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Beta-Fractional Calculus on Time Scales

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Abstract

In this paper, $\beta-\Delta$ fractional derivative and $\beta-\Delta$ fractional integral on time scales are defined and their basic properties are obtained. Then, $\beta-\nabla$ fractional calculus on arbitrary time scales is introduced.

Keywords: Beta-fractional derivative, beta-fractional integral, time scales.
2010 MSC: 26A33, 26E70".

1. Introduction

The calculus on time scales was introduced by Aulbach and Hilger [1], in order to unify and generalize continuous and discrete analysis. A nonempty closed subset of real numbers $\mathbb{R}$ is called a time scale $\mathbb{T}$. Some basic definitions and theorems on time scales can be found in the book [2] and another excellent source on time scales is the book [3]. The study of time scales has led to many important applications, e.g. in the study of epidemic models, insect population models, heat transfer, and neural networks [4]. Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc.; see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Abdon Atangana suggested the $\beta-$fractional derivative recently in [17, 18, 19]. The suggested version fulfills many characteristics that have been utilized to simulate various physical issues and have served as limitations for fractional derivatives. The beta derivative of f of order α is defined as

$$D_t^{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f[t+\varepsilon(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}]-f(t)}{\varepsilon}, \ \alpha\in(0,1], \ t>0.$$

The idea to join the fractional calculus and the calculus on time scales was born with the Ph.D. thesis of Bastos [20]. After the inception of the topic, a number of papers were published see [21, 22, 23, 24, 25, 26]. In this study, we will give the concept of $\beta-$ fractional derivative and integral on time scales. Our new calculus unifies and generalizes the time scale calculus and the $\beta-$ fractional calculus.

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2. $\beta-\Delta-$ Fractional Derivative

Definition 2.1. Assume that $\alpha \in (0,1]$, $\mathbb{T}$ is a time scale and $f : \mathbb{T} \to \mathbb{R}$ is a function. For all $\varepsilon > 0$, if there is a neighborhood U of $t \in \mathbb{T}^k$ $(t > 0)$ such that

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s|, \ \ \forall s \in U,$$

the $\beta-\Delta-$ fractional derivative of f of order α at t is defined by the number ${}^{\beta}T_{\alpha}^{\Delta}(f)(t)$. $\beta-\Delta-$ fractional derivative of f of order α at 0 is defined by ${}^{\beta}T_{\alpha}^{\Delta}(f)(0) = \lim_{t\to 0^+} {}^{\beta}T_{\alpha}^{\Delta}(f)(t)$.

Note that when $\alpha = 1$, we have ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = f^{\Delta}(t)$ and if $\mathbb{T} = \mathbb{R}$, then ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = D_t^{\alpha}(f)(t)$ is the $\beta-$ fractional derivative of f of order α.

Theorem 2.2. *Suppose that $\alpha \in (0,1]$, $\mathbb{T}$ is a time scale, $t \in \mathbb{T}^k$ $(t > 0)$ and $f : \mathbb{T} \to \mathbb{R}$ is a function. Then, we have the following properties.*

(i) If f has $\beta-\Delta-$ fractional derivative of order α at t, then f is continuous at t.

(ii) Let f be continuous at t and t be right-scattered. Then, f has $\beta-\Delta-$ fractional derivative of order α at t and we have ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

(iii) When t is right-dense, f has $\beta-\Delta-$ fractional derivative of order α at t iff the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$ exists as a finite number. Then, we have

$$ {}^{\beta}T_{\alpha}^{\Delta}(f)(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

(iv) If f has $\beta-\Delta-$ fractional derivative of order α at t, then we get

$$f(\sigma(t)) = f(t) + \mu(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \, {}^{\beta}T_{\alpha}^{\Delta}(f)(t).$$

Proof. (i) If f has $\beta-\Delta-$ fractional derivative of order α at t, given any $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(r)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t) - r)| \leqslant \varepsilon^*|\sigma(t) - r|, \ \ \forall r \in U,$$

where $\varepsilon^* = \varepsilon[2\mu(t) + \delta + |{}^{\beta}T_{\alpha}^{\Delta}(f)(t)|]^{-1}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$. For every $r \in U \cap (t - \varepsilon^*, t + \varepsilon^*)$ we have

$$\begin{aligned}|f(t)-f(r)| &\leqslant |f(\sigma(t))-f(r)-\ {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-r)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}| \\ &\quad +|f(\sigma(t))-f(t)-\ {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}| \\ &\quad +|t-r||{}^{\beta}T_{\alpha}^{\Delta}(f)(t)||t+\frac{1}{\Gamma(\alpha)}|^{\alpha-1} \\ &\leqslant \varepsilon^{*}|\sigma(t)-r|(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}+\varepsilon^{*}\mu(t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1} \\ &\quad +\varepsilon^{*}|{}^{\beta}T_{\alpha}^{\Delta}(f)(t)|(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1} \\ &\leqslant \varepsilon^{*}[\mu(t)+|t-r|+\mu(t)+|{}^{\beta}T_{\alpha}^{\Delta}(f)(t)|](t+\frac{1}{\Gamma(\alpha)})^{\alpha-1} \\ &< \varepsilon^{*}[2\mu(t)+\delta+|{}^{\beta}T_{\alpha}^{\Delta}(f)(t)|](t+\frac{1}{\Gamma(\alpha)})^{\alpha-1} \\ &= \varepsilon.\end{aligned}$$

(ii) Suppose that t is right-scattered and f is continuous at t. Since f is continuous at t, we obtain

$$\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}=\frac{f(\sigma(t))-f(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}.$$

Therefore, given $\varepsilon>0$, there exists a neighborhood U of t such that

$$|\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}-\frac{f(\sigma(t))-f(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}|\leqslant\varepsilon$$

for all $s\in U$. Then we have

$$|[f(\sigma(t))-f(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}-\frac{f(\sigma(t))-f(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|$$

and therefore we obtain ${}^{\beta}T_{\alpha}^{\Delta}(f)(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

(iii) Assume t is right-dense and f has $\beta-\Delta-$ fractional derivative of order α at t. For every $\varepsilon>0$, there is a neighborhood U of t such that

$$|[f(\sigma(t))-f(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}-\ {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|,\quad\forall s\in U.$$

Taking $\sigma(t)=t$, for each $s\in U$ and $s\neq t$ we get

$$|\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}-\ {}^{\beta}T_{\alpha}^{\Delta}(f)(t)|\leqslant\varepsilon.$$

Hence, we obtain ${}^{\beta}T_{\alpha}^{\Delta}(f)(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}$. Also, if the limit $\lim_{s\to t}\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}$ exists and is equal to L, then given any $\varepsilon>0$, there is a neighborhood U of t such that

$$|\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}-L|\leqslant\varepsilon$$

for each $s \in U$. Since $\sigma(t) = t$, we get

$$|[f(\sigma(t)) - f(s)](t + \frac{1}{\Gamma(\alpha)})^{1-\alpha} - L(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s|, \;\; \forall s \in U$$

and therefore we obtain ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = L = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

(iv) If $\sigma(t) = t$, then $\mu(t) = 0$ and we get

$$f(\sigma(t)) = f(t) = f(t) + \mu(t)\; {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}.$$

If $\sigma(t) > t$, then we obtain

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t) \frac{f(\sigma(t)) - f(t)}{\mu(t)}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \\ &= f(t) + \mu(t)\; {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1} \end{aligned}$$

from (ii).

□

Example 2.3. If $f : \mathbb{T} \to \mathbb{R}$, $f(t) = c$ for any constant c, then we have ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = 0$.

Example 2.4. If $f : \mathbb{T} \to \mathbb{R}$, $f(t) = t$ then we get ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

Example 2.5. If $h > 0$ and $f : h\mathbb{Z} \to \mathbb{R}$, then we obtain ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = \frac{f(t+h)-f(t)}{h}(t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}$ from Theorem 2.2 (ii).

Theorem 2.6. *Suppose that the functions $f, g : \mathbb{T} \to \mathbb{R}$ have $\beta - \Delta-$ fractional derivatives of order α at $t \in \mathbb{T}^k$. Then, we have the following properties.*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ has $\beta - \Delta-$ fractional derivative of order α at t with ${}^{\beta}T_{\alpha}^{\Delta}(f + g)(t) = {}^{\beta}T_{\alpha}^{\Delta}(f)(t) + {}^{\beta}T_{\alpha}^{\Delta}(g)(t)$.

(ii) The function $cf : \mathbb{T} \to \mathbb{R}$ has $\beta - \Delta-$ fractional derivative of order α at t with ${}^{\beta}T_{\alpha}^{\Delta}(cf)(t) = c\; {}^{\beta}T_{\alpha}^{\Delta}(f)(t)$, where c is any constant.

(iii) The product $f.g : \mathbb{T} \to \mathbb{R}$ has $\beta - \Delta-$ fractional derivative of order α at t with

$${}^{\beta}T_{\alpha}^{\Delta}(f.g)(t) = {}^{\beta}T_{\alpha}^{\Delta}(f)(t)g(t) + f(\sigma(t))\; {}^{\beta}T_{\alpha}^{\Delta}(g)(t) = {}^{\beta}T_{\alpha}^{\Delta}(g)(t)f(t) + g(\sigma(t))\; {}^{\beta}T_{\alpha}^{\Delta}(f)(t).$$

(iv) When $f(t)f(\sigma(t)) \neq 0$, the function $\frac{1}{f}$ has $\beta - \Delta-$ fractional derivative of order α at t with

$${}^{\beta}T_{\alpha}^{\Delta}(\frac{1}{f})(t) = -\frac{{}^{\beta}T_{\alpha}^{\Delta}(f)(t)}{f(t)f(\sigma(t))}.$$

(v) When $g(t)g(\sigma(t)) \neq 0$, the function $\frac{f}{g}$ has $\beta - \Delta-$ fractional derivative of order α at t with

$${}^{\beta}T_{\alpha}^{\Delta}(\frac{f}{g})(t) = \frac{{}^{\beta}T_{\alpha}^{\Delta}(f)(t)g(t) - f(t)\; {}^{\beta}T_{\alpha}^{\Delta}(g)(t)}{g(t)g(\sigma(t))}.$$

Proof. (i) Let $\varepsilon > 0$. Since f and g have $\beta-\Delta-$ fractional derivative of order α at $t \in \mathbb{T}^k$, there are neighborhoods U_1 and U_2 of t such that

$$|[f(\sigma(t)) - f(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-s)| \leqslant \frac{\varepsilon}{2}|\sigma(t)-s|, \ \ \forall s \in U_1,$$

and

$$|[g(\sigma(t)) - g(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(g)(t)(\sigma(t)-s)| \leqslant \frac{\varepsilon}{2}|\sigma(t)-s|, \ \ \forall s \in U_2.$$

If $U = U_1 \cap U_2$, then we obtain

$$\begin{aligned} &|[(f+g)(\sigma(t)) - (f+g)(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - [{}^{\beta}T_{\alpha}^{\Delta}(f)(t) + {}^{\beta}T_{\alpha}^{\Delta}(g)(t)](\sigma(t)-s)| \\ &\leqslant |[f(\sigma(t)) - f(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-s)| \\ &\quad + |[g(\sigma(t)) - g(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(g)(t)(\sigma(t)-s)| \\ &\leqslant \varepsilon|\sigma(t)-s| \end{aligned}$$

for each $s \in U$. Hence, $f+g$ has $\beta-\Delta-$ fractional derivative of order α at t with ${}^{\beta}T_{\alpha}^{\Delta}(f+g)(t) = {}^{\beta}T_{\alpha}^{\Delta}(f)(t) + {}^{\beta}T_{\alpha}^{\Delta}(g)(t)$.

(ii) Because f has $\beta-\Delta-$ fractional derivative of order α at $t \in \mathbb{T}^k$, for any $\varepsilon > 0$ there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-s)| \leqslant \frac{\varepsilon}{|c|}|\sigma(t)-s| \ \ \forall s \in U.$$

Then we get

$$|[(cf)(\sigma(t)) - (cf)(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - c\,{}^{\beta}T_{\alpha}^{\Delta}(f)(t)(\sigma(t)-s)| \leqslant \varepsilon|\sigma(t)-s| \ \ \forall s \in U.$$

Thus, cf has $\beta-\Delta-$ fractional derivative of order α at t with ${}^{\beta}T_{\alpha}^{\Delta}(cf)(t) = c\,{}^{\beta}T_{\alpha}^{\Delta}(f)(t)$.

(iii) When t is right-dense, we have

$$\begin{aligned} {}^{\beta}T_{\alpha}^{\Delta}(f.g)(t) &= \lim_{s\to t} \frac{(f.g)(t)-(f.g)(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= \lim_{s\to t}[\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}]g(t) + \lim_{s\to t}[f(s)\frac{g(t)-g(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}] \\ &= {}^{\beta}T_{\alpha}^{\Delta}(f)(t).g(t) + f(t)\,{}^{\beta}T_{\alpha}^{\Delta}(g)(t) \\ &= {}^{\beta}T_{\alpha}^{\Delta}(f)(t).g(t) + f(\sigma(t))\,{}^{\beta}T_{\alpha}^{\Delta}(g)(t). \end{aligned}$$

When t is right-scattered, we obtain

$$\begin{aligned} {}^{\beta}T_{\alpha}^{\Delta}(f.g)(t) &= \frac{(f.g)(\sigma(t))-(f.g)(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= \frac{f(\sigma(t))-f(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}g(t) + f(\sigma(t))\frac{g(\sigma(t))-g(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \\ &= {}^{\beta}T_{\alpha}^{\Delta}(f)(t)g(t) + f(\sigma(t))\,{}^{\beta}T_{\alpha}^{\Delta}(g)(t). \end{aligned}$$

The other product rule follows by interchanging the functions f and g from this last equation.

(iv) Since $0 = {}^{\beta}T_{\alpha}^{\Delta}(f.\frac{1}{f})(t) = {}^{\beta}T_{\alpha}^{\Delta}(f)(t)\frac{1}{f(t)} + f(\sigma(t))\ {}^{\beta}T_{\alpha}^{\Delta}(\frac{1}{f})(t)$, we have ${}^{\beta}T_{\alpha}^{\Delta}(\frac{1}{f})(t) = -\frac{{}^{\beta}T_{\alpha}^{\Delta}(f)(t)}{f(t)f(\sigma(t))}$.

(v)

$$\begin{aligned}
{}^{\beta}T_{\alpha}^{\Delta}(\frac{f}{g})(t) &= {}^{\beta}T_{\alpha}^{\Delta}(f.\frac{1}{g})(t) \\
&= {}^{\beta}T_{\alpha}^{\Delta}(\frac{1}{g})(t)f(t) + \frac{1}{g(\sigma(t))}\ {}^{\beta}T_{\alpha}^{\Delta}(f)(t) \\
&= -\frac{{}^{\beta}T_{\alpha}^{\Delta}(g)(t)f(t)}{g(t)g(\sigma(t))} + \frac{{}^{\beta}T_{\alpha}^{\Delta}(f)(t)}{g(\sigma(t))} \\
&= \frac{{}^{\beta}T_{\alpha}^{\Delta}(f)(t)g(t) - f(t)\ {}^{\beta}T_{\alpha}^{\Delta}(g)(t)}{g(t)g(\sigma(t))}.
\end{aligned}$$

□

Theorem 2.7. *Let* $m \in \mathbb{N}$, $\alpha \in (0,1]$ *and* c *be a constant.*

(i) ${}^{\beta}T_{\alpha}^{\Delta}((t-c)^m) = (t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum_{i=0}^{m-1}(\sigma(t)-c)^i(t-c)^{m-1-i}$.

(ii) If $(t-c)(\sigma(t)-c) \neq 0$, *then we have* ${}^{\beta}T_{\alpha}^{\Delta}(\frac{1}{(t-c)^m}) = -(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum_{i=0}^{m-1}\frac{1}{(\sigma(t)-c)^{m-i}(t-c)^{i+1}}$.

Proof. (i) If $m = 1$, then ${}^{\beta}T_{\alpha}^{\Delta}(t-c) = (t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}$. Now, we suppose

$$ {}^{\beta}T_{\alpha}^{\Delta}((t-c)^k) = (t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum_{i=0}^{k-1}(\sigma(t)-c)^i(t-c)^{k-1-i}.$$

Since

$$\begin{aligned}
{}^{\beta}T_{\alpha}^{\Delta}((t-c)^{k+1}) &= {}^{\beta}T_{\alpha}^{\Delta}((t-c).(t-c)^k) \\
&= {}^{\beta}T_{\alpha}^{\Delta}((t-c)^k).(t-c) + (\sigma(t)-c)^k\ {}^{\beta}T_{\alpha}^{\Delta}(t-c) \\
&= (t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum_{i=0}^{k-1}(\sigma(t)-c)^i(t-c)^{k-1-i}(t-c) \\
&\quad +(\sigma(t)-c)^k(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \\
&= (t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum_{i=0}^{k}(\sigma(t)-c)^i(t-c)^{k-i},
\end{aligned}$$

this concludes the proof of (i) by mathematical induction.

(ii)

$$
\begin{aligned}
{}^{\beta}T^{\Delta}_{\alpha}\left(\frac{1}{(t-c)^m}\right) &= -\frac{{}^{\beta}T^{\Delta}_{\alpha}((t-c)^m)}{(t-c)^m(\sigma(t)-c)^m} \\
&= -\frac{(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum\limits_{i=0}^{m-1}(\sigma(t)-c)^i(t-c)^{m-1-i}}{(t-c)^m(\sigma(t)-c)^m} \\
&= -(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\sum_{i=0}^{m-1}\frac{1}{(\sigma(t)-c)^{m-i}(t-c)^{i+1}}.
\end{aligned}
$$

□

Theorem 2.8. *(Chain Rule) Let* $g:\mathbb{R}\to\mathbb{R}$ *be continuous and* $\alpha\in(0,1]$. *If* $g:\mathbb{T}\to\mathbb{R}$ *has* $\beta-\Delta-$ *fractional derivative of order* α *at* $t\in\mathbb{T}^k$ $(t>0)$ *and* $f:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable, then there exists* c *in the real interval* $[t,\sigma(t)]$ *such that*

$$
{}^{\beta}T^{\Delta}_{\alpha}(fog)(t) = f'(g(c))\ {}^{\beta}T^{\Delta}_{\alpha}(g)(t). \tag{2.1}
$$

Proof. If $t\in\mathbb{T}^k$ is right-dense, then we have

$$
{}^{\beta}T^{\Delta}_{\alpha}(fog)(t) = \lim_{s\to t}\frac{f(g(t))-f(g(s))}{g(t)-g(s)}\frac{g(t)-g(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}.
$$

There exists ξ_s between $g(s)$ and $g(t)$ such that

$$
{}^{\beta}T^{\Delta}_{\alpha}(fog)(t) = \lim_{s\to t} f'(\xi_s)\frac{g(t)-g(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}
$$

by using the mean value theorem. We get $\lim_{s\to t}\xi_s = g(t)$ from the continuity of g. Thus, we obtain ${}^{\beta}T^{\Delta}_{\alpha}(fog)(t) = f'(g(t))\ {}^{\beta}T^{\Delta}_{\alpha}(g)(t)$.

Let t be right-scattered. Then, we have

$$
{}^{\beta}T^{\Delta}_{\alpha}(fog)(t) = \frac{f(g(\sigma(t)))-f(g(t))}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}.
$$

First, we assume $g(\sigma(t)) = g(t)$. Since we have ${}^{\beta}T^{\Delta}_{\alpha}(fog)(t) = 0$ and ${}^{\beta}T^{\Delta}_{\alpha}(g)(t) = 0$, (2.1) is satisfied for all c in the real interval $[t,\sigma(t)]$. Now, we will suppose $g(\sigma(t)) \neq g(t)$. In this case, by using the mean value theorem

$$
\begin{aligned}
{}^{\beta}T^{\Delta}_{\alpha}(fog)(t) &= \frac{f(g(\sigma(t)))-f(g(t))}{g(\sigma(t))-g(t)}\frac{g(\sigma(t))-g(t)}{\mu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} \\
&= f'(\xi)\ {}^{\beta}T^{\Delta}_{\alpha}(g)(t),
\end{aligned}
$$

where $\xi\in[g(t),g(\sigma(t))]$. There exists $c\in[t,\sigma(t)]$ such that $g(c)=\xi$ by the continuity of g. This concludes the proof.

□

Definition 2.9. Suppose that $\alpha\in(n,n+1]$, $n\in\mathbb{N}$, $\mathbb{T}$ is a time scale and f is n times delta differentiable at $t\in\mathbb{T}^{k^n}$ $(t>0)$. $\beta-\Delta-$ fractional derivative of order α of f is defined by ${}^{\beta}T^{\Delta}_{\alpha}(f)(t) = {}^{\beta}T^{\Delta}_{\alpha-n}(f^{\Delta^n})(t)$.

Theorem 2.10. *If* $\alpha \in (n, n+1]$ *for all* $n \in \mathbb{N}$, *we get* ${}^{\beta}T_{\alpha}^{\Delta}(f)(t) = (t + \frac{1}{\Gamma(\alpha)})^{1+n-\alpha} f^{\Delta^{1+n}}(t)$.

Proof. If $\alpha \in (n, n+1]$, then $\alpha - n \in (0,1]$ and we obtain

$$\begin{aligned} {}^{\beta}T_{\alpha}^{\Delta}(f)(t) &= {}^{\beta}T_{\alpha-n}^{\Delta}(f^{\Delta^n})(t) = (t + \frac{1}{\Gamma(\alpha)})^{1-(\alpha-n)}(f^{\Delta^n})^{\Delta}(t) \\ &= (t + \frac{1}{\Gamma(\alpha)})^{1+n-\alpha} f^{\Delta^{1+n}}(t) \end{aligned}$$

by using Definition 2.9 and Theorem 2.2 (ii) and (iii). □

3. $\beta - \Delta-$ Fractional Integral

Definition 3.1. If the function $f : \mathbb{T} \to \mathbb{R}$ is regulated and $\alpha \in (0,1]$, the $\beta - \Delta-$ fractional integral of f of order α is defined by $\int f(t)\Delta^{\alpha} t := \int f(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}\Delta t$.

Note that if $\alpha = 1$, then Definition 3.1 reduces to the indefinite $\Delta-$ integral and if $\mathbb{T} = \mathbb{R}$, then Definition 3.1 reduces to the $\beta-$ fractional integral.

Definition 3.2. If $\alpha \in (0,1]$ and the function $f : \mathbb{T} \to \mathbb{R}$ is regulated, the indefinite $\beta - \Delta-$ fractional integral of f of order α is defined by

$$\int f(t)\Delta^{\alpha} t = F_{\alpha}(t) + c,$$

where c is any constant and ${}^{\beta}T_{\alpha}^{\Delta}(F_{\alpha})(t) = f(t)$ for each $t \in \mathbb{T}^k$. The Cauchy $\beta - \Delta-$ fractional integral is defined by

$$\int_a^b f(t)\Delta^{\alpha} t = F_{\alpha}(b) - F_{\alpha}(a)$$

for all $a, b \in \mathbb{T}$.

Theorem 3.3. *For* $\alpha \in (0,1]$ *and any rd-continuous function* $f : \mathbb{T} \to \mathbb{R}$, *there exists a function* $F_{\alpha} : \mathbb{T} \to \mathbb{R}$ *such that* ${}^{\beta}T_{\alpha}^{\Delta}(F_{\alpha})(t) = f(t)$ *for every* $t \in \mathbb{T}^k$. *The function* F_{α} *is called to be an* $\beta - \Delta-$ *antiderivative of* f.

Proof. Let $\alpha \in (0,1)$. Since f is rd-continuous, f is regulated and therefore we get $\int f(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}\Delta t = F_{\alpha}(t) + c$. Then, for all $t \in \mathbb{T}^k$ we obtain

$${}^{\beta}T_{\alpha}^{\Delta}(F_{\alpha}(t) + c) = (t + \frac{1}{\Gamma(\alpha)})^{1-\alpha}(F_{\alpha}(t) + c)^{\Delta} = f(t)$$

from Theorem 2.10. The case $\alpha = 1$ is proved in [2]. □

Theorem 3.4. *Let* $a, b, c \in \mathbb{T}$, $\alpha \in (0,1]$, $f, g \in C_{rd}$ *and* $\lambda, \mu \in \mathbb{R}$. *We have the following properties.*

(i) $\int_a^b [\lambda f(t) + \mu g(t)]\Delta^{\alpha} t = \lambda \int_a^b f(t)\Delta^{\alpha} t + \mu \int_a^b g(t)\Delta^{\alpha} t$.
(ii) $\int_a^b f(t)\Delta^{\alpha} t = -\int_b^a f(t)\Delta^{\alpha} t$.

(iii) $\int_a^b f(t)\Delta^\alpha t = \int_a^c f(t)\Delta^\alpha t + \int_c^b f(t)\Delta^\alpha t$.
(iv) $\int_a^a f(t)\Delta^\alpha t = 0$.
(v) *If* $|f(t)| \leqslant g(t)$ *for all* $t \in [a,b)$, *then* $|\int_a^b f(t)\Delta^\alpha t| \leqslant \int_a^b |g(t)|\Delta^\alpha t$.
(vi) *If* $f(t) \geqslant 0$ *for all* $t \in [a,b)$, *then* $\int_a^b f(t)\Delta^\alpha t \geqslant 0$.

Proof. The relations follow from the analogous properties of the $\Delta-$ integral, Definition 3.1 and Definition 3.2. □

Theorem 3.5. *If* $t \in \mathbb{T}^k$, $f \in C_{rd}$ *and* $\alpha \in (0,1]$, *then we get*

$$\int_t^{\sigma(t)} f(s)\Delta^\alpha s = \mu(t)f(t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}.$$

Proof. From Definition 3.2 and Theorem 3.3, there is an antiderivative F_α of f and by Theorem 2.2 (iv) we obtain

$$\begin{aligned}\int_t^{\sigma(t)} f(s)\Delta^\alpha s &= F_\alpha(\sigma(t)) - F_\alpha(t) = \mu(t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}\, {}^\beta T_\alpha^\Delta(F_\alpha)(t)\\ &= \mu(t)f(t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}.\end{aligned}$$

□

Theorem 3.6. *If* ${}^\beta T_\alpha^\Delta(f)(t) \geqslant 0$ *for every* $t \in [a,b]$, *the function* f *is nondecreasing on* $[a,b]$.

Proof. If ${}^\beta T_\alpha^\Delta(f)(t) \geqslant 0$ for all $t \in [a,b]$, then from Theorem 3.4 (vii) we have $\int_c^d {}^\beta T_\alpha^\Delta(f)(t)\Delta^\alpha t \geqslant 0$ for $c,d \in \mathbb{T}$ such that $a \leqslant c \leqslant d \leqslant b$. Then we obtain

$$f(d) = f(c) + \int_c^d {}^\beta T_\alpha^\Delta(f)(t)\Delta^\alpha t \geqslant f(c)$$

by Definition 3.2. This concludes the proof. □

4. $\beta - \nabla-$ Fractional Derivative and Integral

Definition 4.1. Assume that $\alpha \in (0,1]$, $\mathbb{T}$ is a time scale and $f : \mathbb{T} \to \mathbb{R}$ is a function. For all $\varepsilon > 0$, if there is a neighborhood U of $t \in \mathbb{T}_k$ $(t > 0)$ such that

$$|[f(\rho(t)) - f(s)](t+\frac{1}{\Gamma(\alpha)})^{1-\alpha} - {}^\beta T_\alpha^\nabla(f)(t)(\rho(t)-s)| \leqslant \varepsilon|\rho(t)-s|, \ \ \forall s \in U,$$

the $\beta - \nabla-$ fractional derivative of f of order α at t is defined by the number ${}^\beta T_\alpha^\nabla(f)(t)$. $\beta - \nabla-$ fractional derivative of f of order α at 0 is defined by ${}^\beta T_\alpha^\nabla(f)(0) = \lim_{t\to 0^+} {}^\beta T_\alpha^\nabla(f)(t)$.

Note that when $\alpha = 1$, we have ${}^\beta T_\alpha^\nabla(f)(t) = f^\nabla(t)$ and if $\mathbb{T} = \mathbb{R}$, then ${}^\beta T_\alpha^\nabla(f)(t) = D_t^\alpha(f)(t)$ is the $\beta-$ fractional derivative of f of order α.

Theorem 4.2. *Suppose that* $\alpha \in (0,1]$, $\mathbb{T}$ *is a time scale,* $t \in \mathbb{T}_k$ $(t > 0)$ *and* $f : \mathbb{T} \to \mathbb{R}$ *is a function. Then, we have the following properties.*

(i) *If* f *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t*, then* f *is continuous at* t.

(ii) *Let* f *be continuous at* t *and* t *be right-scattered. Then,* f *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *and we have* ${}^{\beta}T_{\alpha}^{\nabla}(f)(t)=\frac{f(t)-f(\rho(t))}{\nu(t)}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}$.

(iii) *When* t *is right-dense,* f *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *iff the limit* $\lim_{s\to t}\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}$ *exists as a finite number. Then, we have*

$$ {}^{\beta}T_{\alpha}^{\nabla}(f)(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}(t+\frac{1}{\Gamma(\alpha)})^{1-\alpha}. $$

(iv) *If* f *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t*, then we get*

$$ f(\rho(t))=f(t)-\nu(t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}\,{}^{\beta}T_{\alpha}^{\nabla}(f)(t). $$

Proof. The proof is similar to the proof of Theorem 2.2. □

Theorem 4.3. *Suppose that the functions* $f,g:\mathbb{T}\to\mathbb{R}$ *have* $\beta-\nabla-$ *fractional derivatives of order* α *at* $t\in\mathbb{T}_k$*. Then, we have the following properties.*

(i) *The sum* $f+g:\mathbb{T}\to\mathbb{R}$ *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *with* ${}^{\beta}T_{\alpha}^{\nabla}(f+g)(t)={}^{\beta}T_{\alpha}^{\nabla}(f)(t)+{}^{\beta}T_{\alpha}^{\nabla}(g)(t)$.

(ii) *The function* $cf:\mathbb{T}\to\mathbb{R}$ *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *with* ${}^{\beta}T_{\alpha}^{\nabla}(cf)(t)=c\,{}^{\beta}T_{\alpha}^{\nabla}(f)(t)$*, where* c *is any constant.*

(iii) *The product* $f.g:\mathbb{T}\to\mathbb{R}$ *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *with*

$$ {}^{\beta}T_{\alpha}^{\nabla}(f.g)(t)={}^{\beta}T_{\alpha}^{\nabla}(f)(t)g(t)+f(\rho(t))\,{}^{\beta}T_{\alpha}^{\nabla}(g)(t)={}^{\beta}T_{\alpha}^{\nabla}(g)(t)f(t)+g(\rho(t))\,{}^{\beta}T_{\alpha}^{\nabla}(f)(t). $$

(iv) *When* $f(t)f(\rho(t))\neq 0$*, the function* $\frac{1}{f}$ *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *with*

$$ {}^{\beta}T_{\alpha}^{\nabla}(\frac{1}{f})(t)=-\frac{{}^{\beta}T_{\alpha}^{\nabla}(f)(t)}{f(t)f(\rho(t))}. $$

(v) *When* $g(t)g(\rho(t))\neq 0$*, the function* $\frac{f}{g}$ *has* $\beta-\nabla-$ *fractional derivative of order* α *at* t *with*

$$ {}^{\beta}T_{\alpha}^{\nabla}(\frac{f}{g})(t)=\frac{{}^{\beta}T_{\alpha}^{\nabla}(f)(t)g(t)-f(t)\,{}^{\beta}T_{\alpha}^{\nabla}(g)(t)}{g(t)g(\rho(t))}. $$

Proof. The proof is similar to the proof of Theorem 2.6. □

Definition 4.4. Suppose that $\alpha\in(n,n+1]$, $n\in\mathbb{N}$, $\mathbb{T}$ is a time scale and f is n times delta differentiable at $t\in\mathbb{T}_{k^n}$ $(t>0)$. $\beta-\nabla-$ fractional derivative of order α of f is defined by ${}^{\beta}T_{\alpha}^{\nabla}(f)(t)={}^{\beta}T_{\alpha-n}^{\nabla}(f^{\nabla^n})(t)$.

Theorem 4.5. *If* $\alpha\in(n,n+1]$, $n\in\mathbb{N}$*, we get* ${}^{\beta}T_{\alpha}^{\nabla}(f)(t)=(t+\frac{1}{\Gamma(\alpha)})^{1+n-\alpha}f^{\nabla^{1+n}}(t)$.

Proof. The proof is similar to the proof of Theorem 2.10. □

Definition 4.6. If the function $f:\mathbb{T}\to\mathbb{R}$ is regulated and $\alpha\in(0,1]$, the $\beta-\nabla-$ fractional integral of f of order α is defined by $\int f(t)\nabla^{\alpha}t:=\int f(t)(t+\frac{1}{\Gamma(\alpha)})^{\alpha-1}\nabla t$.

Note that if $\alpha = 1$, then Definition 4.6 reduces to the indefinite $\nabla-$ integral and if $\mathbb{T} = \mathbb{R}$, then Definition 4.6 reduces to the $\beta-$ fractional integral.

Definition 4.7. If $\alpha \in (0, 1]$ and the function $f : \mathbb{T} \to \mathbb{R}$ is regulated, the indefinite $\beta - \nabla-$ fractional integral of f of order α is defined by

$$\int f(t)\nabla^{\alpha}t = F_{\alpha}(t) + c,$$

where c is any constant and ${}^{\beta}T_{\alpha}^{\nabla}(F_{\alpha})(t) = f(t)$ for each $t \in \mathbb{T}_k$. The Cauchy $\beta - \nabla-$ fractional integral is defined by

$$\int_a^b f(t)\nabla^{\alpha}t = F_{\alpha}(b) - F_{\alpha}(a)$$

for all $a, b \in \mathbb{T}$.

Theorem 4.8. *For $\alpha \in (0, 1]$ and any ld-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exists a function $F_{\alpha} : \mathbb{T} \to \mathbb{R}$ such that ${}^{\beta}T_{\alpha}^{\nabla}(F_{\alpha})(t) = f(t)$ for every $t \in \mathbb{T}_k$. The function F_{α} is called to be an $\beta - \nabla-$ antiderivative of f.*

Proof. The proof is similar to the proof of Theorem 3.3. □

Theorem 4.9. *Let $a, b, c \in \mathbb{T}$, $\alpha \in (0, 1]$, $f, g \in C_{ld}$ and $\lambda, \mu \in \mathbb{R}$. We have the following properties.*

(i) $\int_a^b [\lambda f(t) + \mu g(t)]\nabla^{\alpha}t = \lambda \int_a^b f(t)\nabla^{\alpha}t + \mu \int_a^b g(t)\nabla^{\alpha}t.$

(ii) $\int_a^b f(t)\nabla^{\alpha}t = -\int_b^a f(t)\nabla^{\alpha}t.$

(iii) $\int_a^b f(t)\nabla^{\alpha}t = \int_a^c f(t)\nabla^{\alpha}t + \int_c^b f(t)\nabla^{\alpha}t.$

(iv) $\int_a^a f(t)\nabla^{\alpha}t = 0.$

(v) *If* $|f(t)| \leqslant g(t)$ *for all* $t \in [a, b)$, *then* $|\int_a^b f(t)\nabla^{\alpha}t| \leqslant \int_a^b |g(t)|\nabla^{\alpha}t.$

(vi) *If* $f(t) \geqslant 0$ *for all* $t \in [a, b)$, *then* $\int_a^b f(t)\nabla^{\alpha}t \geqslant 0.$

Proof. The relations follow from the analogous properties of the $\nabla-$ integral, Definition 3.1 and Definition 3.2. □

Theorem 4.10. *If $t \in \mathbb{T}_k$, $f \in C_{ld}$ and $\alpha \in (0, 1]$, then we get*

$$\int_{\rho(t)}^t f(s)\nabla^{\alpha}s = \nu(t)f(t)(t + \frac{1}{\Gamma(\alpha)})^{\alpha-1}.$$

Proof. The proof is similar to the proof of Theorem 3.5. □

5. Conclusions

First, we defined $\beta - \Delta -$ fractional derivative of order α. If $\alpha = 1$, then Hilger derivative was obtained and if $\mathbb{T} = \mathbb{R}$, then $\beta -$ fractional derivative of order α was obtained. Some properties of $\beta - \Delta -$ fractional derivative of order α were given. Second, $\beta - \Delta -$ fractional integral of order α was defined. This definition reduces to the indefinite $\Delta -$ integral when $\alpha = 1$ and this definition reduces to the $\beta -$ fractional integral when $\mathbb{T} = \mathbb{R}$. Then, we obtained some properties of $\beta - \Delta -$ fractional integral of order α. Finally, we defined $\beta - \nabla -$ fractional derivative and integral of order α and their properties were introduced. Our new calculus unifies and generalizes the time scale calculus and the $\beta -$ fractional calculus.

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