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Fixed Point Theorem on C^*-algebra-valued Suprametric Spaces

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Abstract

The present study's objective is to propose a C^*-algebra-valued supramatric spaces to provide an appropriate generalization concerning both supramatric spaces and C^*-algebra-valued metric spaces. The concepts of convergence, Cauchy sequence, and completeness are then examined through supramatric space with C^*-algebra and illustrated with an example. Furthermore, the Banach fixed point theorem, established in pursuance of the same metric, is employed to determine the existence and uniqueness of the solution to an integral equation.

Keywords: C^*-algebra, C^*-algebra-valued supramatric spaces, Contractive mapping, Fixed point theorem.
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1. Introduction and Preliminaries

Fixed point theory is still a fascinating area of study even after more than a century of development. Fixed point theory is appealing because it has applications in many different fields. Although widely recognised, the Banach contraction mapping concept has numerous applications in applied mathematics and is a very helpful, straightforward, and traditional technique in current analysis. It is particularly useful for resolving existence issues in a variety of physics and mathematics fields.

One of the foremost essential and effective methods of nonlinear analysis is the Banach contraction principle. It states that in a complete metric space (X, d), if a mapping $f : X \to X$ is a contraction, that is, there exists a constant $c \in [0, 1)$ such that $d(fx, fy) \leqslant cd(x, y)$ for all $x, y \in X$, then f has a unique fixed point. Moreover, the sequence $\{f^n x\}$ of the n-iterate of f converges to this fixed point for any initial point $x \in X$. Numerous studies have focused on extending, generalizing, and enhancing this concept due to its numerous applications in various areas of mathematics.

In certain studies, the metric space has been expanded to encompass broader spaces, such as Matthew's partial metric spaces [13], Branciari's rectangular metric spaces [4],

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or Czerwik's b-metric spaces [6]. A few enhancements were achieved by relaxing the contraction. Krasnoselskii [8], for example, has studied a sort of local contraction. In [7], Edelstein presents a more comprehensive contractive condition, substituting the existence of a nonempty ω-limit set for the metric space's completeness. The following partial list of references contains further findings that expand on the Banach contraction principle and its applications [3, 5, 9, 22, 16, 20, 18, 19, 15].

In 2022, Maher Berzig [2] introduced the concept of suprametric space and examined several significant facts of its topology. Then, if the space is complete or contains a nonempty ω-limit set, the author proved that certain contraction maps in suprametric spaces have a single fixed point.

In [10], Ma proved several fixed point theorems for self maps with contractive or expansive mappings and introduced the idea of C^*-algebra-valued metric spaces. The key idea is to replace the set of real numbers with the set of all positive elements of a unital C^*-algebra. Additionally, in [11], Ma expanded on the idea of C^*-valued metric spaces by introducing the notion of C^*-valued b-metric spaces.

Inspired by all the previously mentioned articles, in this study we brought forth the notion of C^*-algbera-valued suprametric spaces, leading to the extension of suprametric spaces and C^*-algbera-valued metric spaces. In addition, the ideas of convergence, Cauchy sequence, and completeness are explored, including an example presented within the context of C^*-algebra-valued suprametric space. In this metric, the existence and uniqueness of the solution to an integral equation are examined as an application of the proven Banach fixed point theorem.

Czerwik developed the notion of b-metric spaces in [6], as follows:

Definition 1.1. [6] Let X be a nonempty set and $b \geqslant 1$ be any real number. A function $d : X \times X \to [0, \infty)$ is a b-metric iff for each $\varpi, \rho, \sigma \in X$, the preceding conditions are satisfied:

1. $0 \leqslant d(\varpi, \rho)$ for all $\varpi, \rho \in X$ and $d(\varpi, \rho) = 0$ if and only if $\varpi = \rho$;
2. $d(\varpi, \rho) = d(\rho, \varpi)$ for all $\varpi, \rho \in X$;
3. $d(\varpi, \rho) \leqslant b[d(\varpi, \sigma) + d(\sigma, \rho)]$ for all $\varpi, \rho, \sigma \in X$;
   The pair (X, d) is called a b-metric space.

Remark 1.2. It is important to note that compared to metric spaces, the class of b-metric spaces is perhaps larger. It is true that a b-metric is a metric if and only if $b = 1$.

The following gives the definition of suprametric space:

Definition 1.3. [2] Let $X \neq \emptyset$. A function $d_s : X \times X \to [0, \infty)$ is a suprametric if for all $\varpi, \rho, \sigma \in X$, the following conditions hold:

1. $d_s(\varpi, \rho) = 0$ if and only if $\varpi = \rho$;
2. $d_s(\varpi, \rho) = d_s(\rho, \varpi)$ for all $\varpi, \rho \in X$;
3. $d_s(\varpi, \rho) \leqslant d_s(\varpi, \sigma) + d_s(\sigma, \rho) + \rho d_s(\varpi, \sigma)\, d_s(\sigma, \rho)$

for some constant $\rho \in \mathcal{R}^+$. Then d_s is called a suprametric on X and (X, d_s) is named as suprametric space.

Following this, let $\mathcal{A}$ signifies an unital C*-algebra. Set $\mathcal{A}_h = \{\varpi \in \mathcal{A} : \varpi = \varpi^*\}$. We say that $\varpi \in \mathcal{A}$ is a positive element, denote it by $\theta \preceq \varpi$, if $\varpi = \varpi^*$ and $\sigma(\varpi) \subseteq [0, \infty)$, where θ is a zero element in $\mathcal{A}$ and $\sigma(\varpi)$ is the spectrum of ϖ. The partial ordering on $\mathcal{A}_h$ is given by $\varpi \preceq \rho$ if and only if $\theta \preceq \rho - \varpi$. $\mathcal{A}_+$ and $\mathcal{A}'$ will represent the set $\{\varpi \in \mathcal{A} : \theta \preceq \varpi\}$ and the set $\{\varpi \in \mathcal{A} : \varpi\rho = \rho\varpi, \ \forall \rho \in \mathcal{A}\}$ and $|\varpi| = (\varpi^*\varpi)^{\frac{1}{2}}$ respectively.

Employing the idea of positive components in $\mathcal{A}$, the researchers in [10] devised the concept of C*-algebra-valued metric space, as seen below.

Definition 1.4. [10] Let $X \neq \emptyset$ and the mapping $d_{\mathcal{A}} : X \times X \to \mathcal{A}$ satisfies:

1. $\theta \preceq d_{\mathcal{A}}(\varpi, \rho)$ and $d_{\mathcal{A}}(\varpi, \rho) = \theta$ iff $\varpi = \rho$;
2. $d_{\mathcal{A}}(\varpi, \rho) = d_{\mathcal{A}}(\rho, \varpi)$;
3. $d_{\mathcal{A}}(\varpi, \rho) \preceq d_{\mathcal{A}}(\varpi, \sigma) + d_{\mathcal{A}}(\sigma, \rho)$, $\forall\ \varpi, \rho, \sigma \in X$.

Then $d_{\mathcal{A}}$ is said to be a C*-algebra-valued metric on X and $(X, \mathcal{A}, d_{\mathcal{A}})$ is named as C*-algebra-valued metric space.

The notion of metric spaces is enlarged by the class of C*-algebra-valued metric spaces, which adopts $\mathcal{A}_+$ for the set of real numbers.

2. Main Results

This section presents the idea of C*-algebra-valued suprametric spaces, which is a valid extension that includes both suprametric spaces and C*-algebra-valued metric spaces.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to \mathcal{A}$ is called C*-algebra-valued suprametric if for all $\varpi, \rho, \sigma \in X$ the following properties hold:

1. $d(\varpi, \rho) = \theta$ if and only if $\varpi = \rho$;
2. $d(\varpi, \rho) = d(\rho, \varpi)$;
3. $d(\varpi, \rho) \preceq d(\varpi, \sigma) + d(\sigma, \rho) + A d(\varpi, \sigma) d(\sigma, \rho)$ for some $A \in \mathcal{A}_+$

Then d is called a C*-algebra-valued suprametric on X and $(X, \mathcal{A}, d)$ is called a C*-algebra-valued suprametric space.

Example 2.2. Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$. Define

$$d(\varpi, \rho) = \begin{bmatrix} |\varpi - \rho|(|\varpi - \rho| + \beta) & 0 \\ 0 & \alpha|\varpi - \rho|(|\varpi - \rho| + \beta) \end{bmatrix}$$

where $\alpha \geqslant 0$ is a constant. It is very easy to verify that d is a C*-algebra-valued suprametric and $(\mathbb{R}, M_2(\mathbb{R}), d)$ is a C*-algebra-valued suprametric space with $A = \begin{bmatrix} \frac{2}{\beta} & 0 \\ 0 & \frac{2}{\beta} \end{bmatrix}$, where β is a positive real.

Example 2.3. Let $X = L^{\infty}(E)$ and $H = L^2(E)$, wherein E is a Lebesgue measurable set. $L(H)$ represents a set of bounded linear operators on H, a Hilbert space. Define $d : X \times X \to L(H)$ by $d(\varpi, \rho) = \pi_{|\varpi-\rho|(|\varpi-\rho|+\beta)}$, $(\forall \varpi, \rho \in X$ and $\beta > 0)$ in which the multiplication operator $\pi_h : H \to H$ is defined by $\pi_\eta(\phi) = h.\eta$ for $\eta \in H$. It follows that d is a C^*-algebra-valued suprametric and $(X, L(H), d)$ is a C^*-algebra-valued suprametric space.

Definition 2.4. Assume that $(X, \mathcal{A}, d)$ is a C^*-algebra-valued suprametric space. Let $\{\varpi_n\} \subset X$ and $\varpi \in X$. If for every $\epsilon > 0$, there is N so that for all $n > N$, $\|d(\varpi_n, \varpi)\| \leqslant \epsilon$, then $\{\varpi_n\}$ is said to converge to ϖ with regard to $\mathcal{A}$ and we indicate it by $\lim_{n\to\infty} \varpi_n = \varpi$.

If for every $\epsilon > 0$ there is N so that for all $n, m > N$, $\|d(\varpi_n, \varpi_m)\| \leqslant \epsilon$, then $\{\varpi_n\}$ is referred to as a Cauchy sequence in the context of $\mathcal{A}$.

If every Cauchy sequence is convergent in X with regard to $\mathcal{A}$, then $(X, \mathcal{A}, d)$ is said to be a complete C^*-algebra-valued suprametric space.

Definition 2.5. Assume that $(X, \mathcal{A}, d)$ is a C^*-algebra-valued suprametric space. We refer to a mapping $T : X \to X$ a C^*-algebra-valued contractive mapping on X, if there is an element $B \in \mathcal{A}$ with $\|B\| < 1$ so that

$$d(T\varpi, T\rho) \preceq B^* d(\varpi, \rho) B, \quad \forall\, \varpi, \rho \in X. \tag{2.1}$$

3. Fixed Point Theorem

In the context of C^*-algebra-valued suprametric spaces, we demonstrate the Banach fixed point theorem in this section.

Theorem 3.1. *If $(X, \mathcal{A}, d)$ is a complete C^*-algebra-valued suprametric space and $T : X \to X$ is contractive mapping, there exists a unique fixed point in X.*

Proof. Let $(X, \mathcal{A}, d)$ be a complete C^*-algebra-valued suprametric space with $A \succeq \theta$ (The case $A = \theta$ gives the Banach contraction principle). Define the sequence $\{\varpi_n\}$ by $\varpi_{n+1} = T\varpi_n, \forall n$. Consider,

$$\begin{aligned} d(\varpi_n, \varpi_{n+1}) &= d(T\varpi_{n-1}, T\varpi_n) \\ &\preceq B^* d(\varpi_{n-1}, \varpi_n) B \\ \Rightarrow \|d(\varpi_n, \varpi_{n+1})\| &\leqslant \|B\|^2 \|d(\varpi_{n-1}, \varpi_n)\| \\ &< \|d(\varpi_{n-1}, \varpi_n)\|. \end{aligned}$$

Thus, for all given integer k, the sequence $\{d(\varpi_n, \varpi_{n+1})\}$ is decreasing and fulfils

$$d(\varpi_n, \varpi_{n+1}) \preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}, \forall n > k.$$

This implies

$$\lim_{n\to\infty} \|d(\varpi_n, \varpi_{n+1})\| = 0$$

As a result, given $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have $\|d(\varpi_n, \varpi_{n+1})\| \leqslant 1$. We will now show that the sequence $\{\varpi_n\}$ is Cauchy. Consider

$$\begin{aligned} d(\varpi_n, \varpi_m) &\preceq d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_m) + A d(\varpi_n, \varpi_{n+1}) d(\varpi_{n+1}, \varpi_m) \\ &\preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} + d(\varpi_{n+1}, \varpi_m) \\ &\quad + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} d(\varpi_{n+1}, \varpi_m) \\ &= (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} + [1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] \\ &\quad d(\varpi_{n+1}, \varpi_m) \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} d(\varpi_{n+1}, \varpi_m) &\preceq d(\varpi_{n+1}, \varpi_{n+2}) + d(\varpi_{n+2}, \varpi_m) + A d(\varpi_{n+1}, \varpi_{n+2}) d(\varpi_{n+2}, \varpi_m) \\ &\preceq (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} + d(\varpi_{n+2}, \varpi_m) \\ &\quad + A (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} d(\varpi_{n+2}, \varpi_m) \\ &= (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} + [1 + A (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] \\ &\quad d(\varpi_{n+2}, \varpi_m) \end{aligned} \tag{3.2}$$

Consequently, inequality (3.1) becomes

$$\begin{aligned} d(\varpi_n, \varpi_m) &\preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} \\ &\quad + [1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] \big[(B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} \\ &\quad + [1 + A (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] d(\varpi_{n+2}, \varpi_m) \big] \\ &= (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} + [1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] \\ &\quad (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} + [1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] \\ &\quad [1 + A (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] d(\varpi_{n+2}, \varpi_m) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} d(\varpi_{n+2}, \varpi_m) &\preceq d(\varpi_{n+2}, \varpi_{n+3}) + d(\varpi_{n+3}, \varpi_m) + A d(\varpi_{n+2}, \varpi_{n+3}) d(\varpi_{n+3}, \varpi_m) \\ &\preceq (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2} + d(\varpi_{n+3}, \varpi_m) \\ &\quad + A (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2} d(\varpi_{n+3}, \varpi_m) \\ &= (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2} \\ &\quad + [1 + A (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2}] d(\varpi_{n+3}, \varpi_m) \end{aligned} \tag{3.4}$$

Thereby inequality (3.3) becomes,

$$\begin{aligned} d(\varpi_n, \varpi_m) &\preceq (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k} \\ &\quad + [1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1} \\ &\quad + [1 + A (B^*)^{n-k} d(\varpi_k, \varpi_{k+1}) B^{n-k}] [1 + A (B^*)^{n-k+1} d(\varpi_k, \varpi_{k+1}) B^{n-k+1}] \\ &\quad (B^*)^{n-k+2} d(\varpi_k, \varpi_{k+1}) B^{n-k+2} \end{aligned}$$

$$
\begin{aligned}
&+[1+A(B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}][1+A(B^*)^{n-k+1}d(\varpi_k,\varpi_{k+1})B^{n-k+1}]\\
&\quad[1+A(B^*)^{n-k+2}d(\varpi_k,\varpi_{k+1})B^{n-k+2}]d(\varpi_{n+3},\varpi_m)\\
&\quad\vdots\\
&\preceq (B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}\\
&+[1+A(B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}](B^*)^{n-k+1}d(\varpi_k,\varpi_{k+1})B^{n-k+1}\\
&+[1+A(B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}][1+A(B^*)^{n-k+1}d(\varpi_k,\varpi_{k+1})B^{n-k+1}]\\
&\quad(B^*)^{n-k+2}d(\varpi_k,\varpi_{k+1})B^{n-k+2}\\
&\quad\vdots\\
&+[1+A(B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}][1+A(B^*)^{n-k+1}d(\varpi_k,\varpi_{k+1})B^{n-k+1}]\cdots\\
&\quad[1+A(B^*)^{m-k-3}d(\varpi_k,\varpi_{k+1})B^{m-k-3}](B^*)^{m-k-2}d(\varpi_k,\varpi_{k+1})B^{m-k-2}\\
&+[1+A(B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}][1+A(B^*)^{n-k+1}d(\varpi_k,\varpi_{k+1})B^{n-k+1}]\cdots\\
&\quad[1+A(B^*)^{m-k-2}d(\varpi_k,\varpi_{k+1})B^{m-k-2}]d(\varpi_{m-1},\varpi_m)\\
&\preceq (B^*)^{n-k}d(\varpi_k,\varpi_{k+1})B^{n-k}\\
&+\sum_{i=1}^{m-n-1}(B^*)^{n-k+i}d(\varpi_k,\varpi_{k+1})B^{n-k+i}\prod_{j=1}^{i}[1+A(B^*)^{n-k+j-1}d(\varpi_k,\varpi_{k+1})B^{n-k+j-1}]
\end{aligned}\tag{3.5}
$$

Taking $d(\varpi_k,\varpi_{k+1})=B_k$ in inequality (3.5), we get

$$
\begin{aligned}
d(\varpi_n,\varpi_m) &\preceq (B^*)^{n-k}B_kB^{n-k}\\
&\quad+\sum_{i=1}^{m-n-1}(B^*)^{n-k+i}B_kB^{n-k+i}\prod_{j=1}^{i}[1+A(B^*)^{n-k+j-1}B_kB^{n-k+j-1}]\\
&=(B^*)^{n-k}B_k^{1/2}B_k^{1/2}B^{n-k}+\sum_{i=1}^{m-n-1}(B^*)^{n-k+i}B_k^{1/2}B_k^{1/2}B^{n-k+i}\\
&\quad\prod_{j=1}^{i}[1+A^{1/2}(B^*)^{n-k+j-1}B_k^{1/2}A^{1/2}B^{n-k+j-1}B_k^{1/2}]\\
&=\left[B^{n-k}B_k^{1/2}\right]^*\left[B_k^{1/2}B^{n-k}\right]+\sum_{i=1}^{m-n-1}\left[B^{n-k+i}B_k^{1/2}\right]^*\left[B_k^{1/2}B^{n-k+i}\right]\\
&\quad\prod_{j=1}^{i}\left[1+\left[A^{1/2}B^{n-k+j-1}B_k^{1/2}\right]^*\left[B_k^{1/2}B^{n-k+j-1}A^{1/2}\right]\right]\\
&=|B^{n-k}B_k^{1/2}|^2\\
&\quad+\sum_{i=1}^{m-n-1}|B^{n-k+i}B_k^{1/2}|^2\prod_{j=1}^{i}\left[1+|A^{1/2}B^{n-k+j-1}B_k^{1/2}|^2\right]
\end{aligned}
$$

$$\preceq \|B^{n-k}B_k^{1/2}\|^2 I + \sum_{i=1}^{m-n-1} \|B^{n-k+i}B_k^{1/2}\|^2 I \prod_{j=1}^{i}\left[1+\|A^{1/2}B^{n-k+j-1}B_k^{1/2}\|^2 I\right]$$
$$\preceq \|B^{n-k}\|^2\|B_k^{1/2}\|^2 I + \sum_{i=1}^{m-n-1} \|B^{n-k+j}\|^2\|B_k^{1/2}\|^2 I \prod_{j=1}^{i}\left[1+\|A^{1/2}\|^2\|B^{n-k+j-1}\|^2\|B_k^{1/2}\|^2 I\right]$$
$$\preceq \|B\|^{2(n-k)} I + \sum_{i=1}^{m-n-1} \|B\|^{2(n-k+i)} I \prod_{j=1}^{i}\left[1+\|A\|\|B\|^{2(n-k+j-1)} I\right] \tag{3.6}$$

Let

$$y_i = \|B\|^{2(n-k+i)} I \prod_{j=1}^{i}\left(1+\|A\|\|B\|^{2(n-k+j-1)} I\right).$$

By ratio test, the series $\sum_{i=1}^{\infty} y_i$ converges since $\lim_{i\to\infty}\left\|\frac{y_{i+1}}{y_i}\right\| < 1$ and $\|B\| < 1$. Inequality (3.6) thus lead to the conclusion that $d(\varpi_n, \varpi_m)$ tends to θ as n, m tends to infinity. Subsequently $\{\varpi_n\}$ is a Cauchy sequence with regard to $\mathcal{A}$. Due to the condition that $(X, \mathcal{A}, d)$ is a complete C*-algebra-valued suprametric space, we can find an $\varpi \in X$ so that $\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} T\varpi_{n-1} = \varpi$. Because of the fact that $d(T\varpi, \varpi) \succeq \theta$, we get

$$\begin{aligned} \theta &\preceq d(T\varpi, \varpi) \\ &\preceq d(T\varpi, T\varpi_n) + d(T\varpi_n, \varpi) + A d(T\varpi, T\varpi_n) d(T\varpi_n, \varpi) \\ &\preceq B^* d(\varpi, \varpi_n) B + d(\varpi_{n+1}, \varpi) + A\, B^* d(\varpi, \varpi_n) B\, d(\varpi_{n+1}, \varpi) \\ &\to \theta, \text{ when } n \to \infty \end{aligned} \tag{3.7}$$

Accordingly, $d(T\varpi, \varpi) = \theta$, indicating that $T\varpi = \varpi$ i.e., ϖ is a fixed point of T. With ease, we can demonstrate that ϖ is a unique fixed point of T by applying the inequality (2.1). □

4. Existence and Uniqueness of Solutions for an Integral Equation

In this section, we verify the uniqueness and existence of solutions to integral equations.

Example 4.1. Consider the integral equation

$$\varpi(t) = \int_E K(t, \varpi(s))ds + g(t), t \in E$$

wherein E is a Lebesgue measurable set. Assume that

1. $K : E \times \mathbb{R}^+ \to \mathbb{R}^+$ and $g \in L^\infty(E)$;
2. there exists a continuous function $\phi : E \times E \to \mathbb{R}$ and $q \in (0, 1)$ so that $|K(t, \varpi(s)) - K(t, \rho(s))| \leqslant q|\phi(t, s)||\varpi(s) - \rho(s)|$ for $t \in E$ and $\varpi, \rho \in L^\infty(E)$;
3. $\sup_{t\in E} \int_E |\phi(t, s)|ds \leqslant 1$;

Then, ϖ^* is the unique solution of the integral equation in $L^\infty(E)$.

Proof. Let $X = L^\infty(E)$ and $H = L^2(E)$. For $\varpi, \rho \in X$ and $\beta > 0$, we set $d : X \times X \to L(H)$ by

$$d(\varpi, \rho) = \pi_{|\varpi-\rho|(|\varpi-\rho|+\beta)}$$

wherein $\pi_h : H \to H$ is the multiplication operator defined by

$$\pi_\eta(\phi) = h.\eta$$

for $\eta \in H$. Then d is a C^*-algebra-valued suprametric by Example 2.2 and $(X, L(H), d)$ is a complete C^*-algebra-valued suprametric space. Let $T : L^\infty(E) \to L^\infty(E)$ be specified by

$$T\varpi(t) = \int_E K(t, \varpi(s))ds + g(t), \quad t \in E$$

Setting $B = qI$, we see that $B \in L(H)_+$ and $\|B\| = q < 1$. For any $h \in H$, we have

$$\begin{aligned}
\|d(T\varpi, T\rho)\| &= \sup_{\|h\|=1} (\pi_{|T\varpi-T\rho|(|T\varpi-T\rho|+\beta)}h, h) \\
&= \sup_{\|h\|=1} \int_E \left[\left| \int_E \Big([K(t,\varpi(s)) - K(t,\rho(s))]\Big([K(t,\varpi(s)) - K(t,\rho(s))] + \beta\Big) \Big) ds \right| \right] h(t)\overline{h(t)}dt \\
&\leqslant \sup_{\|h\|=1} \int_E \left[\int_E \Big(|K(t,\varpi(s)) - K(t,s,\rho(s))|\Big(|K(t,\varpi(s)) - K(t,s,\rho(s))| + \beta\Big) \Big) ds \right] |h(t)|^2 dt \\
&= \sup_{\|h\|=1} \int_E \left[\int_E \Big(|K(t,\varpi(s)) - K(t,\rho(s))|^2 + \beta |K(t,\varpi(s)) - K(t,\rho(s))| \Big) ds \right] |h(t)|^2 dt \\
&\leqslant \sup_{\|h\|=1} \int_E \left[\int_E \Big(q^2 |\phi(t,s)|^2 |\varpi(s) - \rho(s)|^2 + \beta q |\phi(t,s)||\varpi(s) - \rho(s)| \Big) ds \right] |h(t)|^2 dt \\
&\leqslant \left(\sup_{\|h\|=1} \int_E |h(t)|^2 dt \right) \left(q^2 \sup_{t\in E} \int_E |\phi(t,s)|^2 ds . \|\varpi-\rho\|_\infty^2 + \beta q \sup_{t \in E} \int_E |\phi(t,s)| ds . \|\varpi - \rho\|_\infty \right) \\
&\leqslant q^2 \|\varpi - \rho\|_\infty^2 + \beta q \|\varpi - \rho\|_\infty \\
&\leqslant q \|\varpi - \rho\|_\infty \Big[\|\varpi - \rho\|_\infty + \beta \Big] \\
&= \|B\| \|d(\varpi, \rho)\|
\end{aligned}$$

The integral equation has a unique solution ϖ^* in $L^\infty(E)$ due to $\|B\| < 1$. □

5. Conclusion

Extending the class of spaces with stronger theoretical frameworks than metric spaces is a potential way to improve the scope of fixed point theory further. Thus, this study aimed to introduce the new metric space, namely C^*-algebra-valued suprametric spaces with an illustration. Moreover, the Banach type contraction in the setting of suprametric spaces with C^*-algebra algebra is investigated, and a fixed point theorem is proved beneath the related contraction. Finally, by employing the proven fixed point theorem, the existence and uniqueness of solutions to integral equations are verified. The future research will be focused on presenting a number of interesting fixed point results in the setting of generalized suprametric spaces over C^*-algebra with applications to fractional differential equation problems.

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