\
 &+ h^4 \left( \frac{35984341}{9661317120} g_3 - \frac{600560537}{11808276480} g_0 + \frac{38401309}{1686896640} g_1 - \frac{108009689}{2361655296} g_2 \right) \\
 &+ h^3 \left( \frac{961262747}{11808276480} f_2 - \frac{355504573}{28983951360} f_3 - \frac{117134762137}{318823464960} f_0 - \frac{1464917941}{1686896640} f_1 \right) \\
 \\
 &h^2 y_1'' - \frac{4}{5} y_0 + \frac{4}{3} y_1 - \frac{8}{15} y_{\frac{5}{2}} = \\
 &+ h^5 \left( -\frac{10159285}{2361655296} \gamma_2 + \frac{2183921}{35424829440} \gamma_3 + \frac{6953831}{35424829440} \gamma_0 + \frac{2584759}{1686896640} \gamma_1 \right) \\
 &+ h^4 \left( -\frac{582386741}{11808276480} g_1 - \frac{78691801}{106274488320} g_3 + \frac{44449637}{1686896640} g_2 + \frac{28872509}{9661317120} g_0 \right) \\
 &+ h^3 \left( \frac{131524231}{9661317120} f_0 + \frac{264592459}{106274488320} f_3 - \frac{196484659}{1686896640} f_2 - \frac{782804467}{11808276480} f_1 \right) \\
 \\
 &h^2 y_2'' - \frac{4}{5} y_0 + \frac{4}{3} y_1 - \frac{8}{15} y_{\frac{5}{2}} = \\
 &+ h^5 \left( \frac{7534789}{1073479680} \gamma_2 - \frac{65327}{1073479680} \gamma_3 + \frac{871373}{11808276480} \gamma_0 + \frac{151772417}{11808276480} \gamma_1 \right) \\
 &+ h^4 \left( \frac{10835725}{214695936} g_1 + \frac{27434509}{35424829440} g_3 - \frac{867204157}{11808276480} g_2 + \frac{7457251}{5060689920} g_0 \right) \\
 &+ h^3 \left( \frac{889659341}{106274488320} f_0 - \frac{292514741}{106274488320} f_3 + \frac{4590646427}{11808276480} f_2 + \frac{471203143}{1073479680} f_1 \right) \\
 \\
 &h^2 y_3'' - \frac{4}{5} y_0 + \frac{4}{3} y_1 - \frac{8}{15} y_{\frac{5}{2}} = \\
 &+ h^5 \left( \frac{74630271}{1312030720} \gamma_2 + \frac{42317269}{15182069760} \gamma_3 + \frac{48124469}{106274488320} \gamma_0 - \frac{3110417}{787218432} \gamma_1 \right) \\
 &+ h^4 \left( -\frac{255231029}{11808276480} g_1 - \frac{1128067781}{21254897664} g_3 - \frac{16008253}{11808276480} g_2 + \frac{70124647}{11808276480} g_0 \right) \\
 &+ h^3 \left( \frac{7373305703}{318823464960} f_0 + \frac{17228216791}{45546209280} f_3 + \frac{14062267547}{11808276480} f_2 + \frac{2846579213}{11808276480} f_1 \right)
 \end{aligned} \right\} \quad (2.9)$$

### 3. Analysis of the method

This section presents the analysis of the basic properties of the proposed method, including order, error constant, consistency, zero stability, and convergence.

#### 3.1. Local truncation error and order

In line with [1], the linear difference operator L associated with the (MHLMF) be defined as

$$L[y(x_n); h] = \sum_{j=0}^k \left( \alpha_j y(x_n + jh) - h^3 \beta_j y'''(x_n + jh) - h^4 \gamma_j y^{iv}(x_n + jh) - h^5 \mu_j y^v(x_n + jh) \right) \quad (3.1)$$

where  $y(x)$  is an arbitrary test function that is continuously differentiable in the interval  $[a, b]$ ,  $\alpha_j$ ,  $\beta_j$ ,  $\gamma_j$  and  $\mu_j$  are the continuous coefficients. Expanding  $(x_n + jh)$ ,  $y'''(x_n + jh)$ ,  $y^{iv}(x_n + jh)$  and  $y^v(x_n + jh)$  for  $j = 0, 1, \dots, m$  in Taylor's series about  $x_n$  and collecting the like terms in  $h$  and  $y$  gives

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + C_3 h^3 y'''(x) + \dots + C_p h^p y^{(p)}(x) + \dots \quad (3.2)$$

The difference operator  $L$  and the associated multi-derivative linear multistep methods are said to be of order  $p$  if  $C_0 = C_1 = C_2 = \dots = C_p = C_{p+1} = C_{p+2} = 0$  and  $C_{p+3} \neq 0$  while the term  $C_{p+3}$  is called the **error constant** and the **local truncation** is given by  $\tau_k = C_{p+3}h^{(p+3)}y^{(p+3)}(x_n) + O(h^{(p+4)})$ . Thus the order and error constants associated with the methods developed are given below.

Table 1: Order and error (MHLMF)

Scheme	Order	Error Constant
(2.7)	12	$-\frac{7100795478769}{23567430529843200}$
(2.8)	12	$-\frac{251}{57919706}$
	12	$-\frac{149208377707}{117837152649216000}$
	12	$-\frac{140999}{176755728973824000}$
	12	$-\frac{55841953}{176755728973824000}$
(2.9)	12	$-\frac{67782419}{88377864486912000}$
	12	$-\frac{9997549}{88377864486912000}$
	12	$-\frac{4474643}{88377864486912000}$
	12	$-\frac{2932213}{3535114579476480}$

### 3.2. Consistency

MHLMF is said to be consistent if the following conditions are satisfied according to [26] :

1. the order  $\rho \geq 1$
2.  $\sum_{j=0}^k \alpha_j = 0$
3.  $\rho(1) = \rho'(1) = 0$
4.  $\rho'''(1) = 3!\sigma(1)$

The consistency of (MHLMF) examine as follows;

1. The order of MHLMF is  $p = 12$
2. The  $\alpha$ 's are;  $\alpha_0 = -\frac{2}{3}, \alpha_1 = 1, \alpha_{\frac{5}{2}} = -\frac{8}{5}, \alpha_3 = 1,$

$$\sum_{j=0}^k \alpha_j = \alpha_0 + \alpha_1 + \alpha_{\frac{5}{2}} + \alpha_3 = -\frac{2}{3} + 1 - \frac{8}{5} + 1 = 0$$

3. Also,  $\rho(r)$  is the first characteristic polynomial here  
 $\rho(r) = r^3 - \frac{8}{5}r^{\frac{5}{2}} + r - \frac{2}{3} = 0$ , when  $r = 1$   $\rho'(1) = 3r^2 - 4r^{\frac{3}{2}} + 1 = 0$ , when  $r = 1$

4. Again,  $\rho'''(1) = 6 - 3r^{-\frac{1}{2}} = 3,$   
 $\sigma(r) = \frac{22539425}{1634992128}r^3 + \frac{17640923}{60555264}r^2 + \frac{11445965}{60555264}r + \frac{9610663}{1634992128}$ , when  $r = 1, 3!\sigma(1) = 3$ , hence,  
 $\rho'''(1) = 3!\sigma(1)$

Thus, conditions 1 - 4 are satisfied. This implies that MHLMF is consistent.

### 3.3. Zero Stability

**Definition 3.1.** The linear multistep method is said to be zero stable if no root of the first characteristic polynomial  $\rho(r)$  has a modulus greater than one and if every root of modulus one has multiplicity not more significant than one (see [27]).

$$\bar{\rho}(r) = \det[z\bar{A} - \bar{E}] \quad (3.3)$$

satisfies  $|r_s| \leq 1$  and every root with  $|r_s| = 1$  has multiplicity not exceeding three in the limit as  $h \rightarrow 0$ . The first characteristic polynomial for equation (2.7) is given by:

$$\rho(r) = r^3 - \frac{2}{5} + r - \frac{8r^{\frac{5}{2}}}{5} \quad (3.4)$$

equating equation (3.4) to zero and solving for  $r$  gives

$$\begin{aligned} r^3 - \frac{2}{5} + r - \frac{8r^{\frac{5}{2}}}{5} &= 0 \\ \implies r &= 1 \end{aligned}$$

The root  $r$  of equation (3.4) for which  $|r| = 1$  is simple, hence the method is zero stable as  $h \rightarrow 0$  by definition given by equation (3.3).

### 3.4. Region of absolute stability

The region of absolute stability of MHLMF

$$\begin{aligned} y_3 &= \frac{2y_0}{5} - y_1 + \frac{8y_{\frac{5}{2}}}{5} \\ &+ \frac{(48053315f_0 + 1545205275f_1 + 2381524605f_2 + 112697125f_3) h^3}{8174960640} \\ &+ \frac{(-113031531g_2 - 23081463g_3 + 10180113g_0 + 121742541g_1) h^4}{8174960640} \\ &+ \frac{(35221959\gamma_1 + 71425665\gamma_2 + 1416477\gamma_3 + 651867\gamma_0) h^5}{8174960640} \end{aligned}$$

is obtained by first considering the following two characteristic polynomials:

$$P(r) = r^3 + r - \frac{8r^{\frac{5}{2}}}{5} - \frac{2}{5} \quad (3.5)$$

and

$$Q(r) = \frac{8275649}{743178240}r^3 + \frac{86663657}{302776320}r^2 + \frac{12608665}{60555264}r + \frac{1682437}{233570304} \quad (3.6)$$

The stability polynomial of MHLMF is given by

$$\begin{aligned} \Pi(r, z) &= P(r) + zQ(r) \\ &= r^3 + r - \frac{8r^{\frac{5}{2}}}{5} - \frac{2}{5} - z \left( \frac{8275649}{743178240}r^3 + \frac{86663657}{302776320}r^2 + \frac{12608665}{60555264}r + \frac{1682437}{233570304} \right). \end{aligned} \quad (3.7)$$

Using  $r = e^{I\theta}$  in (3.7) and setting to zero yields the following expression for  $z$  after simplification.

$$z = -\frac{1634992128 \left( 8 (e^{I\theta})^{\frac{5}{2}} - 5 (e^{I\theta})^3 - 5 e^{I\theta} + 2 \right)}{91032139 (e^{I\theta})^3 + 2339918739 (e^{I\theta})^2 + 1702169775 e^{I\theta} + 58885295} \quad (3.8)$$

(3.8) is then plotted using the following MapleSoft codes

```
complexplot(z, theta = 0..2pi, filled = true, labels = ["Re", "Im"], color = grey)
```

Figure 1: Region of absolute stability of MHLMF

#### 4. Numerical Examples

In this section, we test some linear and nonlinear numerical examples to illustrate the accuracy of the methods. The maximum absolute error is computed as  $AERR = \max |y(x_i) - y_i|$ ,  $i = 1, \dots, N$ , where  $y(x_i)$  is the exact solution computed at the grid point and  $y_i$  is an approximation to the exact solution using the MHLMF. For each example, we find the absolute errors of the approximate solutions and were compared with various existing methods in the literature. The accuracy of our method is seen in the small error values obtained.

**Problem 1:** The first test example considered is the linear third-order ODE

$$y''' = -y, \quad y(0) = 1, y'(0) = -1, y''(0) = 1, \quad 0 \leq x \leq 1, \quad h = 0.1$$

whose exact solution is  $y(x) = e^{-x}$ . The numerical solution was obtained in the interval  $[0,1]$  over ten iterations. The absolute errors of MHLMF are presented in Table 2 and Figure 2 as compared with those of [27]. Figure 2, illustrates that the absolute errors

Table 2: Comparison of absolute error of problem 1 using  $h = 0.1$

N	AMHLMF	Error in [27]
0.1	1.40E - 25	2.8160E - 24
0.2	9.50E - 25	1.1025E - 23
0.3	2.475E - 24	2.4162E - 23
0.4	5.339E - 24	1797E - 23
0.5	1.0068E - 23	6.3522E - 23
0.6	1.6688E - 23	8.8946E - 23
0.7	2.5649E - 23	1.1768E - 22
0.8	3.7323E - 23	1.4936E - 22
0.9	5.1703E - 23	1.8358E - 22
1	6.9087E - 23	2.1997E - 22

associated with the proposed method, indicated by a blue dotted line, are significantly

Figure 2: Graph of comparison of results in Table 2

smaller than those in [27], represented by a pale yellow line.

**Problem 2:** The second test example considered is the oscillatory problem

$$y''' = 3\sin x, \quad y(0) = 1; \quad y'(0) = 0; \quad y''(0) = -2; \quad h = 0.1$$

with the theoretical solution  $y(x) = 3\cos x + \frac{x^2}{2} - 2$ , The proposed method MHLMF was applied to solve the second example in  $[0,1]$  over ten iterations and the absolute error AERR are compared with those of [26] in the Table 3 and Figure 3.

Table 3: Comparison of absolute error of problem 2 using  $h = 0.1$ 

N	AMHLMF	Error in [26]
0.1	6.8003E - 26	5.5511E - 17
0.2	4.61461E - 25	8.3266E - 17
0.3	1.205241E - 24	5.5511E - 17
0.4	2.794291E - 24	2.7755E - 16
0.5	6.085563E - 24	2.2204E - 16
0.6	1.1151435E - 23	2.2204E - 16
0.7	1.9127036E - 23	6.6613E - 16
0.8	3.1460256E - 23	1.6653E - 15
0.9	4.8264535E - 23	
1	7.1213798E - 23	

Figure 3: Graph of comparison of results in Table 3

Figure 3 indicates that the absolute errors in the proposed method (blue dotted line) are substantially smaller than those in [26] (light yellow).

**Problem 3:** The linear general third-order differential equation

$$y'''(x) - 2y''(x) - 3y'(x) + 10y(x) = 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34, \quad x \in [0, b]$$

subject to the initial conditions  $y(0) = 3$ ,  $y'(0) = 0$ ,  $y''(0) = 0$ ; with the exact solution  $y(x) = x^2e^{-2x} - x^2 + 3$  was considered as the third test example. The solution of test example three was considered within  $[0, N]$  for  $N = 10, 20, 30$  and the absolute errors at  $N$  were reported in Table 4 as compared with those in [24].

**Problem 4:** Application to solve nonlinear Genesio equation as seen in [26]. The chaotic Genesio equation is given as:

$$y''' = -\alpha y'' - \beta y' + f(y(x))$$

Table 4: Comparison of absolute error of problem 3

steps(N)	AERR at N with MHLMF	AERR at N with [24]
10	1.361E – 16	1.18E – 6
20	2.365E – 20	3.92E – 16
30	1.440E – 22	3.73E – 16

where,  $f(y(x)) = -\gamma y(x) + y^2(x)$   $y(0) = 0.2$ ,  $y'(0) = -0.3$ ,  $y'' = 0.1$ ,  $x \in [a, b]$ ,  $\alpha = 1.2$ ,  $\beta = 2.92$  and  $\gamma = 6$  are positive constants that satisfied  $\alpha\beta < \gamma$ . The solution of the Genesio equation is considered in  $[0,1]$  over ten iterations. The results are presented in Table 5 and further in Figure 4.

Table 5: Solution of Genesio equation

N	MHLMF Computed
0.1	0.170608965622204597464290143315
0.2	0.142835716947868036239738312383
0.3	0.117198058396711290474008784883
0.4	0.0941040392842143783332605850347
0.5	0.0738532551967699857088374418123
0.6	0.0566414259515802265052717976815
0.7	0.0425664447856751687838129127262
0.8	0.0316354204306159584767520404984
0.9	0.0237742550763226713930282749865
1.0	0.0188376906220213027033615472421

Figure 4: Graph of results displayed in Table 5

**Problem 5:** Application to a boundary layer problem

$$2y''' + yy'' = 0, y(0) = y'(0) = 0; y''(0) = 1$$

There is no known exact solution for the problem. The solution of the boundary layer problem is considered in  $[0,1]$  over ten iterations. The results are presented in Table 6 and further in Figure 5.

Figure 5: Graph of results displayed in Table 6

**5. Conclusion**

This work has produced a new numerical model achieved by constructing a hybrid linear multistep formula that was implemented in block mode for directly solving third-

Table 6: Solution of Boundary layer problem

N	MHLMF Computed
0.1	-0.0949975003536099778090573608
0.2	-0.179965605999674792269966544
0.3	-0.254834841094583495255704304
0.4	-0.319495344819711343873291037
0.5	-0.373799412984372654284932544
0.6	-0.417565690279847931050392155
0.7	-0.450583729019015522970976176
0.8	-0.472617841878174458075630102
0.9	-0.483411404106754455241600175
1.0	-0.482692361291772161598468436

order ordinary differential equations with high levels of precision and minimum error margin, as seen in the results displayed. The method is zero and p-stable, consistent, and convergent. The numerical results obtained from solved problems show the method's efficiency and accuracy advantages over existing methods in the literature. The results in Tables 2-6 and Figures 2-5 show that the MHLMF is better in accuracy and can compete well with others in the literature.

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## Appendix

$$\begin{aligned}
\alpha_0 &= 1 - \frac{7}{5}t + \frac{2}{5}t^2 \\
\alpha_1 &= \frac{5}{3}t - \frac{2}{3}t^2 \\
\alpha_2 &= -\frac{4}{15}t + \frac{4}{15}t^2 \\
\beta_0 &= -\frac{103}{2830464}t^{14} + \frac{607}{741312}t^{13} - \frac{929}{114048}t^{12} + \frac{149}{3168}t^{11} - \frac{54019}{311040}t^{10} + \frac{30815}{72576}t^9 \\
&\quad - \frac{297221}{435456}t^8 + \frac{12301}{18144}t^7 - \frac{4333}{12960}t^6 + \frac{1}{6}t^3 - \frac{117134762137}{637646929920}t^2 + \frac{8336270981}{127529385984}t \\
\beta_1 &= \frac{3}{11648}t^{14} - \frac{25}{4576}t^{13} + \frac{107}{2112}t^{12} - \frac{71}{264}t^{11} + \frac{1149}{1280}t^{10} - \frac{7835}{4032}t^9 \\
&\quad + \frac{1209}{448}t^8 - \frac{9}{4}t^7 + \frac{9}{10}t^6 - \frac{1464917941}{3373793280}t^2 + \frac{1674766231}{4723310592}t \\
\beta_2 &= -\frac{3}{11648}t^{14} + \frac{49}{9152}t^{13} - \frac{205}{4224}t^{12} + \frac{265}{1056}t^{11} - \frac{1039}{1280}t^{10} + \frac{13625}{8064}t^9 \\
&\quad - \frac{3993}{1792}t^8 + \frac{387}{224}t^7 - \frac{99}{160}t^6 + \frac{961262747}{23616552960}t^2 - \frac{32097311}{4723310592}t \\
\beta_3 &= \frac{103}{2830464}t^{14} - \frac{263}{370656}t^{13} + \frac{343}{57024}t^{12} - \frac{23}{792}t^{11} + \frac{27289}{311040}t^{10} - \frac{6205}{36288}t^9 \\
&\quad + \frac{3299}{15552}t^8 - \frac{353}{2268}t^7 + \frac{43}{810}t^6 - \frac{355504573}{57967902720}t^2 + \frac{448912339}{127529385984}t \\
\mu_0 &= -\frac{11}{943488}t^{14} + \frac{49}{185328}t^{13} - \frac{101}{38016}t^{12} + \frac{277}{17820}t^{11} - \frac{6059}{103680}t^{10} + \frac{2659}{18144}t^9 \\
&\quad - \frac{35521}{145152}t^8 + \frac{733}{2835}t^7 - \frac{103}{720}t^6 + \frac{1}{24}t^4 - \frac{600560537}{23616552960}t^2 + \frac{498637229}{42509795328}t \\
\mu_1 &= \frac{1}{11648}t^{14} - \frac{49}{27456}t^{13} + \frac{17}{1056}t^{12} - \frac{217}{2640}t^{11} + \frac{2987}{11520}t^{10} - \frac{4157}{8064}t^9 \\
&\quad + \frac{561}{896}t^8 - \frac{117}{280}t^7 + \frac{9}{80}t^6 + \frac{38401309}{3373793280}t^2 - \frac{38405519}{4723310592}t \\
\mu_2 &= \frac{1}{11648}t^{14} - \frac{25}{13728}t^{13} + \frac{71}{4224}t^{12} - \frac{233}{2640}t^{11} + \frac{3329}{11520}t^{10} - \frac{2447}{4032}t^9 \\
&\quad + \frac{1443}{1792}t^8 - \frac{351}{560}t^7 + \frac{9}{40}t^6 - \frac{108009689}{4723310592}t^2 + \frac{2364085}{224919552}t \\
\mu_3 &= -\frac{11}{943488}t^{14} + \frac{167}{741312}t^{13} - \frac{1}{528}t^{12} + \frac{647}{71280}t^{11} - \frac{565}{20736}t^{10} + \frac{3833}{72576}t^9 \\
&\quad - \frac{4733}{72576}t^8 + \frac{1081}{22680}t^7 - \frac{7}{432}t^6 + \frac{35984341}{19322634240}t^2 - \frac{15164521}{14169931776}t
\end{aligned}$$

$$\begin{aligned}
v_0 &= -\frac{1}{943488}t^{14} + \frac{1}{41184}t^{13} - \frac{47}{190080}t^{12} + \frac{7}{4752}t^{11} - \frac{589}{103680}t^{10} + \frac{179}{12096}t^9 \\
&\quad - \frac{545}{20736}t^8 + \frac{157}{5040}t^7 - \frac{11}{480}t^6 + \frac{1}{120}t^5 - \frac{281826763}{212548976640}t^2 + \frac{4098617}{6072827904}t \\
v_1 &= \frac{1}{34944}t^{14} - \frac{17}{27456}t^{13} + \frac{31}{5280}t^{12} - \frac{23}{720}t^{11} + \frac{1261}{11520}t^{10} - \frac{1961}{8064}t^9 \\
&\quad + \frac{309}{896}t^8 - \frac{81}{280}t^7 + \frac{9}{80}t^6 - \frac{190232149}{7872184320}t^2 + \frac{3703591}{224919552}t \\
v_2 &= -\frac{1}{34944}t^{14} + \frac{1}{1716}t^{13} - \frac{109}{21120}t^{12} + \frac{103}{3960}t^{11} - \frac{943}{11520}t^{10} + \frac{167}{1008}t^9 \\
&\quad - \frac{381}{1792}t^8 + \frac{9}{56}t^7 - \frac{9}{160}t^6 + \frac{2343369}{374865920}t^2 - \frac{2242973}{674758656}t \\
v_3 &= \frac{1}{943488}t^{14} - \frac{5}{247104}t^{13} + \frac{1}{5940}t^{12} - \frac{19}{23760}t^{11} + \frac{247}{103680}t^{10} - \frac{37}{8064}t^9 \\
&\quad + \frac{409}{72576}t^8 - \frac{31}{7560}t^7 + \frac{1}{720}t^6 - \frac{33730349}{212548976640}t^2 + \frac{3881737}{42509795328}t
\end{aligned}$$