

Beta-Fractional Calculus on Time Scales

İSMAİL YASLAN 

Pamukkale University, Department of Mathematics, 20070 Denizli, Turkey

• Received: 20 October 2023

• Accepted: 10 November 2023

• Published Online: 27 December 2023

Abstract

In this paper, $\beta - \Delta$ fractional derivative and $\beta - \Delta$ fractional integral on time scales are defined and their basic properties are obtained. Then, $\beta - \nabla$ fractional calculus on arbitrary time scales is introduced.

Keywords: Beta-fractional derivative, beta-fractional integral, time scales.

2010 MSC: 26A33, 26E70.

1. Introduction

The calculus on time scales was introduced by Aulbach and Hilger [1], in order to unify and generalize continuous and discrete analysis. A nonempty closed subset of real numbers $\mathbb{R}$ is called a time scale $\mathbb{T}$. Some basic definitions and theorems on time scales can be found in the book [2] and another excellent source on time scales is the book [3]. The study of time scales has led to many important applications, e.g. in the study of epidemic models, insect population models, heat transfer, and neural networks [4]. Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc.; see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Abdon Atangana suggested the β -fractional derivative recently in [17, 18, 19]. The suggested version fulfills many characteristics that have been utilized to simulate various physical issues and have served as limitations for fractional derivatives. The beta derivative of f of order α is defined as

$$D_t^\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f[t + \varepsilon(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}] - f(t)}{\varepsilon}, \quad \alpha \in (0, 1], \quad t > 0.$$

The idea to join the fractional calculus and the calculus on time scales was born with the Ph.D. thesis of Bastos [20]. After the inception of the topic, a number of papers were published see [21, 22, 23, 24, 25, 26]. In this study, we will give the concept of β -fractional derivative and integral on time scales. Our new calculus unifies and generalizes the time scale calculus and the β -fractional calculus.

*Corresponding author: iyaslan@pau.edu.tr

2. $\beta - \Delta$ - Fractional Derivative

Definition 2.1. Assume that $\alpha \in (0, 1]$, $\mathbb{T}$ is a time scale and $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function. For all $\varepsilon > 0$, if there is a neighborhood $\mathcal{U}$ of $t \in \mathbb{T}^k$ ($t > 0$) such that

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in \mathcal{U},$$

the $\beta - \Delta$ - fractional derivative of f of order α at t is defined by the number ${}^\beta T_\alpha^\Delta(f)(t)$. $\beta - \Delta$ - fractional derivative of f of order α at 0 is defined by ${}^\beta T_\alpha^\Delta(f)(0) = \lim_{t \rightarrow 0^+} {}^\beta T_\alpha^\Delta(f)(t)$.

Note that when $\alpha = 1$, we have ${}^\beta T_\alpha^\Delta(f)(t) = f^\Delta(t)$ and if $\mathbb{T} = \mathbb{R}$, then ${}^\beta T_\alpha^\Delta(f)(t) = D_t^\alpha(f)(t)$ is the β - fractional derivative of f of order α .

Theorem 2.2. Suppose that $\alpha \in (0, 1]$, $\mathbb{T}$ is a time scale, $t \in \mathbb{T}^k$ ($t > 0$) and $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function. Then, we have the following properties.

- (i) If f has $\beta - \Delta$ - fractional derivative of order α at t , then f is continuous at t .
- (ii) Let f be continuous at t and t be right-scattered. Then, f has $\beta - \Delta$ - fractional derivative of order α at t and we have ${}^\beta T_\alpha^\Delta(f)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.
- (iii) When t is right-dense, f has $\beta - \Delta$ - fractional derivative of order α at t iff the limit $\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$ exists as a finite number. Then, we have

$${}^\beta T_\alpha^\Delta(f)(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

- (iv) If f has $\beta - \Delta$ - fractional derivative of order α at t , then we get

$$f(\sigma(t)) = f(t) + \mu(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} {}^\beta T_\alpha^\Delta(f)(t).$$

Proof. (i) If f has $\beta - \Delta$ - fractional derivative of order α at t , given any $\varepsilon > 0$, there is a neighborhood $\mathcal{U} = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(r)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - r)| \leq \varepsilon^* |\sigma(t) - r|, \quad \forall r \in \mathcal{U},$$

where $\varepsilon^* = \varepsilon[2\mu(t) + \delta + |{}^\beta T_\alpha^\Delta(f)(t)|]^{-1}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$. For every $r \in \mathcal{U} \cap (t - \varepsilon^*, t + \varepsilon^*)$ we have

$$\begin{aligned}
|f(t) - f(r)| &\leq |f(\sigma(t)) - f(r) - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - r)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}| \\
&\quad + |f(\sigma(t)) - f(t) - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}| \\
&\quad + |t - r| |{}^\beta T_\alpha^\Delta(f)(t)| |t + \frac{1}{\Gamma(\alpha)}|^{\alpha-1} \\
&\leq \varepsilon^* |\sigma(t) - r| (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} + \varepsilon^* \mu(t) (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \\
&\quad + \varepsilon^* |{}^\beta T_\alpha^\Delta(f)(t)| (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \\
&\leq \varepsilon^* [\mu(t) + |t - r| + \mu(t) + |{}^\beta T_\alpha^\Delta(f)(t)|] (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \\
&< \varepsilon^* [2\mu(t) + \delta + |{}^\beta T_\alpha^\Delta(f)(t)|] (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \\
&= \varepsilon.
\end{aligned}$$

(ii) Suppose that t is right-scattered and f is continuous at t . Since f is continuous at t , we obtain

$$\lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} = \frac{f(\sigma(t)) - f(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

Therefore, given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}| \leq \varepsilon$$

for all $s \in U$. Then we have

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} (\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

and therefore we obtain ${}^\beta T_\alpha^\Delta(f)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

(iii) Assume t is right-dense and f has $\beta - \Delta$ - fractional derivative of order α at t . For every $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U.$$

Taking $\sigma(t) = t$, for each $s \in U$ and $s \neq t$ we get

$$|\frac{f(t) - f(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)| \leq \varepsilon.$$

Hence, we obtain ${}^\beta T_\alpha^\Delta(f)(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$. Also, if the limit $\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$ exists and is equal to L , then given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|\frac{f(t) - f(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - L| \leq \varepsilon$$

for each $s \in \mathbb{U}$. Since $\sigma(t) = t$, we get

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - L(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in \mathbb{U}$$

and therefore we obtain ${}^\beta T_\alpha^\Delta(f)(t) = L = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

(iv) If $\sigma(t) = t$, then $\mu(t) = 0$ and we get

$$f(\sigma(t)) = f(t) = f(t) + \mu(t) {}^\beta T_\alpha^\Delta(f)(t) (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}.$$

If $\sigma(t) > t$, then we obtain

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t) \frac{f(\sigma(t)) - f(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \\ &= f(t) + \mu(t) {}^\beta T_\alpha^\Delta(f)(t) (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \end{aligned}$$

from (ii).

□

Example 2.3. If $f : \mathbb{T} \rightarrow \mathbb{R}$, $f(t) = c$ for any constant c , then we have ${}^\beta T_\alpha^\Delta(f)(t) = 0$.

Example 2.4. If $f : \mathbb{T} \rightarrow \mathbb{R}$, $f(t) = t$ then we get ${}^\beta T_\alpha^\Delta(f)(t) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

Example 2.5. If $h > 0$ and $f : h\mathbb{Z} \rightarrow \mathbb{R}$, then we obtain ${}^\beta T_\alpha^\Delta(f)(t) = \frac{f(t+h) - f(t)}{h} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$ from Theorem 2.2 (ii).

Theorem 2.6. Suppose that the functions $f, g : \mathbb{T} \rightarrow \mathbb{R}$ have $\beta - \Delta$ - fractional derivatives of order α at $t \in \mathbb{T}^k$. Then, we have the following properties.

- (i) The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \Delta$ - fractional derivative of order α at t with ${}^\beta T_\alpha^\Delta(f + g)(t) = {}^\beta T_\alpha^\Delta(f)(t) + {}^\beta T_\alpha^\Delta(g)(t)$.
- (ii) The function $cf : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \Delta$ - fractional derivative of order α at t with ${}^\beta T_\alpha^\Delta(cf)(t) = c {}^\beta T_\alpha^\Delta(f)(t)$, where c is any constant.
- (iii) The product $f.g : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \Delta$ - fractional derivative of order α at t with

$${}^\beta T_\alpha^\Delta(f.g)(t) = {}^\beta T_\alpha^\Delta(f)(t)g(t) + f(\sigma(t)) {}^\beta T_\alpha^\Delta(g)(t) = {}^\beta T_\alpha^\Delta(g)(t)f(t) + g(\sigma(t)) {}^\beta T_\alpha^\Delta(f)(t).$$

- (iv) When $f(t)f(\sigma(t)) \neq 0$, the function $\frac{1}{f}$ has $\beta - \Delta$ - fractional derivative of order α at t with

$${}^\beta T_\alpha^\Delta(\frac{1}{f})(t) = -\frac{{}^\beta T_\alpha^\Delta(f)(t)}{f(t)f(\sigma(t))}.$$

- (v) When $g(t)g(\sigma(t)) \neq 0$, the function $\frac{f}{g}$ has $\beta - \Delta$ - fractional derivative of order α at t with

$${}^\beta T_\alpha^\Delta(\frac{f}{g})(t) = \frac{{}^\beta T_\alpha^\Delta(f)(t)g(t) - f(t) {}^\beta T_\alpha^\Delta(g)(t)}{g(t)g(\sigma(t))}.$$

Proof. (i) Let $\varepsilon > 0$. Since f and g have $\beta - \Delta$ - fractional derivative of order α at $t \in \mathbb{T}^k$, there are neighborhoods U_1 and U_2 of t such that

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{2} |\sigma(t) - s|, \quad \forall s \in U_1,$$

and

$$|[g(\sigma(t)) - g(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(g)(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{2} |\sigma(t) - s|, \quad \forall s \in U_2.$$

If $U = U_1 \cap U_2$, then we obtain

$$\begin{aligned} & |[(f+g)(\sigma(t)) - (f+g)(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - [{}^\beta T_\alpha^\Delta(f)(t) + {}^\beta T_\alpha^\Delta(g)(t)](\sigma(t) - s)| \\ & \leq |[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - s)| \\ & \quad + |[g(\sigma(t)) - g(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(g)(t)(\sigma(t) - s)| \\ & \leq \varepsilon |\sigma(t) - s| \end{aligned}$$

for each $s \in U$. Hence, $f+g$ has $\beta - \Delta$ - fractional derivative of order α at t with ${}^\beta T_\alpha^\Delta(f+g)(t) = {}^\beta T_\alpha^\Delta(f)(t) + {}^\beta T_\alpha^\Delta(g)(t)$.

(ii) Because f has $\beta - \Delta$ - fractional derivative of order α at $t \in \mathbb{T}^k$, for any $\varepsilon > 0$ there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - s)| \leq \frac{\varepsilon}{|c|} |\sigma(t) - s| \quad \forall s \in U.$$

Then we get

$$|[(cf)(\sigma(t)) - (cf)(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - c {}^\beta T_\alpha^\Delta(f)(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \forall s \in U.$$

Thus, cf has $\beta - \Delta$ - fractional derivative of order α at t with ${}^\beta T_\alpha^\Delta(cf)(t) = c {}^\beta T_\alpha^\Delta(f)(t)$.

(iii) When t is right-dense, we have

$$\begin{aligned} {}^\beta T_\alpha^\Delta(f.g)(t) &= \lim_{s \rightarrow t} \frac{(f.g)(t) - (f.g)(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= \lim_{s \rightarrow t} [\frac{f(t) - f(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}] g(t) + \lim_{s \rightarrow t} [f(s) \frac{g(t) - g(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}] \\ &= {}^\beta T_\alpha^\Delta(f)(t).g(t) + f(t) {}^\beta T_\alpha^\Delta(g)(t) \\ &= {}^\beta T_\alpha^\Delta(f)(t).g(t) + f(\sigma(t)) {}^\beta T_\alpha^\Delta(g)(t). \end{aligned}$$

When t is right-scattered, we obtain

$$\begin{aligned} {}^\beta T_\alpha^\Delta(f.g)(t) &= \frac{(f.g)(\sigma(t)) - (f.g)(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= \frac{f(\sigma(t)) - f(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} g(t) + f(\sigma(t)) \frac{g(\sigma(t)) - g(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= {}^\beta T_\alpha^\Delta(f)(t)g(t) + f(\sigma(t)) {}^\beta T_\alpha^\Delta(g)(t). \end{aligned}$$

The other product rule follows by interchanging the functions f and g from this last equation.

(iv) Since $0 = {}^\beta T_\alpha^\Delta(f \cdot \frac{1}{f})(t) = {}^\beta T_\alpha^\Delta(f)(t) \frac{1}{f(t)} + f(\sigma(t)) {}^\beta T_\alpha^\Delta(\frac{1}{f})(t)$, we have ${}^\beta T_\alpha^\Delta(\frac{1}{f})(t) = -\frac{{}^\beta T_\alpha^\Delta(f)(t)}{f(t)f(\sigma(t))}$.

(v)

$$\begin{aligned} {}^\beta T_\alpha^\Delta(\frac{f}{g})(t) &= {}^\beta T_\alpha^\Delta(f \cdot \frac{1}{g})(t) \\ &= {}^\beta T_\alpha^\Delta(\frac{1}{g})(t)f(t) + \frac{1}{g(\sigma(t))} {}^\beta T_\alpha^\Delta(f)(t) \\ &= -\frac{{}^\beta T_\alpha^\Delta(g)(t)f(t)}{g(t)g(\sigma(t))} + \frac{{}^\beta T_\alpha^\Delta(f)(t)}{g(\sigma(t))} \\ &= \frac{{}^\beta T_\alpha^\Delta(f)(t)g(t) - f(t) {}^\beta T_\alpha^\Delta(g)(t)}{g(t)g(\sigma(t))}. \end{aligned}$$

□

Theorem 2.7. Let $m \in \mathbb{N}$, $\alpha \in (0, 1]$ and c be a constant.

(i) ${}^\beta T_\alpha^\Delta((t-c)^m) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^{m-1} (\sigma(t)-c)^i (t-c)^{m-1-i}.$

(ii) If $(t-c)(\sigma(t)-c) \neq 0$, then we have ${}^\beta T_\alpha^\Delta(\frac{1}{(t-c)^m}) = -(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^{m-1} \frac{1}{(\sigma(t)-c)^{m-i} (t-c)^{i+1}}.$

Proof. (i) If $m = 1$, then ${}^\beta T_\alpha^\Delta(t-c) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$. Now, we suppose

$${}^\beta T_\alpha^\Delta((t-c)^k) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^{k-1} (\sigma(t)-c)^i (t-c)^{k-1-i}.$$

Since

$$\begin{aligned} {}^\beta T_\alpha^\Delta((t-c)^{k+1}) &= {}^\beta T_\alpha^\Delta((t-c) \cdot (t-c)^k) \\ &= {}^\beta T_\alpha^\Delta((t-c)^k) \cdot (t-c) + (\sigma(t)-c)^k {}^\beta T_\alpha^\Delta(t-c) \\ &= (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^{k-1} (\sigma(t)-c)^i (t-c)^{k-1-i} (t-c) \\ &\quad + (\sigma(t)-c)^k (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^k (\sigma(t)-c)^i (t-c)^{k-i}, \end{aligned}$$

this concludes the proof of (i) by mathematical induction.

(ii)

$$\begin{aligned}
{}^\beta T_\alpha^\Delta\left(\frac{1}{(t-c)^m}\right) &= -\frac{{}^\beta T_\alpha^\Delta((t-c)^m)}{(t-c)^m(\sigma(t)-c)^m} \\
&= -\frac{(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^{m-1} (\sigma(t)-c)^i (t-c)^{m-1-i}}{(t-c)^m(\sigma(t)-c)^m} \\
&= -(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \sum_{i=0}^{m-1} \frac{1}{(\sigma(t)-c)^{m-i} (t-c)^{i+1}}.
\end{aligned}$$

□

Theorem 2.8. (Chain Rule) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $\alpha \in (0, 1]$. If $g : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \Delta$ - fractional derivative of order α at $t \in \mathbb{T}^k$ ($t > 0$) and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, then there exists c in the real interval $[t, \sigma(t)]$ such that

$${}^\beta T_\alpha^\Delta(fog)(t) = f'(g(c)) {}^\beta T_\alpha^\Delta(g)(t). \quad (2.1)$$

Proof. If $t \in \mathbb{T}^k$ is right-dense, then we have

$${}^\beta T_\alpha^\Delta(fog)(t) = \lim_{s \rightarrow t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \frac{g(t) - g(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

There exists ξ_s between $g(s)$ and $g(t)$ such that

$${}^\beta T_\alpha^\Delta(fog)(t) = \lim_{s \rightarrow t} f'(\xi_s) \frac{g(t) - g(s)}{t - s} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$$

by using the mean value theorem. We get $\lim_{s \rightarrow t} \xi_s = g(t)$ from the continuity of g . Thus, we obtain ${}^\beta T_\alpha^\Delta(fog)(t) = f'(g(t)) {}^\beta T_\alpha^\Delta(g)(t)$.

Let t be right-scattered. Then, we have

$${}^\beta T_\alpha^\Delta(fog)(t) = \frac{f(g(\sigma(t))) - f(g(t))}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

First, we assume $g(\sigma(t)) = g(t)$. Since we have ${}^\beta T_\alpha^\Delta(fog)(t) = 0$ and ${}^\beta T_\alpha^\Delta(g)(t) = 0$, (2.1) is satisfied for all c in the real interval $[t, \sigma(t)]$. Now, we will suppose $g(\sigma(t)) \neq g(t)$. In this case, by using the mean value theorem

$$\begin{aligned}
{}^\beta T_\alpha^\Delta(fog)(t) &= \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} \frac{g(\sigma(t)) - g(t)}{\mu(t)} (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \\
&= f'(\xi) {}^\beta T_\alpha^\Delta(g)(t),
\end{aligned}$$

where $\xi \in [g(t), g(\sigma(t))]$. There exists $c \in [t, \sigma(t)]$ such that $g(c) = \xi$ by the continuity of g . This concludes the proof.

□

Definition 2.9. Suppose that $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $\mathbb{T}$ is a time scale and f is n times delta differentiable at $t \in \mathbb{T}^{k^n}$ ($t > 0$). $\beta - \Delta$ - fractional derivative of order α of f is defined by ${}^\beta T_\alpha^\Delta(f)(t) = {}^\beta T_{\alpha-n}^\Delta(f^{\Delta^n})(t)$.

Theorem 2.10. If $\alpha \in (n, n+1]$ for all $n \in \mathbb{N}$, we get ${}^\beta T_\alpha^\Delta(f)(t) = (t + \frac{1}{\Gamma(\alpha)})^{1+n-\alpha} f^{\Delta^{1+n}}(t)$.

Proof. If $\alpha \in (n, n+1]$, then $\alpha - n \in (0, 1]$ and we obtain

$$\begin{aligned} {}^\beta T_\alpha^\Delta(f)(t) &= {}^\beta T_{\alpha-n}^\Delta(f^{\Delta^n})(t) = (t + \frac{1}{\Gamma(\alpha)})^{1-(\alpha-n)} (f^{\Delta^n})^\Delta(t) \\ &= (t + \frac{1}{\Gamma(\alpha)})^{1+n-\alpha} f^{\Delta^{1+n}}(t) \end{aligned}$$

by using Definition 2.9 and Theorem 2.2 (ii) and (iii). $\square$

3. $\beta - \Delta$ - Fractional Integral

Definition 3.1. If the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is regulated and $\alpha \in (0, 1]$, the $\beta - \Delta$ - fractional integral of f of order α is defined by $\int f(t) \Delta^\alpha t := \int f(t) (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \Delta t$.

Note that if $\alpha = 1$, then Definition 3.1 reduces to the indefinite Δ - integral and if $\mathbb{T} = \mathbb{R}$, then Definition 3.1 reduces to the β - fractional integral.

Definition 3.2. If $\alpha \in (0, 1]$ and the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is regulated, the indefinite $\beta - \Delta$ - fractional integral of f of order α is defined by

$$\int f(t) \Delta^\alpha t = F_\alpha(t) + c,$$

where c is any constant and ${}^\beta T_\alpha^\Delta(F_\alpha)(t) = f(t)$ for each $t \in \mathbb{T}^k$. The Cauchy $\beta - \Delta$ - fractional integral is defined by

$$\int_a^b f(t) \Delta^\alpha t = F_\alpha(b) - F_\alpha(a)$$

for all $a, b \in \mathbb{T}$.

Theorem 3.3. For $\alpha \in (0, 1]$ and any rd-continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$, there exists a function $F_\alpha : \mathbb{T} \rightarrow \mathbb{R}$ such that ${}^\beta T_\alpha^\Delta(F_\alpha)(t) = f(t)$ for every $t \in \mathbb{T}^k$. The function F_α is called to be an $\beta - \Delta$ - antiderivative of f .

Proof. Let $\alpha \in (0, 1)$. Since f is rd-continuous, f is regulated and therefore we get $\int f(t) (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \Delta t = F_\alpha(t) + c$. Then, for all $t \in \mathbb{T}^k$ we obtain

$${}^\beta T_\alpha^\Delta(F_\alpha(t) + c) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} (F_\alpha(t) + c)^\Delta = f(t)$$

from Theorem 2.10. The case $\alpha = 1$ is proved in [2]. $\square$

Theorem 3.4. Let $a, b, c \in \mathbb{T}$, $\alpha \in (0, 1]$, $f, g \in C_{rd}$ and $\lambda, \mu \in \mathbb{R}$. We have the following properties.

- (i) $\int_a^b [\lambda f(t) + \mu g(t)] \Delta^\alpha t = \lambda \int_a^b f(t) \Delta^\alpha t + \mu \int_a^b g(t) \Delta^\alpha t$.
- (ii) $\int_a^b f(t) \Delta^\alpha t = - \int_b^a f(t) \Delta^\alpha t$.

$$(iii) \int_a^b f(t) \Delta^\alpha t = \int_a^c f(t) \Delta^\alpha t + \int_c^b f(t) \Delta^\alpha t.$$

$$(iv) \int_a^a f(t) \Delta^\alpha t = 0.$$

$$(v) \text{ If } |f(t)| \leq g(t) \text{ for all } t \in [a, b], \text{ then } \left| \int_a^b f(t) \Delta^\alpha t \right| \leq \int_a^b |g(t)| \Delta^\alpha t.$$

$$(vi) \text{ If } f(t) \geq 0 \text{ for all } t \in [a, b], \text{ then } \int_a^b f(t) \Delta^\alpha t \geq 0.$$

Proof. The relations follow from the analogous properties of the Δ - integral, Definition 3.1 and Definition 3.2. $\square$

Theorem 3.5. If $t \in \mathbb{T}^k$, $f \in C_{rd}$ and $\alpha \in (0, 1]$, then we get

$$\int_t^{\sigma(t)} f(s) \Delta^\alpha s = \mu(t) f(t) \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}.$$

Proof. From Definition 3.2 and Theorem 3.3, there is an antiderivative F_α of f and by Theorem 2.2 (iv) we obtain

$$\begin{aligned} \int_t^{\sigma(t)} f(s) \Delta^\alpha s &= F_\alpha(\sigma(t)) - F_\alpha(t) = \mu(t) \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha-1} {}^\beta T_\alpha^\Delta(F_\alpha)(t) \\ &= \mu(t) f(t) \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}. \end{aligned}$$

$\square$

Theorem 3.6. If ${}^\beta T_\alpha^\Delta(f)(t) \geq 0$ for every $t \in [a, b]$, the function f is nondecreasing on $[a, b]$.

Proof. If ${}^\beta T_\alpha^\Delta(f)(t) \geq 0$ for all $t \in [a, b]$, then from Theorem 3.4 (vii) we have $\int_c^d {}^\beta T_\alpha^\Delta(f)(t) \Delta^\alpha t \geq 0$ for $c, d \in \mathbb{T}$ such that $a \leq c \leq d \leq b$. Then we obtain

$$f(d) = f(c) + \int_c^d {}^\beta T_\alpha^\Delta(f)(t) \Delta^\alpha t \geq f(c)$$

by Definition 3.2. This concludes the proof. $\square$

4. $\beta - \nabla$ - Fractional Derivative and Integral

Definition 4.1. Assume that $\alpha \in (0, 1]$, $\mathbb{T}$ is a time scale and $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function. For all $\varepsilon > 0$, if there is a neighborhood $\mathcal{U}$ of $t \in \mathbb{T}_k$ ($t > 0$) such that

$$|[f(\rho(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\nabla(f)(t)(\rho(t) - s)| \leq \varepsilon |\rho(t) - s|, \quad \forall s \in \mathcal{U},$$

the $\beta - \nabla$ - fractional derivative of f of order α at t is defined by the number ${}^\beta T_\alpha^\nabla(f)(t)$. $\beta - \nabla$ - fractional derivative of f of order α at 0 is defined by ${}^\beta T_\alpha^\nabla(f)(0) = \lim_{t \rightarrow 0^+} {}^\beta T_\alpha^\nabla(f)(t)$.

Note that when $\alpha = 1$, we have ${}^\beta T_\alpha^\nabla(f)(t) = f^\nabla(t)$ and if $\mathbb{T} = \mathbb{R}$, then ${}^\beta T_\alpha^\nabla(f)(t) = D_t^\alpha(f)(t)$ is the β - fractional derivative of f of order α .

Theorem 4.2. Suppose that $\alpha \in (0, 1]$, $\mathbb{T}$ is a time scale, $t \in \mathbb{T}_k$ ($t > 0$) and $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function. Then, we have the following properties.

- (i) If f has $\beta - \nabla$ - fractional derivative of order α at t , then f is continuous at t .
- (ii) Let f be continuous at t and t be right-scattered. Then, f has $\beta - \nabla$ - fractional derivative of order α at t and we have ${}^\beta T_\alpha^\nabla(f)(t) = \frac{f(t) - f(\rho(t))}{v(t)}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.
- (iii) When t is right-dense, f has $\beta - \nabla$ - fractional derivative of order α at t iff the limit $\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$ exists as a finite number. Then, we have

$${}^\beta T_\alpha^\nabla(f)(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

- (iv) If f has $\beta - \nabla$ - fractional derivative of order α at t , then we get

$$f(\rho(t)) = f(t) - v(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} {}^\beta T_\alpha^\nabla(f)(t).$$

Proof. The proof is similar to the proof of Theorem 2.2. □

Theorem 4.3. Suppose that the functions $f, g : \mathbb{T} \rightarrow \mathbb{R}$ have $\beta - \nabla$ - fractional derivatives of order α at $t \in \mathbb{T}_k$. Then, we have the following properties.

- (i) The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \nabla$ - fractional derivative of order α at t with ${}^\beta T_\alpha^\nabla(f + g)(t) = {}^\beta T_\alpha^\nabla(f)(t) + {}^\beta T_\alpha^\nabla(g)(t)$.
- (ii) The function $cf : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \nabla$ - fractional derivative of order α at t with ${}^\beta T_\alpha^\nabla(cf)(t) = c {}^\beta T_\alpha^\nabla(f)(t)$, where c is any constant.
- (iii) The product $f.g : \mathbb{T} \rightarrow \mathbb{R}$ has $\beta - \nabla$ - fractional derivative of order α at t with

$${}^\beta T_\alpha^\nabla(f.g)(t) = {}^\beta T_\alpha^\nabla(f)(t)g(t) + f(\rho(t)) {}^\beta T_\alpha^\nabla(g)(t) = {}^\beta T_\alpha^\nabla(g)(t)f(t) + g(\rho(t)) {}^\beta T_\alpha^\nabla(f)(t).$$

- (iv) When $f(t)f(\rho(t)) \neq 0$, the function $\frac{1}{f}$ has $\beta - \nabla$ - fractional derivative of order α at t with

$${}^\beta T_\alpha^\nabla(\frac{1}{f})(t) = -\frac{{}^\beta T_\alpha^\nabla(f)(t)}{f(t)f(\rho(t))}.$$

- (v) When $g(t)g(\rho(t)) \neq 0$, the function $\frac{f}{g}$ has $\beta - \nabla$ - fractional derivative of order α at t with

$${}^\beta T_\alpha^\nabla(\frac{f}{g})(t) = \frac{{}^\beta T_\alpha^\nabla(f)(t)g(t) - f(t) {}^\beta T_\alpha^\nabla(g)(t)}{g(t)g(\rho(t))}.$$

Proof. The proof is similar to the proof of Theorem 2.6. □

Definition 4.4. Suppose that $\alpha \in (n, n + 1]$, $n \in \mathbb{N}$, $\mathbb{T}$ is a time scale and f is n times delta differentiable at $t \in \mathbb{T}_k$ ($t > 0$). $\beta - \nabla$ - fractional derivative of order α of f is defined by ${}^\beta T_\alpha^\nabla(f)(t) = {}^\beta T_{\alpha-n}^\nabla(f^{\nabla^n})(t)$.

Theorem 4.5. If $\alpha \in (n, n + 1]$, $n \in \mathbb{N}$, we get ${}^\beta T_\alpha^\nabla(f)(t) = (t + \frac{1}{\Gamma(\alpha)})^{1+n-\alpha} f^{\nabla^{1+n}}(t)$.

Proof. The proof is similar to the proof of Theorem 2.10. □

Definition 4.6. If the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is regulated and $\alpha \in (0, 1]$, the $\beta - \nabla$ - fractional integral of f of order α is defined by $\int f(t) \nabla^\alpha t := \int f(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \nabla t$.

Note that if $\alpha = 1$, then Definition 4.6 reduces to the indefinite ∇ - integral and if $\mathbb{T} = \mathbb{R}$, then Definition 4.6 reduces to the β - fractional integral.

Definition 4.7. If $\alpha \in (0, 1]$ and the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is regulated, the indefinite $\beta - \nabla$ - fractional integral of f of order α is defined by

$$\int f(t) \nabla^\alpha t = F_\alpha(t) + c,$$

where c is any constant and ${}^\beta T_\alpha^\nabla(F_\alpha)(t) = f(t)$ for each $t \in \mathbb{T}_k$. The Cauchy $\beta - \nabla$ - fractional integral is defined by

$$\int_a^b f(t) \nabla^\alpha t = F_\alpha(b) - F_\alpha(a)$$

for all $a, b \in \mathbb{T}$.

Theorem 4.8. For $\alpha \in (0, 1]$ and any ld-continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$, there exists a function $F_\alpha : \mathbb{T} \rightarrow \mathbb{R}$ such that ${}^\beta T_\alpha^\nabla(F_\alpha)(t) = f(t)$ for every $t \in \mathbb{T}_k$. The function F_α is called to be an $\beta - \nabla$ - antiderivative of f .

Proof. The proof is similar to the proof of Theorem 3.3. $\square$

Theorem 4.9. Let $a, b, c \in \mathbb{T}$, $\alpha \in (0, 1]$, $f, g \in C_{ld}$ and $\lambda, \mu \in \mathbb{R}$. We have the following properties.

- (i) $\int_a^b [\lambda f(t) + \mu g(t)] \nabla^\alpha t = \lambda \int_a^b f(t) \nabla^\alpha t + \mu \int_a^b g(t) \nabla^\alpha t.$
- (ii) $\int_a^b f(t) \nabla^\alpha t = - \int_b^a f(t) \nabla^\alpha t.$
- (iii) $\int_a^b f(t) \nabla^\alpha t = \int_a^c f(t) \nabla^\alpha t + \int_c^b f(t) \nabla^\alpha t.$
- (iv) $\int_a^a f(t) \nabla^\alpha t = 0.$
- (v) If $|f(t)| \leq g(t)$ for all $t \in [a, b)$, then $|\int_a^b f(t) \nabla^\alpha t| \leq \int_a^b |g(t)| \nabla^\alpha t.$
- (vi) If $f(t) \geq 0$ for all $t \in [a, b)$, then $\int_a^b f(t) \nabla^\alpha t \geq 0.$

Proof. The relations follow from the analogous properties of the ∇ - integral, Definition 3.1 and Definition 3.2. $\square$

Theorem 4.10. If $t \in \mathbb{T}_k$, $f \in C_{ld}$ and $\alpha \in (0, 1]$, then we get

$$\int_{\rho(t)}^t f(s) \nabla^\alpha s = \nu(t) f(t) (t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}.$$

Proof. The proof is similar to the proof of Theorem 3.5. $\square$

5. Conclusions

First, we defined $\beta - \Delta$ - fractional derivative of order α . If $\alpha = 1$, then Hilger derivative was obtained and if $\mathbb{T} = \mathbb{R}$, then β - fractional derivative of order α was obtained. Some properties of $\beta - \Delta$ - fractional derivative of order α were given. Second, $\beta - \Delta$ - fractional integral of order α was defined. This definition reduces to the indefinite Δ - integral when $\alpha = 1$ and this definition reduces to the β - fractional integral when $\mathbb{T} = \mathbb{R}$. Then, we obtained some properties of $\beta - \Delta$ - fractional integral of order α . Finally, we defined $\beta - \nabla$ - fractional derivative and integral of order α and their properties were introduced. Our new calculus unifies and generalizes the time scale calculus and the β - fractional calculus.

References

- [1] Hilger S (1990). *Analysis on measure chains-A unified approach to continuous and discrete calculus*. Results Math. **18**: 18–56.
- [2] Bohner M and Peterson A (2001). "Dynamic Equations on Time Scales: An Introduction with Applications". Birkhäuser, Boston.
- [3] Bohner M and Peterson A (2003). "Advances in Dynamic Equations on Time Scales". Birkhäuser, Boston.
- [4] Agarwal RP, Bohner M and Li WT (2004). "Nonoscillation and oscillation theory for functional differential equations". Pure and Applied Mathematics Series, Marcel Dekker, New York.
- [5] Carpinteri A, Cornetti P and Saporita A (2014). *Nonlocal elasticity: an approach based on fractional calculus*. Meccanica **49**(11): 2551-2569.
- [6] Herrmann R (2014). "Fractional Calculus: An Introduction for Physicists". World Scientific, Singapore.
- [7] Kilbas AA, Srivastava HM and Trujillo JJ (2006). "Theory and Applications of Fractional Differential Equations". Elsevier, Amsterdam.
- [8] Meilanov RP and Magomedov RA (2014). *Thermodynamics in Fractional Calculus*. Journal of Engineering Physics and Thermophysics **87**(6): 1521-1531.
- [9] Oldham KB and Spanier J (1974). "Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order". Academic Press, New York, NY, USA.
- [10] O.P. Sabatier OP, Agrawal JA and Machado T (2007). "Advances in Fractional Calculus". Springer, Dordrecht, The Netherlands.
- [11] Samko SG, Kilbas AA and Marichev OI (1993). "Fractional Integral and Derivatives: Theory and Applications". Gordon and Breach, Yverdon, Switzerland.
- [12] Shah K, Ali G, Ansari KJ, Abdeljawad T, Meganathan M and Abdalla B (2023). *On qualitative analysis of boundary value problem of variable order fractional delay differential equations*. Bound. Value Probl. **2023**(55): 1–15.
- [13] Shah K, Abdalla B, Abdeljawad T and Gul R (2023). *Analysis of multipoint impulsive problem of fractional-order differential equations*. Bound. Value Probl. **2023**(1): 1–17.
- [14] Shah K (2020). *Nonlocal boundary value problems for nonlinear toppled system of fractional differential equations*. Hacet. J. Math. Stat. **49**: 316–337.
- [15] Shah K, Abdeljawad T, Jarad F and Al-Mdallal Q (2023). *On nonlinear conformable fractional order dynamical system via differential transform method*. CMES-Computer Modeling in Engineering Sciences **136**: 1457-1472.
- [16] Khan ZA, Shah K, Abdalla B and Abdeljawad T (2023). *A numerical study of complex dynamics of a chemostat model under fractal-fractional derivative*. Fractals, **31**(2340181): 1-14.
- [17] Atangana A and Goufo EFD (2014). *Extension of matched asymptotic method to fractional boundary layers problems*. Math. Probl. Eng. **2014**(107535): 1–7.
- [18] Atangana A (2015). *A novel model for the lassa hemorrhagic fever: Deathly disease for pregnant women*. Neural Comput. Appl. **26**: 1895-1903.
- [19] Atangana A, Baleanu D and Alsaedi A (2016). *Analysis of time fractional Hunter-Saxton equation: A model of neumatic liquid crystal*. Open Phys. **14**(1): 145-149.
- [20] Bastos NRO (2012). "Fractional calculus on time scales". Ph.D. thesis, University of Aveiro.

- [21] Bendouma B and Hammoudi A (2019). *A Nabla Conformable Fractional Calculus on Time Scales*. Electr. J. Math. Anal. Appl. **7**(1): 202-216.
- [22] Benkhettou N, Brito da Cruz AMC and Torres DFM (2015). *A fractional calculus on arbitrary time scales: Fractional differentiation and fractional Integration*. Signal Process. **107**(1): 230-237.
- [23] Benkhettou N, Hassani S and Torres DFM (2016). *A conformable fractional calculus on arbitrary time scales*. Journal of King Saud University **28**(1): 93–98.
- [24] Rahmat MRS and Noorani MSM (2021). *A new conformable nabla derivative and its application on arbitrary time scales*. Adv. Diff. Equ. **2021**(238): 1–27.
- [25] Zhao DF and You XX (2016). *A new fractional derivative on time scales*. Adv. Appl. Math. Anal. **11**(1): 1-9.
- [26] Zhu J and Zhu Y (2013). *Fractional Cauchy problem with Riemann-Liouville fractional delta derivative on time scales*. Abst. Appl. Anal. **2013**(401596): 1-19.