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Some Chebyshev type inequalities on time scales

STEPHEN N. AJEGA-AKEM [*,a,c], MOHAMMED M. IDDRISU[b] AND KWARA NANTOMAH[a]

[a] Department of Mathematics, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, Navrongo , Ghana

[b] Department of Mathematics, Faculty of Physical Sciences, University for Development Studies, Tamale , Ghana

[c] Department of Mathematics, St. John Bosco College of Education, Navrongo, Ghana

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Abstract

The extension of classical inequalities into time scale aims to generalize these inequalities to make them applicable to dynamic systems that exhibit both continuous and discrete characteristics. Time-scale calculus is becoming an unavoidable area as it is gaining ground in most areas of study. In this work, we presented some Chebyshev and other type inequalities on time scales by applying the Cauchy-Schwarz and Arithmetic-Geometric Mean inequality.

Keywords: : Chebyshev inequality, Cauchy-Schwarz inequality, time scales.
2010 MSC: 2010 (MSC): 26D15, 26E70, 26A33.

1. Introduction

The inequality

$$\frac{1}{b-a}\int_a^b x(t)dt\frac{1}{b-a}\int_a^b y(t)dt \leqslant \frac{1}{b-a}\int_a^b x(t)y(t)dt, \tag{1.1}$$

where x, y are both non-increasing or non-decreasing function on $[a, b]$ is the Chebyshev inequality[8]. Inequality (1.1) is well known in literature as one of the foundational inequality in statistical estimation and in mathematical analysis. There are some other literature on (1.1) in terms of applications, extensions and generalizations [2, 7, 10, 11, 14, 15, 16, 17, 18, 20, 21].

According to[19], we need to develop and refine our capabilities to generalize recent results directly related to the topic of fractional calculus because fractional calculus is interesting and has many applications in modeling natural phenomena in the world. Time

*Corresponding author: akemesteve@gmail.com

Scale calculus is also another area of importance that we should also consider. A time scale is an arbitrary nonempty closed subset of the real numbers and is usually denoted by $\mathbb{T}$. Time scale calculus gives an efficient tool to unify continuous and discrete problems in one theory. Time scale calculus provides an integrated framework that can handle a broader collection of systems and data types and this makes time scale calculus more valuable in various applications where traditional calculus falls short. When $\mathbb{T} = \mathbb{R}$, Time Scale reduces to real analysis and when $\mathbb{T} = \mathbb{Z}$, Time Scale reduces to discrete analysis. Time scale calculus is an area of mathematics that has recently received a lot of attention from researchers. This area was born by Stefan Hilger in 1988 [13]. Since the discovery of time scale calculus till date, time scale has made a lot of steady inroads in explaining the interconnections that exist among the various calculi. The birth of time scale by Stefan Hilger was therefore to unify discrete and continuous analysis. The extension of classical inequalities into time scale in recent times among researchers is on the increase due to influence of time scale in mathematical analysis. The extension of classical inequalities into time scale is an aim of generalizing these inequalities and well-established mathematical concepts to make them applicable to dynamic systems that exhibit both continuous and discrete characteristics. Time scale calculus is becoming unavoidable as it is gaining grounds in most areas of study. Since its inception, time scale calculus has found applications in entomology, computer science, medical sciences, macroeconomics and other key areas of life [5].

2. Preliminaries

Definition 2.1. [1] Let $t \in \mathbb{T}$, then t is called right dense if $\sigma(t) = t$. It is called right scattered if $\sigma(t) > t$. It is called left scattered if $\rho(t) < t$. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are said to be dense.

Definition 2.2. Let $\sup \mathbb{T}$ be finite and left-scattered, then we defined $\mathbb{T}^{\star} = \mathbb{T} \backslash \{\sup \mathbb{T}\}$.

Lemma 2.3. *[1] Let $\mathbb{T}$ be an arbitrarily given time scale. Also, let f be delta differentiable at $t \in \mathbb{T}^{\star}$. Then*

$$f^{\Delta}(t) = \frac{f(\sigma(t) - f(t))}{\mu(t)}. \tag{2.1}$$

Definition 2.4. Let $f : f \in \mathbb{T}$. When $\mathbb{T} = \mathbb{R}$, then we have

$$f^{\Delta} = f^{\nabla} = f' \tag{2.2}$$

and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt. \tag{2.3}$$

Definition 2.5. [12] Let a and b be two arbitrary points in $\mathbb{T}$. Then every constant function $f(t) = c$, for $t \in \mathbb{T}$ is Δ- integrable from a to b and is given by

$$\int_a^b c\Delta t = c(b-a). \tag{2.4}$$

Definition 2.6. [4] The graininess function $\mu\colon \mathbb{T} \longrightarrow [0,\infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{2.5}$$

Definition 2.7. [4] Let t be an arbitrary point in $\mathbb{T}$. Every function f defined on $\mathbb{T}$ is Δ-integrable from t to $\sigma(t)$ and is expressed as

$$\int_t^{\sigma(t)} f(x)\Delta x = (\sigma(t) - t)f(t). \tag{2.6}$$

Definition 2.8. [1] Let f, g be two differentiable functions on $\mathbb{T}$, then

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t). \tag{2.7}$$

Theorem 2.9. *[9] Let $\mathbb{T}$ be a bounded time scales. Let also $f, g : \mathbb{T} \to \mathbb{R}$ be integrable functions, and $\alpha \in \mathbb{R}$ a constant. Then the Constant Multiple Rule, Sum rule and Combination rule are defined respectively by (2.8), (2.9) and (2.10).*

$$\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t, \tag{2.8}$$

$$\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t \tag{2.9}$$

and

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t. \tag{2.10}$$

Definition 2.10. [6] A function $F\colon \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f\colon \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. The delta indefinite and definite integrals are defined by the following equations respectively.

$$\int f(t)\Delta t = F(t) + C, \tag{2.11}$$

$$\int_a^b f(s)\Delta s = F(b) - F(a). \tag{2.12}$$

Definition 2.11. Let x, y be random variables defined to assumed values in $\mathbb{I} \in \mathbb{R}$. Then the two variable x, y are identically distributed if

$$F(x) = F(y), \tag{2.13}$$

for all $x, y \in \mathbb{I}$ and $F\colon \mathbb{T} \to \mathbb{R}$.

Lemma 2.12. *Let f be continuous on $[a, b]$ and differentiable on (a, b), then there exist $c \in (a, b))$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{2.14}$$

The integral form of (2.14) is given by

$$\int_a^b f(t)dt = f(c)(b - a). \tag{2.15}$$

The classical discrete and integral forms of the Cauchy-Schwarz inequality are defined respectively below by the following definitions.

Definition 2.13. Let $a_1, a_2 ... a_n$ and $b_1, b_2 ... b_n$ be real numbers then,

$$\left(\sum_{i=1}^{n} |a_i b_i|\right)^2 \leqslant \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2. \tag{2.16}$$

Definition 2.14. Let f, g be continuous functions on $[a, b]$ then

$$\int_a^b |f(t)g(t)|dt \leqslant \sqrt{\int_a^b |f(t)|^2 dt. \int_a^b |g(t)|^2 dt.} \tag{2.17}$$

Lemma 2.15. *Let ξ and μ positve real numbers. Then the Arithmetic-Geometric Mean Inequality is defined as*

$$\frac{\xi + \mu}{2} \geqslant \sqrt{\xi\mu}, \tag{2.18}$$

for $\xi > \mu$, equality occurs when $\xi = \mu$.

Lemma 2.16. *[3] For the positive integrable functions η, ζ on $[0, \infty)$ and for the positive integrable functions G_1, G_2, H_1, H_2, let $\nu \in (0, 1]$, $\gamma \in \mathbb{C}$, $\Re(\gamma) \geqslant 0$, and ψ be a strictly increasing continuous function. If the following conditions: $(Z_1) 0 < G_1(\mu) \leqslant \eta(\mu) \leqslant G_2(\mu), 0 < H_1(\mu) \leqslant \zeta(\mu) \leqslant H_2(\mu), (\mu \in [0, \omega], \omega > 0)$, hold, then the following inequality also holds:*

$$\frac{{}^{(k,\psi)}\mathbb{I}^{\gamma,\nu}\left\{H_1H_2\eta^2\right\}(\omega)\,{}^{(\gamma,\nu)}\mathbb{I}^{\gamma,\nu}\left\{G_1G_2\zeta^2\right\}(\omega)}{\left({}^{(k,\psi)}\mathbb{I}^{\gamma,\nu}\{(G_1H_1+G_2H_2)\eta\zeta\}(\omega)\right)^2} \leqslant \frac{1}{4}. \tag{2.19}$$

The above inequality was obtained through the application of Lemma 2.15.

3. Results and Discussion

Theorem 3.1. *Let f and g are real-valued integrable functions over $[a, b]$ and that both are non-increasing or non-decreasing, $h \geqslant 0$ for all $a, b \in \mathbb{R}$. Then*

$$\begin{aligned}(b-a)\int_a^b f(s)g(s)h(s)\Delta s + \int_a^b f(s)g(s)\Delta s\int_a^b h(s)\Delta s \geqslant \\ \int_a^b f(s)h(s)\Delta s\int_a^b g(s)\Delta s + \int_a^b f(s)\Delta s\int_a^b g(s)h(s)\Delta s,\end{aligned} \tag{3.1}$$

with the inequality reversed if one is non-increasing and the other is non-decreasing.

Proof. Since f, g are both non-increasing or non-decreasing, then for all $s, t \geqslant 0$ we have

$$[f(s)-f(t)][g(s)-g(t)] \geqslant 0 \tag{3.2}$$

and

$$h(s)+h(t) \geqslant 0 \tag{3.3}$$

since $h \geqslant 0$.
Using (3.2) and (3.3) we have

$$\begin{aligned}\int_a^b\int_a^b [f(s)g(s)h(s)+f(s)g(s)h(t)+f(t)g(t)h(s)+f(t)g(t)h(t)- \\ f(s)g(t)h(s)-f(t)g(s)h(t)-f(s)g(t)h(t)-f(t)g(s)h(t)]\Delta s\Delta t \geqslant 0.\end{aligned} \tag{3.4}$$

Which further gives

$$\begin{aligned}(b-a)\int_a^b f(s)g(s)h(s)\Delta s + \int_a^b f(s)g(s)\Delta s\int_a^b h(t)\Delta t + \int_a^b f(t)g(t)\Delta t\int_a^b h(s)\Delta s \\ +(b-a)\int_a^b f(t)g(t)h(t)\Delta t - \int_a^b f(s)h(s)\Delta s\int_a^b g(t)\Delta t - \int_a^b f(t)h(t)\Delta t\int_a^b g(s)\Delta s \\ -\int_a^b f(s)\Delta s\int_a^b g(t)h(t)\Delta t - \int_a^b f(t)h(t)\Delta t\int_a^b g(s)\Delta s \geqslant 0.\end{aligned} \tag{3.5}$$

Applying (2.13) to (3.5) and simplifying, yields the required result.

□

Corollary 3.2. *Let* f, g *be two positive non-decreasing functions on the interval* $\mathbb{I} = [0, a]_{\mathbb{T}}$ *such that* f, g *are time scale integrable and* s, t *are identically distributed. Then*

$$\left(\int_0^a f(s)\Delta s\right)\left(\int_0^a g(s)\Delta s\right) \leqslant a\int_0^a f(s)g(s)\Delta s. \tag{3.6}$$

Proof. Let f, g be synchronous functions for all $s, t \in [0, a]_{\mathbb{T}}$. Then

$$\int_0^a\int_0^a [f(s) - f(t)][g(s) - g(t)]\Delta s\Delta t \geqslant 0.$$

Thus

$$\int_0^a\int_0^a [f(s)g(s) - f(t)g(s) - f(s)g(t) + f(t)g(t)]\Delta s\Delta t \geqslant 0,$$

and

$$a\int_0^a f(s)g(s)\Delta s - \int_0^a f(t)\Delta t\int_0^a g(s)\Delta s - \int_0^a f(s)\Delta s\int_0^a g(t)\Delta t + a\int_0^a f(t)g(t)\Delta t \geqslant 0.$$

Since s, t are identically distributed, then by (2.13) we have

$$a\int_0^a f(s)g(s)\Delta s \geqslant \left(\int_0^a f(s)\Delta s\right)\left(\int_0^a g(s)\Delta s\right) \tag{3.7}$$

□

Corollary 3.3. *If* f *and* g *are real-valued integrable functions over* $[a, b]$, *both non-increasing or non-decreasing and* $h > 0$ *for all* a, b *in* $\mathbb{R}$. *Then*

$$\int_a^b f(s)g(s)\Delta s \geqslant \frac{1}{(b-a)}\int_a^b f(s)\Delta s\frac{1}{(b-a)}\int_a^b g(s)\Delta s \tag{3.8}$$

Proof. From (3.1) when $h(s) = 1$ and by (2.4),we have

$$\begin{aligned} &\int_a^b f(s)g(s)\Delta s + \int_a^b f(s)g(s)\Delta s \geqslant \\ &\frac{1}{(b-a)}\int_a^b f(s)\Delta s\frac{1}{(b-a)}\int_a^b g(s)\Delta s + \frac{1}{(b-a)}\int_a^b f(s)\Delta s\frac{1}{(b-a)}\int_a^b g(s)\Delta s. \end{aligned} \tag{3.9}$$

Hence we obtained (3.8) from (3.9) by simplification. □

Remark 3.4. Let $\mathbb{T} = \mathbb{R}$. Then (3.8) becomes

$$\frac{1}{(b-a)}\int_a^b f(s)g(s)ds \geqslant \frac{1}{(b-a)}\int_a^b f(s)ds\frac{1}{(b-a)}\int_a^b g(s)ds \tag{3.10}$$

which is the classical Chebyshev inequality in (1.1).

Theorem 3.5. *Let* f *and* g *be two real-valued integrable functions over* $[a, b]$. *Also, let* $\phi \geqslant 0$ *for all* $[t, \sigma(t)]$ *in* $\mathbb{T}$ *such that* $\phi(t) \leqslant \frac{f(x)}{g(x)} \leqslant \phi(\sigma(t)), \quad g(x) \neq 0$. *Then*

$$\frac{1}{(\phi(t)+\phi(\sigma(t)))}\int_a^b \left(f^2(x) + \frac{\phi(\sigma(t))\phi(t)}{(b-a)}g^2(x)\right)\Delta x \leqslant \left(\int_a^b f^2(x)\Delta x\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\Delta x\right)^{\frac{1}{2}}. \tag{3.11}$$

Proof. Since

$$\phi(t) \leqslant \frac{f(x)}{g(x)} \leqslant \phi(\sigma(t)),$$

we have

$$\left[\frac{f(x)}{g(x)} - \phi(t)\right]\left[\phi(\sigma(t)) - \frac{f(x)}{g(x)}\right]g^2(x) \geqslant 0. \tag{3.12}$$

Which further yields

$$\int_a^b \left(f(x)g(x)\phi(\sigma(t)) - f^2(x) - \phi(\sigma(t))\phi(t)g^2(x) + \phi(t)f(x)g(x)\right)\Delta x \geqslant 0,$$

and

$$\frac{\phi(t)+\phi(\sigma(t))}{b-a}\int_a^b f(x)g(x)\Delta x \geqslant \frac{1}{b-a}\int_a^b f^2(x)\Delta x + \frac{\phi(\sigma(t))\phi(t)}{b-a}\int_a^b g^2(x)\Delta x.$$

By (2.17)we have

$$\frac{\phi(t)+\phi(\sigma(t))}{b-a}\left(\int_a^b f^2(x)\Delta x\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\Delta x\right)^{\frac{1}{2}} \geqslant \frac{1}{b-a}\int_a^b f^2(x)\Delta x + \frac{\phi(\sigma(t))\phi(t)}{b-a}\int_a^b g^2(x)\Delta x, \tag{3.13}$$

$$\frac{\phi(t)+\phi(\sigma(t))}{b-a}\left(\int_a^b f^2(x)\Delta x\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\Delta x\right)^{\frac{1}{2}} \geqslant \frac{1}{b-a}\int_a^b \left(f^2(x) + \frac{\phi(\sigma(t))\phi(t)}{(b-a)}g^2(x)\right)\Delta x,$$

and

$$\left(\int_a^b f^2(x)\Delta x\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\Delta x\right)^{\frac{1}{2}} \geqslant \frac{1}{\phi(t)+\phi(\sigma(t))}\int_a^b \left(f^2(x) + \frac{\phi(\sigma(t))\phi(t)}{(b-a)}g^2(x)\right)\Delta x.$$

□

Corollary 3.6. *Let* f *and* g *be two real-valued integrable functions over* $[a, b]$. *Also, let* $\phi \geqslant 0$ *for all* $[t, \sigma(t)]$ *in* $\mathbb{T}$ *such that* $\phi(t) \leqslant \frac{f(x)}{g(x)} \leqslant \phi(\sigma(t)), g(x) \neq 0$. *Then*

$$\frac{\phi(t)+\phi(\sigma(t))}{b-a}\left(\int_a^b f^2(x)\Delta x\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)\Delta x\right)^{\frac{1}{2}} \geqslant \frac{2\phi(\sigma(t))\phi(t)\int_a^b f^2(x)\Delta x\int_a^b g^2(x)\Delta x}{(b-a)\left(\phi(\sigma(t))\phi(t)\int_a^b g^2(x)\Delta x+\int_a^b f^2(x)\Delta x\right)}. \tag{3.14}$$

Proof. Recalling (3.13) and applying AM-GM inequality yields the desired result. □

Theorem 3.7. *Let* $f\colon \mathbb{T} \to \mathbb{R}$ *be differentiable and* $\mathbb{T}$ *a given time scale such that* $\sigma(t)$ *is forward jump operator and* f *is* Δ *differentiable at* $t \in \mathbb{T}^{\star}$*. Then*

$$\left(\int_0^1 \frac{f^{\Delta}(t^2)^{\Delta}\sigma(t)_{\Delta t}}{4}\right)^2 \leqslant f(\mu(t))f(t). \tag{3.15}$$

and

$$\left(\frac{\int_0^1 f^{\Delta}\left(\int_t^{\sigma(t)} f(\tau)\Delta\tau\right)\sigma(t)_{\Delta\tau}}{4}\right)^2 \leqslant f(\mu(t))f(t). \tag{3.16}$$

Proof. Let $\phi(\tau) = f\,(t+\tau\sigma(t))$.

When $\tau = 0$, we have

$$\phi(0) = f(t). \tag{3.17}$$

When $\tau = 1$ we have

$$\phi(1) = f(t+\sigma(t)). \tag{3.18}$$

Subtracting (3.17) from (3.18) yields

$$\phi(1)-\phi(0) = f(t+\sigma(t)) - f(t) = \int_0^1 \phi^{\Delta}(\tau)_{\Delta\tau}. \tag{3.19}$$

But

$$\phi^{\Delta}(\tau) = f^{\Delta}(t+\tau\sigma(t))\sigma(t). \tag{3.20}$$

Thus

$$f(t+\sigma(t)) - f(t) = \int_0^1 f^{\Delta}(t+\tau\sigma(t))\sigma(t)_{\Delta\tau},$$

which further becomes

$$f(\sigma(t)) = \int_0^1 f^{\Delta}(t+\tau\sigma(t))\sigma(t)_{\Delta\tau}. \tag{3.21}$$

When $\tau = 1$, (3.21) becomes

$$f(\mu(t)+t) = \int_0^1 f^\Delta (t^2)^\Delta \sigma(t)_{\Delta\tau}$$

which further yields

$$\int_0^1 f^\Delta (t^2)^\Delta \sigma(t)_{\Delta\tau} \leqslant 2\sqrt{f(\mu(t))f(t)}. \tag{3.22}$$

That ends the proof for 3.15.

Let $\tau = \frac{[\sigma(t)-t]f(t)-t}{\sigma(t)}$ in (3.21). Then we have

$$\int_0^1 f^\Delta \left(\int_t^{\sigma(t)} f(t)\Delta t \right) f(\sigma(t))_{\Delta\tau} \leqslant 2\sqrt{f(\mu(t))f(t)}. \tag{3.23}$$

□

4. Conclusion

The extension of classical inequalities into time scale is aiming at generalizing such well-established mathematical concepts to make them applicable to dynamic systems that exhibit both continuous and discrete characteristics. This paper presented some Chebyshev type integral inequalities on time scales by applying Cauchy-Schwarz inequality and arithmetic-geometric inequality. The Chebyshev type integral inequalities presented in this work reduces to the classical Chebyshev type inequalities when the time scale ($\mathbb{T}$) is on the real ($\mathbb{R}$). In future work, we are going consider studying Hermite–Hadamard-type inequalities on time scale which are equally important in the research community.

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