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Fixed Point Results in Archimedean Type Neutrosophic $\mathfrak{b}$-Metric Spaces

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Abstract

We defined the terms Archimedean, Caristi-Krirk balls, and Neutrosophic b metric space (Nb-MS) in this paper. Using the Archimedean idea and complete Nb-MS, we have shown the existence of a shared fixed point using two self-mappings and an upper semicontinuous function. We have demonstrated the existence of a fixed point by employing a k-continuous self-map and an upper semi-continuous function in conjunction with the Archimedean notion and complete Nb-MS. Furthermore, we have proven the completeness of the space by using Archimedean, Nb-MS, and a k-continuous self-map.

Keywords: b-metric space, Neutrosophic b metric spaces, common fixed point.
2010 MSC: 47H10.

1. Introduction

In 1965, Zadeh introduced fuzzy set theory [21], offering a groundbreaking approach to represent situations involving vague or uncertain knowledge. Fuzzy set theory allows for partial membership of elements in a set, providing a flexible mathematical framework to model imprecise information. In 1986, Atanassov [1] expanded this concept by introducing the Neutrosophic set, which categorized imprecise data based on two properties: the degree of membership (belongingness property) and the degree of non-membership (non-belongingness property).

The development of fuzzy metric spaces began with the work of Kelava and Seikkala [12], which was later refined by Kramosil and Michalek [13], and George and Veeramani

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[8]. These advancements extended the classical metric space by incorporating fuzziness, allowing for more accurate representations of real-world phenomena where precise data may be unavailable.

Similarly, the concept of b-metric spaces, introduced by Bakhtin [2] and further generalized by Czerwik [4], extended the triangle inequality, a fundamental property of metric spaces. Heilpern [10] established fixed point theory within the realm of fuzzy metric spaces, opening new directions for research. Several notable studies have contributed to this field, including Jeyaraman and Poovaragavan [11], who explored fixed point results in generalized b-fuzzy metric spaces, and Poovaragavan and Jeyaraman [15], who presented fixed point theorems for proximal contraction in these spaces. Additionally, Poovaragavan, Pandiselvi, and Jeyaraman [16] investigated coupled common fixed point theorems in generalized b-fuzzy metric spaces, further enriching the theoretical foundations of this topic. Many subsequent works [5, 17, 20] have explored and expanded upon these concepts, with notable contributions from Shakila and Jeyaraman et al. [18, 19], who introduced the idea of Neutrosophic b-metric spaces (NbMS).

More recently, R. A. Epsin-Andrade et al. [6, 7] discussed Archimedean logic, providing further generalization within the context of NbMS. This paper presents the notion of Neutrosophic b-metric spaces of Archimedean type (NbMS). It also includes the classification of Caristi-Kirk balls within this context and establishes several recent fixed point theorems, with an emphasis on common fixed points in Archimedean-type NbMS. Furthermore, we extend the results of previous work by generalizing these theorems.

2. Some Basic Definitions

Definition 2.1. Let Ξ be a set that is non-empty. Additionally in Ξ we specify a set $\mathfrak{F}$ where $\mathfrak{F} = \{(\kappa, \dot{\mathfrak{h}}_{\mathfrak{F}}(\kappa), \ddot{\mathfrak{p}}_{\mathfrak{F}}(\kappa), \ddot{\dot{\mathfrak{q}}}_{\mathfrak{F}}(\kappa) : \kappa \in \Xi, 0 \leqslant \dot{\mathfrak{h}}_I + \ddot{\mathfrak{p}}_I + \ddot{\dot{\mathfrak{q}}}_1 \leqslant 1\}$ from which the mappings $\dot{\mathfrak{h}}_{\mathfrak{F}} : \Xi \to [0,1] \subseteq \mathbb{R}$ follows the membership degree in addition both $\ddot{\mathfrak{p}}_{\mathfrak{F}} : \Xi \to [0,1] \subseteq \mathbb{R}$ and also $\ddot{\dot{\mathfrak{q}}}_{\mathfrak{F}} : \Xi \to [0,1] \subseteq \mathbb{R}$ follows the non-membership degree among function associated with an element $\kappa \in \Xi$, thus $\mathfrak{F}$ is known as Neutrosophic Set (abbreviated "NS").

Definition 2.2. Let Ξ be a set that is non-empty in addition that $*$ is a t-norm with continuous. Consider $\dot{\mathfrak{h}}$ is a fuzzy set on $\Xi^2 \times (0, \infty)$. If the following requirements are met with respect to a given number $\mathfrak{b} \geqslant 1$ and $\forall \hat{\jmath}_1, \hat{\jmath}_2, \hat{\jmath}_3 \in \Xi$ along with $\sigma, \vartheta > 0$, then the three-tuple $(\Xi, \dot{\mathfrak{h}}, *)$ is known as fuzzy $\mathfrak{b}$-metric space.

1. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) > 0$,
2. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) = 1$ iff $\hat{\jmath}_1 = \hat{\jmath}_2$,
3. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) = \dot{\mathfrak{h}}(\hat{\jmath}_2, \hat{\jmath}_1, \vartheta)$,
4. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_3, \vartheta + \sigma) \geqslant \left(\dot{\mathfrak{h}}\left(\hat{\jmath}_1, \hat{\jmath}_2, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \left(\dot{\mathfrak{h}}\left(\hat{\jmath}_2, \hat{\jmath}_3, \frac{\sigma}{\mathfrak{b}}\right)\right)$,
5. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \cdot) : (0, \infty) \to [0,1]$ and it is continuous.

Definition 2.3. Let Ξ become a set that is non-empty, in addition $*$ denote as a t-norm with continuous, $\circ$ and $\star$ denote as a t-conorm with continuous along with $\dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \ddot{\dot{\mathfrak{q}}}$ represent the sets with fuzzy upon $\Xi^2 \times (0, \infty)$. If the following conditions are met with respect to a given number $\mathfrak{b} \geqslant 1$ in addition $\forall \hat{\jmath}_1, \hat{\jmath}_2, \hat{\jmath}_3 \in \Xi$ along with $\mathfrak{s}, > 0$, then a seven-tuple $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \ddot{\dot{\mathfrak{q}}}, *, \circ, \star)$ is known to be a Neutrosophic $\mathfrak{b}$-metric space (abbreviated, N$\mathfrak{b}$-MS).

1. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) + \ddot{\mathfrak{p}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) + \ddot{\dot{\mathfrak{q}}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) \leqslant 1$,
2. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) > 0$,
3. $\dot{\mathfrak{h}}(\hat{\jmath}_1, \hat{\jmath}_2, \vartheta) = 1$ iff $\hat{\jmath}_1 = \hat{\jmath}_2$,

4. $\dot{\mathfrak{h}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = \dot{\mathfrak{h}}(\hat{\mathfrak{j}}_2, \hat{\mathfrak{j}}_1, \vartheta)$,
5. $\dot{\mathfrak{h}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_3, \vartheta + \sigma) \geqslant \left(\dot{\mathfrak{h}}\left(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \left(\dot{\mathfrak{h}}\left(\hat{\mathfrak{j}}_2, \hat{\mathfrak{j}}_3, \frac{\sigma}{\mathfrak{b}}\right)\right)$,
6. $\dot{\mathfrak{h}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \cdot)$ can be a nondecreasing function among $\mathbb{R}^+$ and $\lim_{t\to\infty} \dot{\mathfrak{h}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = 1$.
7. $\ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) < 1$,
8. $\ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = 0$ iff $\hat{\mathfrak{j}}_1 = \hat{\mathfrak{j}}_2$,
9. $\ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = \ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_2, \hat{\mathfrak{j}}_1, \vartheta)$,
10. $\ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_3, \vartheta + \sigma) \leqslant \left(\ddot{\mathfrak{p}}\left(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \left(\ddot{\mathfrak{p}}\left(\hat{\mathfrak{j}}_2, \hat{\mathfrak{j}}_3, \frac{\sigma}{\mathfrak{b}}\right)\right)$,
11. $\ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \cdot)$ can be non-increasing function among $\mathbb{R}^+$ and $\lim_{t\to\infty} \ddot{\mathfrak{p}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = 1$.
12. $\dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) < 1$,
13. $\dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = 0$ iff $\hat{\mathfrak{j}}_1 = \hat{\mathfrak{j}}_2$,
14. $\dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = \dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_2, \hat{\mathfrak{j}}_1, \vartheta)$,
15. $\dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_3, \vartheta + \sigma) \leqslant \left(\dddot{\mathfrak{q}}\left(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \left(\dddot{\mathfrak{q}}\left(\hat{\mathfrak{j}}_2, \hat{\mathfrak{j}}_3, \frac{\sigma}{\mathfrak{b}}\right)\right)$,
16. $\dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \cdot)$ can be non-increasing function among $\mathbb{R}^+$ and $\lim_{t\to\infty} \dddot{\mathfrak{q}}(\hat{\mathfrak{j}}_1, \hat{\mathfrak{j}}_2, \vartheta) = 1$.

Proposition 2.4. *Examine the sequence* $\{\kappa_n\} \in [0,1]$ *where* $\lim_{n\to\infty} \kappa_n = 1$ *and evaluate an* $\mathfrak{H}$*-type t-norm* $\mathfrak{F}$. *After that* $\lim_{n\to\infty} \tilde{\mathfrak{J}}_{i=1}^{\infty} \kappa_i = \lim_{n\to\infty} \tilde{\mathfrak{J}}_{i=1}^{\infty} \kappa_{n+i} = 1$.

3. Main Results

Listed below are a few theorems of fixed point in addition associated statements within Archimedean type N$\mathfrak{b}$MS.

Definition 3.1. Let $\check{\xi} : [0,1] \to [0,1]$ like that
1. If $\check{\xi}^{-1}(\{1\}) = \{1\}$ then $\check{\xi}$ is called amenable.
2. For any $\vartheta, \sigma \in [0,1]$, if $\check{\xi}(\vartheta * \sigma) \geqslant \check{\xi}(\vartheta) * \check{\xi}(\sigma)$ then $\check{\xi}$ is called $*$-superadditive.
3. For any $\vartheta, \sigma \in [0,1]$, if $\check{\xi}(\vartheta \circ \sigma) \leqslant \check{\xi}(\vartheta) \circ \check{\xi}(\sigma)$ in addition $\check{\xi}(\vartheta \star \sigma) \leqslant \check{\xi}(\vartheta) \star \check{\xi}(\sigma)$ then $\check{\xi}$ is called $\circ, \star$-co-superadditive.

Lemma 3.2. *Let a function* $\check{\xi} : [0,1] \to [0,1]$ *that is continuous as well as nondecreasing. If certain* $\vartheta \in (0,1), \check{\xi}(\vartheta) = 1$ *in addition* $(*, \circ, \star)$ *be an Archimedean, after that* $\forall \sigma \in [0,1], \check{\xi}(\sigma) = 1$.

Definition 3.3. Let $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{q}}, *, \circ, \star)$ be an N$\mathfrak{b}$-MS in addition consider $\tilde{\varphi} : \Xi \to [0,1]$ as well as $\check{\xi} : [0,1] \to [0,1]$. For any $\acute{\gamma} \in \Xi$ like that $\varphi(\acute{\gamma}) \neq 0$ is defined as the Caristi-Krirk balls

$$\mathfrak{C}_{\mathfrak{k}}(\acute{\gamma}) = \left\{ \begin{array}{c} \acute{\omega} \in \Xi : \check{\xi}\left(\dot{\mathfrak{h}}\left(\acute{\gamma}, \acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\acute{\omega}) \geqslant \tilde{\varphi}(\acute{\gamma}), \check{\xi}\left(\ddot{\mathfrak{p}}\left(\acute{\gamma}, \acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\acute{\omega}) \leqslant 1 - \tilde{\varphi}(\acute{\gamma}) \\ \text{and } \check{\xi}\left(\dddot{\mathfrak{q}}\left(\acute{\gamma}, \acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\acute{\omega}) \leqslant 1 - \tilde{\varphi}(\acute{\gamma}), \forall \vartheta > 0 \end{array} \right\}.$$

Theorem 3.4. *Assume that* $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{q}}, *, \circ, \star)$ *is a complete N*$\mathfrak{b}$*-MS in addition* $(*, \circ, \star)$ *is Archimedean as well as continuous. With reference to self maps let* $\tilde{\mathfrak{J}}, \breve{\mathfrak{S}} : \Xi \to \Xi$*, consider the function* $\tilde{\varphi} : \Xi \to [0,1]$ *that is upper semi-continuous where* $\kappa \in \Xi$ *exist, then* $\tilde{\varphi}(\breve{\mathfrak{S}}\kappa) \neq 0$.
Consider the following scenario $\check{\xi} : [0,1] \to [0,1]$ *is a nondecreasing, continuous function that yields* $\check{\xi}(\vartheta * \sigma) \geqslant \check{\xi}(\vartheta) * \check{\xi}(\sigma), \check{\xi}(\vartheta \circ \sigma) \leqslant \check{\xi}(\vartheta) \circ \check{\xi}(\sigma)$ *along with* $\check{\xi}(\vartheta \star \sigma) \leqslant \check{\xi}(\vartheta) \star \check{\xi}(\sigma)$ *in addition* $\check{\xi}^{-1}(\{1\}) = \{1\}$ *and it also meeting the requirements:*
$\check{\xi}\left(\dot{\mathfrak{h}}\left(\breve{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa)$, $\check{\xi}\left(\ddot{\mathfrak{p}}\left(\breve{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa)$ *and*
$\check{\xi}\left(\dddot{\mathfrak{q}}\left(\breve{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa) \forall \kappa \in \Xi, \vartheta > 0$.
Following that $\tilde{\mathfrak{J}}$ *and* $\breve{\mathfrak{S}}$ *having a common fixed point in* Ξ *when* $\breve{\mathfrak{S}}(\Xi)$ *belongs to complete.*

Proof. We get, $\tilde{\varphi}(\breve{\mathfrak{S}}\kappa) \neq 0$, for specific $\kappa \in \Xi$ that meet the conditions,

$$\mathfrak{C}_{\mathfrak{k}}(\kappa) = \left\{ \begin{array}{c} \mathfrak{y} \in \Xi : \check{\xi}\left(\dot{\mathrm{h}}\left(\kappa, \mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\mathfrak{y}) \geqslant \tilde{\varphi}(\kappa), \check{\xi}\left(\ddot{\mathrm{p}}\left(\kappa, \mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\mathfrak{y}) \leqslant 1 - \tilde{\varphi}(\kappa) \\ \check{\xi}\left(\dddot{\mathrm{q}}\left(\kappa, \mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\mathfrak{y}) \leqslant 1 - \tilde{\varphi}(\kappa), \forall \vartheta > 0 \end{array} \right\}.$$

and $\underbrace{\sup}_{\mathfrak{y} \in \mathfrak{C}_{\mathfrak{k}}(\kappa)} \tilde{\varphi}(\mathfrak{y}) = \breve{\sigma}(\kappa)$. It suggests that $1 \geqslant \breve{\sigma}(\kappa) \geqslant \tilde{\varphi}(\mathfrak{y})$, for any $\mathfrak{y} \in \mathfrak{C}_{\mathfrak{k}}(\kappa)$.
Given that, for every $\kappa, \tilde{\mathfrak{J}}\kappa \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa)$, as a result $\mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa) \neq \emptyset$.
Consider $\kappa_1 = \kappa$ in addition assume for every $\vartheta \geqslant 0, \kappa_{n+1}$ which means $\breve{\mathfrak{S}}\kappa_{n+1} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa_n)$ along with $\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \geqslant \breve{\sigma}(\breve{\mathfrak{S}}\kappa_n) - \frac{1}{n}$. Given that $\breve{\mathfrak{S}}\kappa_{n+1} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa_n)$,
$\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \geqslant \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_{n+1}, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n)$,
$1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \leqslant \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_{n+1}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \leqslant 1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n)$ and
$1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \leqslant \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_{n+1}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \leqslant 1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n)$
for every $\vartheta > 0$. Thus $\{\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n)\}$ indicate a convergent increasing sequence.
Before that, $\breve{\sigma}(\breve{\mathfrak{S}}\kappa_n) \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_{n+1}) \geqslant \breve{\sigma}(\breve{\mathfrak{S}}\kappa_n) - \frac{1}{n}$.
Thus, $\lim_{n \to \infty} \breve{\sigma}(\breve{\mathfrak{S}}\kappa_n) = \lim_{n \to \infty} \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n)$ exists.
Let

$$\mathfrak{k} = \lim_{n \to \infty} \breve{\sigma}(\breve{\mathfrak{S}}\kappa_n) = \lim_{n \to \infty} \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \tag{3.1}$$

Next consider for $n \in \mathbb{N}$.

$$\check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m) \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n), \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m) \leqslant 1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n)$$

$$\text{and } \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m) \leqslant 1 - \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \forall \vartheta > 0. \tag{3.2}$$

Now by using mathematical induction, we get.
Based on $m = n + 1$, given that $\breve{\mathfrak{S}}\kappa_{n+1} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa_n)$ hence the before inequality (3.2) represent true.
Assume that for $m > n$, true can be holds by the inequality.

Following this, verify that the form $\mathfrak{m}+1$,

$$\begin{aligned}
\check{\xi}\left(\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1},\frac{\vartheta}{\mathfrak{b}}\right)\right)*\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1}) &\geqslant \check{\xi}\left(\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{2\mathfrak{b}}\right)\right)* \\
&\quad \check{\xi}\left(\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1},\frac{\vartheta}{2\mathfrak{b}}\right)\right)*\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1}) \\
&\geqslant \check{\xi}\left(\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{2\mathfrak{b}}\right)\right)*\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}}) \\
&\geqslant \tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{n}}), \\
\check{\xi}\left(\ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1},\frac{\vartheta}{\mathfrak{b}}\right)\right)\circ\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1}) &\leqslant \check{\xi}\left(\ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{2\mathfrak{b}}\right)\right)\circ \\
&\quad \check{\xi}\left(\ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1},\frac{\vartheta}{2\mathfrak{b}}\right)\right)\circ\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1}) \\
&\leqslant \check{\xi}\left(\ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{2\mathfrak{b}}\right)\right)\circ\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}}) \\
&\leqslant 1-\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{n}}) \text{ and} \\
\check{\xi}\left(\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1},\frac{\vartheta}{\mathfrak{b}}\right)\right)\star\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1}) &\leqslant \check{\xi}\left(\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{2\mathfrak{b}}\right)\right)\star \\
&\quad \check{\xi}\left(\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1},\frac{\vartheta}{2\mathfrak{b}}\right)\right)\star\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}+1}) \\
&\leqslant \check{\xi}\left(\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{2\mathfrak{b}}\right)\right)\star\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{m}}) \\
&\leqslant 1-\tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{n}})
\end{aligned}$$

Consequently, for $\mathfrak{m}+1$ the inequality (3.2) holds valid. Therefore for all $\mathfrak{m} > \mathfrak{n}$ the inequality (3.2) holds.

We must then demonstrate that $\check{\mathfrak{S}}\kappa_{\mathfrak{n}}$ is a Cauchy sequence.

If at all possible, that $\check{\mathfrak{S}}\kappa_{\mathfrak{n}}$ cannot be a Cauchy sequence. Thus $\exists 0 < \check{\rho} < 1$ along with $\vartheta > 0$ which means $\forall\, \mathfrak{n} \in \mathbb{N}\, \exists\, \mathfrak{m} \in \mathbb{N}$ which means $\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{\mathfrak{b}}\right) \leqslant 1-\check{\varepsilon}, \ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{\mathfrak{b}}\right) \geqslant \check{\varepsilon}$ along with $\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa_{\mathfrak{n}},\check{\mathfrak{S}}\kappa_{\mathfrak{m}},\frac{\vartheta}{\mathfrak{b}}\right) \geqslant \check{\varepsilon}$.

Repeatly for everyone $0 < \check{\varepsilon}' < 1\, \exists\, \mathfrak{N} \in \mathbb{N}$ that gives $\mathfrak{k} \geqslant \tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{n}}) \geqslant \mathfrak{k}(1-\check{\varepsilon}')$ in addition $\mathfrak{k} \leqslant \tilde{\varphi}(\check{\mathfrak{S}}\kappa_{\mathfrak{n}}) \leqslant \mathfrak{k}(\check{\varepsilon}')$, for every $\mathfrak{n} > \mathbb{N}$.

We can determine from this that

$$\begin{aligned}
\mathfrak{k} * \check{\xi}((1-\check{\varepsilon})) &\geqslant \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \mathfrak{k} \geqslant \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m) \\
&\geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \\
&\geqslant \mathfrak{k}(1-\check{\varepsilon}'), \\
\mathfrak{k} \circ \check{\xi}((\check{\varepsilon})) &\leqslant \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \mathfrak{k} \leqslant \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m) \\
&\leqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \\
&\leqslant \mathfrak{k}(\check{\varepsilon}') \text{ and} \\
\mathfrak{k} \star \check{\xi}((\check{\varepsilon})) &\leqslant \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \mathfrak{k} \leqslant \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m) \\
&\leqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \\
&\leqslant \mathfrak{k}(\check{\varepsilon}')
\end{aligned}$$

it is valid $\forall \mathfrak{m} > \mathfrak{n} > \mathbb{N}$.
This suggests, $\mathfrak{k} * \check{\xi}((1-\check{\varepsilon})) \geqslant \mathfrak{k}(1-\check{\varepsilon}'), \mathfrak{k} \circ \check{\xi}((\check{\varepsilon})) \leqslant \mathfrak{k}(\check{\varepsilon}')$ in addition $\mathfrak{k} \star \check{\xi}((\check{\varepsilon})) \leqslant \mathfrak{k}(\check{\varepsilon}')$.
However, there is an inconsistency with the Archimedean condition.
Thus, $\breve{\mathfrak{S}}\kappa_n$ be a Cauchy sequence exists.
Given that $\breve{\mathfrak{S}}(\Xi)$ which is complete, the sequence $\breve{\mathfrak{S}}\kappa_n$ which is converges towards $\acute{\gamma} = \breve{\mathfrak{S}}\acute{\omega} \in \breve{\mathfrak{S}}(\Xi)$ along with $\tilde{\varphi}$ be an upper semi-conditions $\mathfrak{k} = \lim_{n\to\infty} \sup \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \leqslant \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$.
Now putting a limit on, we have
$\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \leqslant \lim_{m\to\infty} \sup \left(\check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m)\right) \leqslant \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$,
$1-\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \geqslant \lim_{m\to\infty} \inf \left(\check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m)\right) \geqslant \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$ and
$1-\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \geqslant \lim_{m\to\infty} \inf \left(\check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\kappa_m, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_m)\right) \geqslant \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$
Then $\breve{\mathfrak{S}}\acute{\omega} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa_n)$. Therefore $\breve{\sigma}(\breve{\mathfrak{S}}\kappa_n) \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$.
Hence, $\mathfrak{k} \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$ implies $\mathfrak{k} \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) = \tilde{\varphi}(\acute{\gamma})$ based on the inequality (3.2), given that $\breve{\mathfrak{S}}\acute{\omega} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa_n)$ and $\tilde{\mathfrak{J}}\acute{\omega} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\acute{\omega})$, we have

$$\begin{aligned}
\check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) &\geqslant \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) * \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) \\
&\geqslant \check{\xi}\left(\dot{\mathrm{h}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) * \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) \\
&\geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n), \\
\check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) &\leqslant \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \circ \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) \\
&\leqslant \check{\xi}\left(\ddot{\mathrm{p}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) \\
&\leqslant 1-\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n) \text{ and} \\
\check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) &\leqslant \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \star \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) \\
&\leqslant \check{\xi}\left(\dddot{\mathrm{q}}\left(\breve{\mathfrak{S}}\kappa_n, \breve{\mathfrak{S}}\acute{\omega}, \frac{\vartheta}{2\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) \\
&\leqslant 1-\tilde{\varphi}(\breve{\mathfrak{S}}\kappa_n), \forall \vartheta > 0
\end{aligned}$$

As a result $\tilde{\mathfrak{J}}\acute{\omega} \in \mathfrak{C}_{\mathfrak{k}}(\breve{\mathfrak{S}}\kappa_n) \forall n \in \mathbb{N}$.
Let us now consider, $\tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) \leqslant \breve{\sigma}_n(\kappa_n) \forall n \in \mathbb{N}$.
Thus we possess $\tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) \leqslant \mathfrak{k}$. Therefore, $\tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) = \mathfrak{k} \geqslant \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) \geqslant \tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega})$.
Therefore $\tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) = \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\omega}) = \mathfrak{k}$ additionally for every $\vartheta > 0$,
$\mathfrak{k} * \check{\xi}\left(\dot{\mathfrak{h}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \geqslant \mathfrak{k}$, $\mathfrak{k} \circ \check{\xi}\left(\ddot{\mathfrak{p}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \leqslant 1 - \mathfrak{k}$ along with
$\mathfrak{k} \star \check{\xi}\left(\dddot{\mathfrak{q}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) \leqslant 1 - \mathfrak{k}$ and it gives that
$\check{\xi}\left(\dot{\mathfrak{h}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) = 1, \check{\xi}\left(\ddot{\mathfrak{p}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) = 1$ along with $\check{\xi}\left(\dddot{\mathfrak{q}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right)\right) = 1$.
Thus, for every $\vartheta > 0, \dot{\mathfrak{h}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right) = 1, \ddot{\mathfrak{p}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right) = 1$ in addition
$\dddot{\mathfrak{q}}\left(\breve{\mathfrak{S}}\acute{\omega}, \tilde{\mathfrak{J}}\acute{\omega}, \frac{\vartheta}{\mathfrak{b}}\right) = 1$.
And we get $\breve{\mathfrak{S}}\acute{\omega} = \tilde{\mathfrak{J}}\acute{\omega}$. □

Corollary 3.5. *Assume that* $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{q}}, *, \circ, \star)$ *is an N𝔟-MS with complete,* $(*, \circ, \star)$ *is an Archimedean along with continuous.* $\tilde{\mathfrak{J}}, \breve{\mathfrak{S}} : \Xi \to \Xi$ *denote two self-maps where* $\breve{\mathfrak{S}}$ *is an identity in addition* $\tilde{\varphi} : \Xi \to [0,1]$ *be a function of upper semi-continuous in which* $\kappa \in \Xi$ *exist,* $\tilde{\varphi}(\breve{\mathfrak{S}}\kappa) \neq 0$. *Let* $\check{\xi} : [0,1] \to [0,1]$ *be a function which is nondecreasing continuous that gives* $\check{\xi}(\vartheta * \sigma) \geqslant \check{\xi}(\vartheta) * \check{\xi}(\sigma), \check{\xi}(\vartheta \circ \sigma) \leqslant \check{\xi}(\vartheta) \circ \check{\xi}(\sigma)$ *along with* $\check{\xi}(\vartheta \star \sigma) \leqslant \check{\xi}(\vartheta) \star \check{\xi}(\sigma)$ *in addition* $\check{\xi}^{-1}(\{1\}) = \{1\}$ *and also satisfies the following requirements*
$\check{\xi}\left(\dot{\mathfrak{h}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \geqslant \tilde{\varphi}(\kappa)$, $\check{\xi}\left(\ddot{\mathfrak{p}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\kappa)$ *and*
$\check{\xi}\left(\dddot{\mathfrak{q}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\kappa), \forall \kappa \in \Xi, \vartheta > 0$.
In Ξ, *follows that* $\tilde{\mathfrak{J}}$ *having a fixed point.*

Corollary 3.6. *Assume that* $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{q}}, *, \circ, \star)$ *is a N𝔟-MS with complete,* $(*, \circ, \star)$ *is an Archimedean along with continuous. Let* $\tilde{\mathfrak{J}}, \breve{\mathfrak{S}} : \Xi \to \Xi$ *be two self-maps where the identity that is* $\breve{\mathfrak{S}}$ *in addition* $\tilde{\varphi} : \Xi \to [0,1]$ *be a function that is an upper semi-continuous in which* $\kappa \in \Xi$ *exist,* $\tilde{\varphi}(\breve{\mathfrak{S}}\kappa) \neq 0$. *Let* $\check{\xi} : [0,1] \to [0,1]$ *be a identity map satisfies the following requirements:*
$\dot{\mathfrak{h}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \geqslant \tilde{\varphi}(\kappa)$, $\check{\xi}\left(\ddot{\mathfrak{p}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\kappa)$ *and*
$\check{\xi}\left(\dddot{\mathfrak{q}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\kappa), \forall \kappa \in \Xi, \vartheta > 0$.
In Ξ, *follows that* $\tilde{\mathfrak{J}}$ *having a fixed point.*
Following that, we have to generalize the Theorem (3.7)

Theorem 3.7. *Assume that* $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{q}}, *, \circ, \star)$ *be an N𝔟-MS with complete,* $(*, \circ, \star)$ *is Archimedean along with continuous. Let* $\tilde{\mathfrak{J}}, \breve{\mathfrak{S}} : \Xi \to \Xi$ *be a self-map that is* $\mathfrak{k}$*-continuous. Consider a function* $\tilde{\varphi} : \Xi \to [0,1]$ *which is an upper semi-continuous in which* $\kappa \in \Xi$ *exists,* $\tilde{\varphi}(\breve{\mathfrak{S}}\kappa) \neq 0$, *and satisfies the following condition:*

$$\begin{aligned} &\dot{\mathfrak{h}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \geqslant \tilde{\varphi}(\kappa), \ddot{\mathfrak{p}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\kappa) \\ &\textit{and}\ \dddot{\mathfrak{q}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\kappa) \end{aligned} \tag{3.3}$$

if $\kappa \in \Xi$ *and* $\vartheta > \vartheta_0$, *any* $\vartheta_0 > 0$.
There is a fixed point in $\tilde{\mathfrak{J}}$ *in* Ξ.

Proof. We can arrive at the conclusion that $\{\kappa_n\}$ is a Cauchy sequence by applying the same reasons from Theorem (3.4) proof in addition to the assumption that $\check{\xi}$ is an identity map. Given that Ξ represents complete, thus for everyone $p \geqslant 1$, $\exists \mathfrak{y}_0 \in \Xi$ be an element, which means $\lim_{n\to\infty}(\kappa_n) = \mathfrak{y}_0$ and $\lim_{n\to\infty}(\tilde{\mathfrak{J}}^p\kappa_n) = \mathfrak{y}_0$. Next, we have $\lim_{n\to\infty}(\tilde{\mathfrak{J}}^k\kappa_n) \to \tilde{\mathfrak{J}}\mathfrak{y}_0$ from the $\mathfrak{k}$-continuity of $\tilde{\mathfrak{J}}$. Consequently as $\lim_{n\to\infty}(\tilde{\mathfrak{J}}^k\kappa_n) \to \mathfrak{y}_0, \tilde{\mathfrak{J}}\mathfrak{y}_0 \to \mathfrak{y}_0$. It suggest that $\mathfrak{y}_0$ represents

a fixed point according to $\tilde{\mathfrak{J}}$.
Next, we introduce a theorem that describes the completeness property of the Archimedean N$\mathfrak{b}$-MS type. □

Theorem 3.8. *Consider* $(\Xi, \dot{\mathfrak{h}}, \dot{\mathfrak{p}}, \ddot{\mathfrak{q}}, *, \circ, \star)$ *be an N$\mathfrak{b}$-MS with complete,* $(*, \circ, \star)$ *is Archimedean along with continuous. Take that* $\forall \kappa \neq \tilde{\mathfrak{J}}\kappa$ *and* $\vartheta > 0$, *any self-map of* Ξ *with* $\mathfrak{k}$*-continuous contain a fixed point that satisfies every requirement of Theorem (3.7) also satisfies the following condition:*

$\dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}^2\kappa, \frac{\vartheta}{\mathfrak{b}}\right) > \dot{\mathfrak{h}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \Rightarrow \dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}^2\kappa, \frac{\vartheta}{\mathfrak{b}}\right)^2 \geqslant \dot{\mathfrak{h}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)$
$\dot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}^2\kappa, \frac{\vartheta}{\mathfrak{b}}\right) < \dot{\mathfrak{p}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \Rightarrow \dot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}^2\kappa, \frac{\vartheta}{\mathfrak{b}}\right)^2 \leqslant \dot{\mathfrak{p}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)$ *and*
$\ddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}^2\kappa, \frac{\vartheta}{\mathfrak{b}}\right) < \ddot{\mathfrak{q}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \Rightarrow \ddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}^2\kappa, \frac{\vartheta}{\mathfrak{b}}\right)^2 \leqslant \ddot{\mathfrak{q}}\left(\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right)$
In the above case, Ξ *represent complete.*

Proof. Let us examine every self-map of Ξ with $\mathfrak{k}$-continuous contain a fixed point which meet every requirements in Theorem (3.7) are given. Suppose that Ξ represents a not complete when possible. Next, let $\{\kappa_n\} = \{\acute{\omega}_1, \acute{\omega}_2, \acute{\omega}_3, \ldots.\}$ be a Cauchy sequence that is non convergence inside Ξ composed of points with distinct.
Assume that $\acute{\gamma} \in \Xi$ which is a component of $\{\kappa_n\}$, is not a limit point. Subsequently, $\exists$ an integer representing the least positive number $\acute{\gamma}_0$ so that $\acute{\gamma} \neq \acute{\omega}_{\acute{\gamma}_0}$.
Now we get $\tilde{\varphi}(\breve{\mathfrak{S}}\acute{\omega}) = \tilde{\varphi}(\acute{\gamma})$ for all $\mathfrak{m} \geqslant \acute{\gamma}_0$ and $\vartheta > 0$,
$\dot{\mathfrak{h}}\left(\acute{\gamma}, \acute{\omega}_{\acute{\gamma}_0}, \frac{\vartheta}{\mathfrak{b}}\right) < \dot{\mathfrak{h}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\gamma}_{\mathfrak{m}}, \frac{\vartheta}{\mathfrak{b}}\right)$, $\dot{\mathfrak{p}}\left(\acute{\gamma}, \acute{\omega}_{\acute{\gamma}_0}, \frac{\vartheta}{\mathfrak{b}}\right) > \dot{\mathfrak{p}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\gamma}_{\mathfrak{m}}, \frac{\vartheta}{\mathfrak{b}}\right)$ and
$\ddot{\mathfrak{q}}\left(\acute{\gamma}, \acute{\omega}_{\acute{\gamma}_0}, \frac{\vartheta}{\mathfrak{b}}\right) > \ddot{\mathfrak{q}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\gamma}_{\mathfrak{m}}, \frac{\vartheta}{\mathfrak{b}}\right)$
Additionally create a function $\tilde{\mathfrak{J}} : \Xi \to \Xi$ that gives $\tilde{\mathfrak{J}}(\acute{\gamma}) = \acute{\omega}_{\acute{\gamma}_0}$. Next, for each and every $\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma} \neq \acute{\gamma}$. From (3.3), for any $\vartheta > 0$ and $\acute{\gamma} \in \Xi$, we have
$\dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right) = \dot{\mathfrak{h}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\omega}_{T(\acute{\gamma}_0)}, \frac{\vartheta}{\mathfrak{b}}\right) > \dot{\mathfrak{h}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\gamma}_{\mathfrak{m}}, \frac{\vartheta}{\mathfrak{b}}\right) = \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)$
$\dot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right) = \dot{\mathfrak{p}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\omega}_{T(\acute{\gamma}_0)}, \frac{\vartheta}{\mathfrak{b}}\right) < \dot{\mathfrak{p}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\gamma}_{\mathfrak{m}}, \frac{\vartheta}{\mathfrak{b}}\right) = \dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)$ and
$\ddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right) = \ddot{\mathfrak{q}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\omega}_{T(\acute{\gamma}_0)}, \frac{\vartheta}{\mathfrak{b}}\right) < \ddot{\mathfrak{q}}\left(\acute{\omega}_{\acute{\gamma}_0}, \acute{\gamma}_{\mathfrak{m}}, \frac{\vartheta}{\mathfrak{b}}\right) = \ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)$
Follows that $\dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)^2 \geqslant \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)$, $\dot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)^2 \leqslant \dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)$
along with $\ddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)^2 \leqslant \ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right)$.
Consider $\tilde{\varphi}(\acute{\gamma}) = \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right)^2$, $1 - \tilde{\varphi}(\acute{\gamma}) = \dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right)^2$ along with
$1 - \tilde{\varphi}(\acute{\gamma}) = \ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right)^2$, we have
$\dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) = \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) * \dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right)^2 \geqslant \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) * \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) = \tilde{\varphi}(\acute{\gamma})$,
$\dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) = \dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) \circ \dot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right)^2 \leqslant \dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) \circ \dot{\mathfrak{p}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) = 1 - \tilde{\varphi}(\acute{\gamma})$ and
$\ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) = \ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) \star \ddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\acute{\gamma}, \tilde{\mathfrak{J}}^2\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right)^2 \leqslant \ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) \star \ddot{\mathfrak{q}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) = 1 - \tilde{\varphi}(\acute{\gamma})$
Moreover
$\dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \geqslant \dot{\mathfrak{h}}\left(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \geqslant \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma})$,
$\dot{\mathfrak{p}}(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \leqslant \dot{\mathfrak{p}}(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \leqslant 1 - \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma})$ and
$\ddot{\mathfrak{q}}(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta}{\mathfrak{b}}) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \leqslant \ddot{\mathfrak{q}}(\acute{\gamma}, \tilde{\mathfrak{J}}\acute{\gamma}, \frac{\vartheta_0}{\mathfrak{b}}) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \leqslant 1 - \tilde{\varphi}(\tilde{\mathfrak{J}}\acute{\gamma}) \forall \vartheta > \vartheta_0$.
Thus, the function $\tilde{\mathfrak{J}}$ satisfies the contractive one along with all other requirements according to Theorem (3.7). Furthermore $\tilde{\mathfrak{J}}$ indicate a function without fixed points in addition

its range is a part of $\{\kappa_n\}$ be a non-convergent Cauchy sequence.
Consequently, there is no sequence $\{\kappa_n\}$ within Ξ that $\{\tilde{\mathfrak{J}}\kappa_n\}$ converges i.e. $\exists$ no sequence $\{\kappa_n\}$ within Ξ that the condition $(\tilde{\mathfrak{J}}\kappa_n) \to \check{\zeta} \Rightarrow (\tilde{\mathfrak{J}}^2\kappa_n) \to \tilde{\mathfrak{J}}\check{\zeta}$ is violated.
As a result, $\tilde{\mathfrak{J}}$ represent a function with 2-continuous. Consequently, we have a self-mapping $\tilde{\mathfrak{J}}$ of Ξ that satisfies all of the conditions of Theorem (3.7) is contradictory, and lacks a fixed point.
Hence Ξ represent a complete.
We then give an additional generalization of the previous theorem. □

Corollary 3.9. *Assume that Nb-MS* $(\Xi, \dot{\mathfrak{h}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{q}}, *, \circ, \star), (*, \circ, \star)$ *provide Archimedean along with continuous. Take* $\tilde{\mathfrak{J}}, \check{\mathfrak{S}} : \Xi \to \Xi$. *Consider that* $\exists$ *a function* $\tilde{\varphi} : \Xi \to [0,1]$ *that gives*
1. $\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) * \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \geqslant \tilde{\varphi}(\check{\mathfrak{S}}\kappa), \ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \circ \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\check{\mathfrak{S}}\kappa)$ *along with* $\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\kappa, \frac{\vartheta}{\mathfrak{b}}\right) \star \tilde{\varphi}(\tilde{\mathfrak{J}}\kappa) \leqslant 1 - \tilde{\varphi}(\check{\mathfrak{S}}\kappa)$, *for every* $\vartheta \geqslant 0$ *in addition* $\kappa \in \Xi$.
2. $\dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)^2 > \min\left\{\dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa, \check{\mathfrak{S}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)^2, \dot{\mathfrak{h}}\left(\check{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right) * \dot{\mathfrak{h}}\left(\tilde{\mathfrak{J}}\kappa, \check{\mathfrak{S}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)\right\}$,
$\ddot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)^2 < \max\left\{\ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa, \check{\mathfrak{S}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)^2, \ddot{\mathfrak{p}}\left(\check{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right) \circ \ddot{\mathfrak{p}}\left(\tilde{\mathfrak{J}}\kappa, \check{\mathfrak{S}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)\right\}$ *and*
$\dddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\kappa, \tilde{\mathfrak{J}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)^2 < \max\left\{\dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa, \check{\mathfrak{S}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)^2, \dddot{\mathfrak{q}}\left(\check{\mathfrak{S}}\kappa, \tilde{\mathfrak{J}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right) \star \dddot{\mathfrak{q}}\left(\tilde{\mathfrak{J}}\kappa, \check{\mathfrak{S}}\mathfrak{y}, \frac{\vartheta}{\mathfrak{b}}\right)\right\}$, *for all* $\kappa \neq \mathfrak{y}$ *and* $\vartheta \geqslant 0$.
3. $\tilde{\mathfrak{J}}(\Xi) \subset \check{\mathfrak{S}}(\Xi)$.
4. $\tilde{\mathfrak{J}}(\Xi)$ *or* $\check{\mathfrak{S}}(\Xi)$ *represent complete.*
Following that $\tilde{\mathfrak{J}}$ *in addition* $\check{\mathfrak{S}}$ *contains a common fixed point within* Ξ.

4. Application

Our primary application of Archimedean Compensatory Neutrosophic Logic (ACNL) is centered around the Archimedean type Neutrosophic b-metric space (Nb-MS), which has substantial implications in real-world problem-solving across multiple disciplines. ACFL (Archimedean Compensatory Fuzzy Logic) enhances traditional fuzzy logic systems by extending t-norm and t-conorm to Archimedean logic. This duality, introduced via the negative operator, has significant practical applications in areas like decision-making, data fusion, and uncertainty modeling—where truth values are often nuanced and require a more flexible approach.

The integration of fixed point theory in Neutrosophic spaces holds vast potential in optimization, control systems, and signal processing. For instance, the work of Bouhadjera et al. [3] which deals with unique common fixed points for occasionally weakly biased maps of type (A) in b-metric-like spaces, offers foundational insights that can be applied in designing algorithms for real-time decision-making systems, such as autonomous navigation and adaptive learning models.

Similarly, the contribution by Hardan et al. [9] which extended fixed point theory to Hardy-Rogers type fractional differential equations, has real-life relevance in fields like telecommunications and finance, where time-fractional models are used to predict behaviors over time, optimize resource allocations, and manage uncertainties in complex systems.

Furthermore, Patil et al. [14] demonstrated the utility of fixed point theorems for generalized contractive type mappings in fractional differential equations. This has a direct application in engineering and physics, particularly in systems where dynamic behavior is influenced by memory or hereditary properties—such as in thermal systems, control mechanisms, or even the modeling of biological processes.

The Generalized Sigmoidal Function and Generalized Linguistic Modifiers offer practical applications in areas such as natural language processing, sentiment analysis, and machine learning. These models provide a flexible framework for developing intelligent systems that require adaptive input-output mappings, such as in AI-driven customer service, recommendation systems, and medical diagnosis tools. The Generalized Continuous Linguistic Variable—which includes parameterized families of shape functions like convex and sigmoid functions—is critical for fine-tuning models in AI and machine learning applications, enabling more accurate predictions and decisions based on imprecise or incomplete information.

Moreover, Knowledge Discovery (KD) models based on Archimedean type N$\mathfrak{b}$-MS have far-reaching implications in big data analytics and optimization. These models help in solving large-scale optimization problems over continuous parameter spaces, which is crucial for industries like supply chain management, logistics, and predictive maintenance in manufacturing. The ability to optimize processes and systems using N$\mathfrak{b}$-MS allows companies to reduce costs, improve efficiency, and enhance decision-making in real time.

By extending the classical notions of fixed point theory and fuzzy logic into Neutrosophic spaces, this work bridges theoretical advances with practical, real-world applications, making it highly relevant for emerging fields such as smart cities, IoT (Internet of Things), and cyber-physical systems where decision-making under uncertainty is paramount.

5. Conclusion

In the present chapter, we have provided the concept of Archimedean type N$\mathfrak{b}$-MS and provided several theorems with common fixed point. The nature of fixed points in this space must be demonstrated. This chapter concentrates on the generalized version of fuzzy metric space. Additional findings are extended by utilizing the self-mapping with $\mathfrak{k}$-continuous idea. We have also included several corollaries of the theorem that was recently proven. These noval discoveries give scholars a plethora of knowledge to further fixed point theory in a noval directions.

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