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Fractional Multiplicative Corrected Dual-Simpson type inequalities

Badreddine Meftah [a,*], Meriem Boucharebb [b], Nadjla Boutelhig [c]

[a] Department of Mathematics, University 8 may 1945 Guelma, Algeria.
[b] Environmental Research Center (C.R.E), Annaba, Algeria.
[c] Laboratoire de Recherches et des Etudes Economiques, Mohamed-Cherif Messaadia University, Souk-ahras, Algeria.

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Abstract

This paper delves into the realm of inequalities, focusing on the corrected dual Simpson-type inequalities for fractional multiplicative integrals. Based on a new identity, we establish some new inequalities via multiplicative s-convexity. Finally, we provide some applications of the results obtained to special means in the frame of multiplicative calculus.

Keywords: Corrected dual-Simpson formula, integral inequalities, Fractional multiplicative integral, multiplicative s-convexity.
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1. Introduction

Convexity is a fundamental mathematical concept with wide-ranging applications across various scientific disciplines. Its significance lies in its ability to capture the essential characteristics of numerous real-world phenomena, making it a potent tool for modeling and analysis. Convex functions, in particular, possess remarkable properties that simplify optimization, economics, and even the understanding of physical systems. In mathematical terms, a function φ is deemed convex on the interval $[a, b]$ if, for all $x, y \in [a, b]$, the subsequent inequality holds for all $t \in [0, 1]$:

$$\varphi(tx + (1-t)y) \leqslant t\varphi(x) + (1-t)\varphi(y).$$

One of the most consequential results associated with convexity is the Hermite-Hadamard (H-H) inequality. For a convex function φ defined on an interval $[a, b]$, the inequality is expressed as:

*Corresponding author: badrimeftah@yahoo.fr

$$\varphi\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \varphi(x)dx \leqslant \frac{\varphi(a)+\varphi(b)}{2}.$$

The concept of convexity serves as the basis for the derivation of important integral inequalities like the Hermite-Hadamard inequality, which have wide-ranging applications in mathematical analysis, physics, and engineering, highlighting its significance in both theoretical and practical mathematical contexts, see [19, 25, 26].

Multiplicative calculus, originally introduced by Grosman and Katz in 1967, emerged as an alternative to classical calculus aimed at addressing issues concerning rates of change and multiplicative processes [13]. This mathematical framework, which exclusively deals with positive functions, underwent formalization in a comprehensive work by Bashirov et al. in 2008 [5]. Its significance lies in its enhanced capacity to handle phenomena involving growth, decay, and proportional relationships more effectively than traditional calculus. Over time, it has found relevance in a range of fields, including physics [29], finance [6], and biology [11], offering a fresh perspective for modeling and analysis in scenarios where conventional calculus may encounter limitations.

In the context of multiplicative calculus, the most suitable form is logarithmic convexity, often referred to as multiplicative convexity. This concept can be articulated as follows:

Definition 1.1 ([22])**.** A positive function φ is said to exhibit multiplicative convexity over the interval I if, for all $x, y \in I$, the subsequent inequality:

$$\varphi\left(tx+(1-t)\,y\right) \leqslant [\varphi\left(x\right)]^{t}\,[\varphi(y)]^{1-t},$$

holds true for all $t \in [0, 1]$.

In [2], Ali et al. established the Hermite-Hadamard inequality for multiplicative integrals as follows

$$\varphi\left(\tfrac{a+b}{2}\right) \leqslant \left(\int_a^b \varphi\left(x\right)^{dx}\right)^{\frac{1}{b-a}} \leqslant \sqrt{\varphi\left(a\right)\varphi\left(b\right)}, \tag{1.1}$$

where φ is multiplicatively convex on $[a, b]$.

In the domain of multiplicative integral inequalities, substantial research has been undertaken. In [7], the authors established The Midpoint and Trapezoid-type inequalities for multiplicatively convex derivatives. The Ostrowski and Simpson-type inequalities were provided by Ali et al. in [4]. The Maclaurin inequalities was provided in [18], and the Dual Simpson-type inequalities in [20]. These findings collectively enhance our understanding of multiplicative integral inequalities. For additional works, readers are referred to [3, 18, 33].

On the other hand, fractional calculus is a mathematical field that deals with differentiation and integration of non-integer order. It provides a powerful framework for modeling complex phenomena that are not always well described by traditional calculus [31, 34, 35]. In the context of fractional integral inequalities, significant progress has been made in understanding the properties of functions, fractional integration operators, and the resulting inequalities. Researchers have explored various types of inequalities, including Hermite-Hadamard, Simpson, and Maclaurin inequalities, using fractional integral operators, see [27, 28].

The multiplicative Riemann-Liouville fractional integrals were introduced by Abdeljawad and Grossman in 2016 as follow:

Definition 1.2. [1] The multiplicative Riemann-Liouville fractional integral operators of order $\alpha \in \mathbb{C}$, where $\mathrm{Re}(\alpha) > 0$, are defined as follows:

$$_{a}I_{*}^{\alpha}\varphi(x) = e^{\left(J_{a^{+}}^{\alpha}(\ln\circ\varphi)\right)(x)}, \quad a < x \tag{1.2}$$

and

$$_{*}I_{b}^{\alpha}\varphi(x) = e^{\left(J_{b^{-}}^{\alpha}(\ln\circ\varphi)\right)(x)}, \quad x < b, \tag{1.3}$$

where $J_{c^{+}}^{\alpha}$ and $J_{c^{-}}^{\alpha}$ represent the classical Riemann-Liouville operators.

In [9], Budak and Özçelik, established the Hermite-Hadamard inequalities pretaining to multiplicative Riemann-Liouville fractional integrals as follow:

$$\varphi\left(\tfrac{a+b}{2}\right) \leqslant \left[{}_{*}I_{\frac{a+b}{2}}^{\alpha}\varphi(a)\ {}_{\frac{a+b}{2}}I_{*}^{\alpha}\varphi(b)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}} \leqslant \sqrt{\varphi(a)\varphi(b)}, \tag{1.4}$$

where φ is multiplicatively convex function defined on the interval $[a, b]$, and $\alpha > 0$.

A generalization of the results presented by Ali et al. in [2] and Budak et al. in [9] was provided by Fu et al. in [12]. The fractional multiplicative Bullen-type inequalities is given in [8], while the Simpson-type inequalities were provided in [21]. Moreover, Peng et al. made a significant contribution to the field through their investigation into fractional multiplicative Maclaurin-type inequalities in [24]. Additional pertinent results are available in [23, 30], as well as in the referenced sources.

The corrected dual-Simpson formula for multiplicative integrals is given by:

$$\left(\int_{a}^{b}\varphi(x)dx\right)^{\frac{1}{b-a}} = \left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right] + \mathcal{R}(\varphi),$$

where $\mathcal{R}(\varphi)$ is the approximation error, see [16].

Motivated by the above cited papers, in this work, we will discuss the corrected dual Simpson's formula via fractional multiplicative integrals. To do so, we first establish a new integral identity. On the basis of this equality, we derive some corrected dual-Simpson type inequalities for functions whose multiplicative derivatives are multiplicatively s-convex in the second sense.

We recall that a function is said to be multiplicatively s-convex in the second sense for some fixed $s \in (0, 1]$ if, for all $x, y \in I$ the following inequality holds true for all $t \in [0, 1]$.

$$\varphi(tx + (1-t)y) \leqslant [\varphi(x)]^{t^{s}}[\varphi(y)]^{(1-t)^{s}}.$$

2. Preliminaries

Definition 2.1 ([5])**.** The definition of the multiplicative derivative for a positive function φ, indicated as φ^*, is provided as follows:

$$\frac{d^*\varphi}{dx} = \varphi^*(x) = \lim_{h\to 0}\left(\frac{\varphi(x+h)}{\varphi(x)}\right)^{\frac{1}{h}}.$$

Remark 2.2. Every positive and differentiable function φ naturally possesses multiplicative differentiability, with the relationship between φ' and φ^* determined by the following connection:

$$\varphi^*(x) = e^{(\ln\varphi(x))'} = e^{\frac{\varphi'(x)}{\varphi(x)}}.$$

Proposition 2.3 ([5])**.** *Suppose φ and ξ are two multiplicatively differentiable functions, and ψ is differentiable. Let k be an arbitrary positive constant. Then, the functions $k\varphi$, $\varphi\xi$, $\varphi+\xi$, φ/ξ, and φ^{ψ} exhibit multiplicative differentiability, and*

- $(k\varphi)^*(x) = \varphi^*(x)$,
- $(\varphi+\xi)^*(x) = \varphi^*(x)^{\frac{\varphi(x)}{\varphi(x)+\xi(x)}}\,\xi^*(x)^{\frac{\xi(x)}{\varphi(x)+\xi(x)}}$,
- $(\varphi\xi)^*(x) = \varphi^*(x)\,\xi^*(x)$,
- $\left(\frac{\varphi}{\xi}\right)^*(x) = \frac{\varphi^*(x)}{\xi^*(x)}$,
- $(\varphi^{\psi})^*(x) = \varphi^*(x)^{\psi(x)}\,\varphi(x)^{\psi'(x)}$,
- $(\varphi\circ\psi)^*(x) = \varphi^*(\psi(x))^{\psi'(x)}$.

Definition 2.4 ([5])**.** The definition of the multiplicative integral for a positive function φ, is provided as follows:

$$\int_a^b (\varphi(x))^{dx} = \exp\left(\int_a^b \ln(\varphi(x))\,dx\right).$$

Proposition 2.5 ([5])**.** *If we have positive and Riemann-integrable functions φ and ξ, then both φ and ξ are considered multiplicative integrable, and the following properties are valid.*

- $\int_a^b ((\varphi(x))^p)^{dx} = \left(\int_a^b (\varphi(x))^{dx}\right)^p, \quad p\in\mathbb{R}$,
- $\int_a^b (\varphi(x))^{dx} = \int_a^c (\varphi(x))^{dx}\int_c^b (\varphi(x))^{dx}, a<c<b$,
- $\int_a^a (\varphi(x))^{dx} = 1$ *and* $\int_a^b (\varphi(x))^{dx} = \left(\int_b^a (\varphi(x))^{dx}\right)^{-1}$,

- $\int_a^b (\varphi(x)\,\xi(x))^{dx} = \int_a^b (\varphi(x))^{dx} \int_a^b (\xi(x))^{dx}$,
- $\int_a^b \left(\frac{\varphi(x)}{\xi(x)}\right)^{dx} = \frac{\int_a^b (\varphi(x))^{dx}}{\int_a^b (\xi(x))^{dx}}$.

The multiplicative integration by parts is as follows:

Lemma 2.6 ([5]). *Take a positive multiplicative differentiable function* φ *defined on the interval* $[a, b]$. *Also, let* $\psi : J \subset \mathbb{R} \to \mathbb{R}$ *and* $\xi : [a, b] \to \mathbb{R}$ *be two separate differentiable functions. Consequently, we can state:*

$$\int_a^b \left(\varphi^*(\psi(x))^{\psi'(x)\xi(x)}\right)^{dx} = \frac{\varphi(\psi(b))^{\xi(b)}}{\varphi(\psi(a))^{\xi(a)}} \times \frac{1}{\int_a^b \left(\varphi(\psi(x))^{\xi'(x)}\right)^{dx}}.$$

Now, let's recall particular functions that will be used later in our inquiry.

Definition 2.7. [15] The beta function is defined for any complex numbers x, y such that $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$ by

$$B(x, y) = \int_0^1 v^{x-1}(1-v)^{y-1}\,dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(.)$ denotes the Euler gamma function.

Definition 2.8. [15] For any complex numbers x, y, z and a such that $\mathrm{Re}(z) > \mathrm{Re}(y) > 0$ and $|a| < 1$, the hypergeometric function is defined as follows

$${}_2F_1(x, y, z; y) = \frac{1}{B(y, z-y)} \int_0^1 v^{y-1}(1-v)^{z-y-1}(1-av)^{-x}\,dv,$$

where $B(., .)$ is the beta function.

Here is an example from biology that allows us to assess the usefulness of the concepts of multiplicative differentiation and integration.

Example 2.9. In biology, a common example that can be modeled using an ordinary differential equation (ODE) involves the growth of a population of bacteria. Consider a population of bacteria in an ideal environment with sufficient resources for their growth. We can use an ODE to represent the change in the population size of bacteria over time.

One of the simplest population growth equations for bacteria is the exponential growth equation:

$$\varphi'(t) = \mathcal{R}(t) \times \varphi(t), \tag{2.1}$$

where:

- φ is the size of the bacterial population.
- φ' represents the rate of change in the bacterial population with respect to time.
- $\mathcal{R}(t)$ is the intrinsic growth rate of the bacterial population.

This equation models bacterial population growth exponentially, meaning that the population increases proportionally to its current size. However, in reality, bacterial growth is subject to limiting factors, such as the availability of finite resources, which can be accounted for using more complex growth equations like the logistic growth equation.

It is important to note that the time variable, t, is always positive and the growth rate of the bacterial population is positive.

Using multiplicative calculus, equation (2.1) can be reformulated as $e^{(\ln(\varphi(t)))'} = e^{\mathcal{R}(t)}$, or $\varphi^*(t) = e^{\mathcal{R}(t)}$, whose solution is given in the form of a multiplicative integral, as follows:

$$\varphi(t) = \lambda \int_{t_0}^{t} \left(e^{\mathcal{R}(t)}\right)^{dt}, \quad \text{with } \lambda = \varphi\left(t_0\right).$$

Hence, through this example, we have assessed the usefulness and connection between the concepts of multiplicative differentiation and integration and their application to differential equations.

3. Main results

In ordre to establish our results, we need the following lemma.

Lemma 3.1. *Assuming that φ is a positive multiplicatively differentiable function on $[a,b]$, and φ^* is multiplicatively integrable over $[a,b]$, then for $\alpha > 0$, $\rho \geqslant 0$ and $x \in \left[a, \frac{a+b}{2}\right]$, the following fractional multiplicative integral identity holds:*

$$\begin{aligned}
&\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\mathcal{J}\left(a,b;\varphi\right)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}} \\
&= \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\,a + t\tfrac{3a+b}{4}\right)\right)^{t^{\alpha}}\right)^{dt}\right)^{\frac{b-a}{16}} \\
&\times \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\,\tfrac{3a+b}{4} + t\tfrac{a+b}{2}\right)\right)^{-\frac{2}{15}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{b-a}{16}} \\
&\times \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\,\tfrac{a+b}{2} + t\tfrac{a+3b}{4}\right)\right)^{t^{\alpha}+\frac{2}{15}}\right)^{dt}\right)^{\frac{b-a}{16}} \\
&\times \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\,\tfrac{a+3b}{4} + tb\right)\right)^{-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{b-a}{16}},
\end{aligned}$$

where $\mathcal{J}$ is defined as follows:

$$\mathcal{J}(a,b;\varphi)=\left(\left({}_{*}I^{\alpha}_{\left(\frac{3a+b}{4}\right)}\varphi\right)(a)\left({}_{\left(\frac{a+3b}{4}\right)}I^{\alpha}_{*}\varphi\right)(b)\right.$$
$$\left.\times\left({}_{\left(\frac{3a+b}{4}\right)}I^{\alpha}_{*}\varphi\right)\left(\frac{a+b}{2}\right)\left({}_{*}I^{\alpha}_{\left(\frac{a+3b}{4}\right)}\varphi\right)\left(\frac{a+b}{2}\right)\right)^{\frac{1}{(b-a)^{\alpha-1}}}. \tag{3.1}$$

Proof. Let

$$I_1=\left(\int_0^1\left(\left(\varphi^*\left((1-t)a+t\frac{3a+b}{4}\right)\right)^{t^\alpha}\right)^{dt}\right)^{\frac{b-a}{16}},$$

$$I_2=\left(\int_0^1\left(\left(\varphi^*\left((1-t)\frac{3a+b}{4}+t\frac{a+b}{2}\right)\right)^{-\frac{2}{15}-(1-t)^\alpha}\right)^{dt}\right)^{\frac{b-a}{16}},$$

$$I_3=\left(\int_0^1\left(\left(\varphi^*\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)\right)^{t^\alpha+\frac{2}{15}}\right)^{dt}\right)^{\frac{b-a}{16}}$$

and

$$I_4=\left(\int_0^1\left(\left(\varphi^*\left((1-t)\frac{a+3b}{4}+tb\right)\right)^{-(1-t)^\alpha}\right)^{dt}\right)^{\frac{(x-a)^2}{b-a}}.$$

Using the multiplicative integration by parts, from I_1 we have

$$\begin{aligned}
I_1&=\left(\int_0^1\left(\left(\varphi^*\left((1-t)a+t\frac{3a+b}{4}\right)\right)^{t^\alpha}\right)^{dt}\right)^{\frac{b-a}{16}}\\
&=\int_0^1\left(\left(\varphi^*\left((1-t)a+t\frac{3a+b}{4}\right)\right)^{\frac{b-a}{16}(t^\alpha)}\right)^{dt}\\
&=\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{b-a}{4}}\frac{1}{\int_0^1\left(\left(\varphi\left((1-t)a+t\frac{3a+b}{4}\right)\right)^{\alpha\frac{b-a}{4}t^{\alpha-1}}\right)^{dt}}\\
&=\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{b-a}{4}}\frac{1}{\exp\left\{\alpha\frac{b-a}{4}\int_0^1\left(t^{\alpha-1}\left(\ln\varphi\left((1-t)a+t\frac{3a+b}{4}\right)\right)\right)dt\right\}}\\
&=\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{b-a}{4}}\frac{1}{\exp\left\{\frac{\alpha4^{\alpha-1}}{(b-a)^\alpha}\int_a^{\frac{3a+b}{4}}\left((y-a)^{\alpha-1}(\ln\varphi(y))\right)dy\right\}}\\
&=\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{b-a}{4}}\frac{1}{\left(\exp\left\{\left(\frac{1}{\Gamma(\alpha)}\int_a^{\frac{3a+b}{4}}\left((y-a)^{\alpha-1}(\ln\varphi(y))\right)dy\right)\right\}\right)^{\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}
\end{aligned}$$

$$= \left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{b-a}{4}} \left(\left({}_{*}I^{\alpha}_{\left(\frac{3a+b}{4}\right)}\varphi\right)(a)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}. \tag{3.2}$$

Similarly,

$$I_2 = \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\frac{3a+b}{4} + t\frac{a+b}{2}\right)\right)^{-\frac{2}{15}-(1-t)^{\alpha}}\right)^{dt}\right)^{\frac{b-a}{16}}$$

$$= \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\frac{3a+b}{4} + t\frac{a+b}{2}\right)\right)^{\frac{b-a}{16}\left(-\frac{2}{15}-(1-t)^{\alpha}\right)}\right)^{dt}\right)$$

$$= \frac{\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{17}{60}}}{\left(\varphi\left(\frac{a+b}{2}\right)\right)^{\frac{1}{30}}} \frac{1}{\int_0^1 \left(\left(\varphi\left((1-t)\frac{3a+b}{4}+t\frac{a+b}{2}\right)\right)^{\frac{\alpha}{4}(1-t)^{\alpha-1}}\right)^{dt}}$$

$$= \frac{\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{17}{60}}\left(\varphi\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{30}}}{\exp\left\{\frac{\alpha}{4}\int_0^1 (1-t)^{\alpha-1}\ln\left(\varphi\left((1-t)\frac{3a+b}{4}+t\frac{a+b}{2}\right)\right)dt\right\}}$$

$$= \frac{\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{17}{60}}\left(\varphi\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{30}}}{\exp\left\{\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-y\right)^{\alpha-1}\ln(\varphi(y))dy\right)\right\}}$$

$$= \left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{17}{60}}\left(\varphi\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{30}}\left(\left({}_{\left(\frac{3a+b}{4}\right)}I^{\alpha}_{*}\varphi\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}, \tag{3.3}$$

$$I_3 = \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\frac{a+b}{2} + t\frac{a+3b}{4}\right)\right)^{t^{\alpha}+\frac{2}{15}}\right)^{dt}\right)^{\frac{b-a}{16}}$$

$$= \int_0^1 \left(\left(\varphi^*\left((1-t)\frac{a+b}{2} + t\frac{3a+b}{4}\right)\right)^{\frac{b-a}{16}\left(t^{\alpha}+\frac{2}{15}\right)}\right)^{dt}$$

$$= \frac{\left(\varphi\left(\frac{a+3b}{4}\right)\right)^{\frac{17}{60}}}{\left(\varphi\left(\frac{a+b}{2}\right)\right)^{\frac{1}{30}}} \frac{1}{\int_0^1 \left(\left(\varphi\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)\right)^{\frac{\alpha}{4}t^{\alpha-1}}\right)^{dt}}$$

$$= \frac{\left(\varphi\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{30}}\left(\varphi\left(\frac{a+3b}{4}\right)\right)^{\frac{17}{60}}}{\exp\left\{\frac{\alpha}{4}\int_0^1 t^{\alpha-1}\ln\left(\varphi\left((1-t)\frac{a+b}{2}+t\frac{a+3b}{4}\right)\right)dt\right\}}$$

$$= \frac{\left(\varphi\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{30}}\left(\varphi\left(\frac{a+3b}{4}\right)\right)^{\frac{17}{60}}}{\left(\exp\left\{\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(y-\frac{a+b}{2}\right)^{\alpha-1}\ln(\varphi(y))dy\right\}\right)^{\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}$$

$$= \left(\varphi\left(\frac{a+b}{2}\right)\right)^{-\frac{1}{30}}\left(\varphi\left(\frac{a+3b}{4}\right)\right)^{\frac{17}{60}}\left(\left({}_{*}I^{\alpha}_{\left(\frac{a+3b}{4}\right)}\varphi\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}} \tag{3.4}$$

and

$$\begin{aligned}
I_4 &= \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\tfrac{a+3b}{4}+tb\right)\right)^{-(1-t)^\alpha}\right)^{dt}\right)^{\frac{b-a}{16}} \\
&= \left(\int_0^1 \left(\left(\varphi^*\left((1-t)\tfrac{a+3b}{4}+tb\right)\right)^{-\frac{b-a}{16}(1-t)^\alpha}\right)^{dt}\right) \\
&= \left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{1}{4}} \frac{1}{\int_0^1 \left(\left(\varphi\left((1-t)\frac{a+3b}{4}+tb\right)\right)^{\frac{\alpha}{4}(1-t)^{\alpha-1}}\right)^{dt}} \\
&= \left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{1}{4}} \frac{1}{\exp\left\{\frac{\alpha}{4}\int_0^1 (1-t)^{\alpha-1}\ln\left(\varphi\left((1-t)\frac{a+3b}{4}+tb\right)\right)dt\right\}} \\
&= \left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{1}{4}} \frac{1}{\exp\left\{\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_{\frac{a+3b}{4}}^{b}(b-y)^{\alpha-1}\ln(\varphi(y))dy\right)\right\}} \\
&= \left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{1}{4}} \left(\left({}_{\left(\frac{a+3b}{4}\right)}I_*^\alpha \varphi\right)(b)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}. \qquad (3.5)
\end{aligned}$$

By multiplying (3.2)-(3.5) we obtain the desired result. This completes the proof. □

Theorem 3.2. *Let* $\varphi : [a,b] \to \mathbb{R}^+$ *be an increasing and multiplicatively differentiable function on* $[a,b]$. *If* φ^* *is multiplicative* s*-convex on* $[a,b]$, *then we have*

$$\begin{aligned}
&\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right](\mathcal{J}(a,b;\varphi))^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}\right| \\
&\leqslant \left[(\varphi^*(a)\varphi^*(b))^{B(\alpha+1,s+1)}\varphi^*\left(\tfrac{a+b}{2}\right)^{\left(\frac{4}{15(s+1)}+2B(s+1,\alpha+1)\right)}\right. \\
&\quad \left.\times\left(\varphi^*\left(\tfrac{3a+b}{4}\right)\varphi^*\left(\tfrac{a+3b}{4}\right)\right)^{\left(\frac{32(s+1)+2\alpha}{15(\alpha+s+1)(s+1)}\right)}\right]^{\frac{b-a}{16}}.
\end{aligned}$$

where $\mathcal{J}$ *is defined as* (3.1), *and* B *is the beta function.*

Proof. From Lemma 3.1, properties of multiplicative integrals and the multiplicative s-convexity of φ^*, we have

$$\begin{aligned}
&\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right](\mathcal{J}(a,b;\varphi))^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}\right| \\
&\leqslant \exp\left(\frac{b-a}{16}\int_0^1 t^\alpha \ln\left(\varphi^*\left((1-t)a+t\tfrac{3a+b}{4}\right)\right)dt\right) \\
&\quad \times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 \left(\tfrac{2}{15}+(1-t)^\alpha\right)\ln\varphi^*\left((1-t)\tfrac{3a+b}{4}+t\tfrac{a+b}{2}\right)dt\right)
\end{aligned}$$

$$\times \exp\left(\frac{b-a}{16}\int_0^1 \left(t^{\alpha}+\tfrac{2}{15}\right)\ln \varphi^*\left((1-t)\tfrac{a+b}{2}+t\tfrac{a+3b}{4}\right)dt\right)$$

$$\times \exp\left(\frac{b-a}{16}\int_0^1 (1-t)^{\alpha}\ln \varphi^*\left((1-t)\tfrac{a+3b}{4}+tb\right)dt\right)$$

$$\leqslant \exp\left(\frac{b-a}{16}\int_0^1 t^{\alpha}\left((1-t)^s\ln \varphi^*(a)+t^s\ln\varphi^*\left(\tfrac{3a+b}{4}\right)\right)dt\right)$$

$$\times \exp\left(\frac{b-a}{16}\int_0^1 \left(\tfrac{2}{15}+(1-t)^{\alpha}\right)\left((1-t)^s\ln \varphi^*\left(\tfrac{3a+b}{4}\right)+t^s\ln\varphi^*\left(\tfrac{a+b}{2}\right)\right)dt\right)$$

$$\times \exp\left(\frac{b-a}{16}\int_0^1 \left(t^{\alpha}+\tfrac{2}{15}\right)\left((1-t)^s\ln \varphi^*\left(\tfrac{a+b}{2}\right)+t^s\ln\varphi^*\left(\tfrac{a+3b}{4}\right)\right)dt\right)$$

$$\times \exp\left(\frac{b-a}{16}\int_0^1 (1-t)^{\alpha}\left((1-t)^s\ln \varphi^*\left(\tfrac{a+3b}{4}\right)+t^s\ln\varphi^*(b)\right)dt\right)$$

$$=\exp\left(\frac{b-a}{16}\left(\int_0^1 t^{\alpha}(1-t)^s\,dt\right)\ln\varphi^*(a)+\frac{b-a}{16}\left(\int_0^1 t^{\alpha+s}dt\right)\ln\varphi^*\left(\tfrac{3a+b}{4}\right)\right)$$

$$\times \exp\left(\frac{b-a}{16}\left(\int_0^1 \left(\tfrac{2}{15}+(1-t)^{\alpha}\right)(1-t)^s\,dt\right)\ln\varphi^*\left(\tfrac{3a+b}{4}\right)\right.$$

$$\left.+\frac{b-a}{16}\left(\int_0^1 \left(\tfrac{2}{15}+(1-t)^{\alpha}\right)t^s dt\right)\ln\varphi^*\left(\tfrac{a+b}{2}\right)\right)$$

$$\times \exp\left(\frac{b-a}{16}\left(\int_0^1 \left(t^{\alpha}+\tfrac{2}{15}\right)(1-t)^s\,dt\right)\ln\varphi^*\left(\tfrac{a+b}{2}\right)\right.$$

$$\left.+\frac{b-a}{16}\left(\int_0^1 \left(t^{\alpha}+\tfrac{2}{15}\right)t^s dt\right)\ln\varphi^*\left(\tfrac{a+3b}{4}\right)dt\right)$$

$$\times \exp\left(\frac{b-a}{16}\left(\int_0^1 (1-t)^{\alpha+s}\,dt\right)\ln\varphi^*\left(\tfrac{a+3b}{4}\right)+\frac{b-a}{16}\left(\int_0^1 (1-t)^{\alpha}t^s dt\right)\ln\varphi^*(b)\right)$$

$$=\left[\left(\varphi^*(a)\,\varphi^*(b)\right)^{B(\alpha+1,s+1)}\varphi^*\left(\tfrac{a+b}{2}\right)^{\left(\frac{4}{15(s+1)}+2B(s+1,\alpha+1)\right)}\right.$$

$$\left.\times\left(\varphi^*\left(\tfrac{3a+b}{4}\right)\varphi^*\left(\tfrac{a+3b}{4}\right)\right)^{\left(\frac{32(s+1)+2\alpha}{15(\alpha+s+1)(s+1)}\right)}\right]^{\frac{b-a}{16}},$$

where we have used

$$\int_0^1 t^{\alpha}(1-t)^{s}\,dt=\int_0^1 (1-t)^{\alpha}t^{s}dt=B(\alpha+1,s+1), \tag{3.6}$$

$$\int_0^1 t^{\alpha+s}dt=\int_0^1 (1-t)^{\alpha+s}\,dt=\frac{1}{\alpha+s+1}, \tag{3.7}$$

$$\int_0^1 \left(\tfrac{2}{15}+(1-t)^{\alpha}\right)(1-t)^{s}\,dt=\int_0^1 \left(t^{\alpha}+\tfrac{2}{15}\right)t^{s}dt=\frac{17(s+1)+2\alpha}{15(\alpha+s+1)(s+1)} \tag{3.8}$$

and

$$\int_0^1 \left(t^{\alpha}+\tfrac{2}{15}\right)(1-t)^{s}\,dt=\int_0^1 \left(\tfrac{2}{15}+(1-t)^{\alpha}\right)t^{s}dt=\frac{2}{15(s+1)}+B(s+1,\alpha+1). \tag{3.9}$$

The proof is completed. □

Corollary 3.3. *In Theorem 3.2, if we take $s=1$, then we get the following fractional corrected dual-Simpson type inequality for multiplicative convex functions*

$$\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\mathcal{J}(a,b;\varphi)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|$$
$$\leqslant\left[\left(\varphi^{*}(a)\,\varphi^{*}(b)\right)^{\frac{1}{(\alpha+1)(\alpha+2)}}\varphi^{*}\left(\tfrac{a+b}{2}\right)^{\left(\frac{2(\alpha+1)(\alpha+2)+30}{15(\alpha+1)(\alpha+2)}\right)}\right.$$
$$\left.\times\left(\varphi^{*}\left(\tfrac{3a+b}{4}\right)\varphi^{*}\left(\tfrac{a+3b}{4}\right)\right)^{\left(\frac{32+\alpha}{15(\alpha+2)}\right)}\right]^{\frac{b-a}{16}}.$$

Corollary 3.4. *By setting $\alpha=1$ in Theorem 3.2, we get*

$$\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\int_a^b\varphi(x)dx\right)^{\frac{1}{a-b}}\right|$$
$$\leqslant\left[\left(\varphi^{*}(a)\,\varphi^{*}(b)\right)^{\frac{1}{(s+1)(s+2)}}\varphi^{*}\left(\tfrac{a+b}{2}\right)^{\left(\frac{4(s+2)+30}{15(s+1)(s+2)}\right)}\right.$$
$$\left.\times\left(\varphi^{*}\left(\tfrac{3a+b}{4}\right)\varphi^{*}\left(\tfrac{a+3b}{4}\right)\right)^{\left(\frac{32(s+1)+2}{15(s+1)(s+2)}\right)}\right]^{\frac{b-a}{16}}.$$

Corollary 3.5. *In Theorem 3.2, if we take $\alpha=s=1$, then we get*

$$\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\int_a^b\varphi(x)dx\right)^{\frac{1}{a-b}}\right|$$
$$\leqslant\left[\left(\varphi^{*}(a)\,\varphi^{*}(b)\right)^{\frac{1}{6}}\varphi^{*}\left(\tfrac{a+b}{2}\right)^{\frac{7}{15}}\left(\varphi^{*}\left(\tfrac{3a+b}{4}\right)\varphi^{*}\left(\tfrac{a+3b}{4}\right)\right)^{\left(\frac{11}{15}\right)}\right]^{\frac{b-a}{16}}.$$

Theorem 3.6. *Let $\varphi : [a,b] \to \mathbb{R}^{+}$ be an increasing and multiplicatively differentiable over* $[a,b]$*. If for* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*,* $(\ln\varphi^{*})^{q}$ *is* s*-convex on* $[a,b]$*, then we have*

$$\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\mathcal{J}(a,b;\varphi)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|$$
$$\leqslant\left[\left(\varphi^{*}(a)\,\varphi^{*}\left(\tfrac{3a+b}{4}\right)\varphi^{*}\left(\tfrac{a+3b}{4}\right)\varphi^{*}(b)\right)^{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}\right.$$
$$\left.\times\left(\varphi^{*}\left(\tfrac{3a+b}{4}\right)\left(\varphi^{*}\left(\tfrac{a+b}{2}\right)\right)^{2}\varphi^{*}\left(\tfrac{a+3b}{4}\right)\right)^{\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{17}{15}\right)\left({}_{2}F_{1}\left(-p,1,\frac{\alpha+1}{\alpha};\frac{15}{17}\right)\right)^{\frac{1}{p}}}\right]^{\frac{b-a}{16}},$$

where $\mathcal{J}$ *is defined as* (3.1)*, and* ${}_{2}F_{1}$ *is the hypergeometric function.*

Proof. From Lemma 3.1, Hölder inequality and s-convexity of $(\ln\varphi^{*})^{q}$, we have:

$$\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\mathcal{J}(a,b;\varphi)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|$$
$$\leqslant\exp\left(\frac{b-a}{16}\left(\int_{0}^{1}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\ln\left(\varphi^{*}\left((1-t)a+t\tfrac{3a+b}{4}\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{b-a}{16}\left(\int_{0}^{1}\left((1-t)^{\alpha}+\tfrac{2}{15}\right)^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\ln\varphi^{*}\left((1-t)\tfrac{3a+b}{4}+t\tfrac{a+b}{2}\right)\right|^{q}dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{b-a}{16}\left(\int_{0}^{1}\left(t^{\alpha}+\tfrac{2}{15}\right)^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\ln\varphi^{*}\left((1-t)\tfrac{a+b}{2}+t\tfrac{a+3b}{4}\right)\right|^{q}dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{b-a}{16}\left(\int_{0}^{1}(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\ln\varphi^{*}\left((1-t)\tfrac{a+3b}{4}+tb\right)\right|^{q}dt\right)^{\frac{1}{q}}\right)$$
$$\leqslant\exp\left(\frac{b-a}{16}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\int_{0}^{1}\left((1-t)^{s}\left(\ln\left(\varphi^{*}(a)\right)\right)^{q}+t^{s}\left(\ln\left(\varphi^{*}\left(\tfrac{3a+b}{4}\right)\right)\right)^{q}\right)dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{17(b-a)}{240}\left({}_{2}F_{1}\left(-p,1,\frac{\alpha+1}{\alpha};\frac{15}{17}\right)\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\int_{0}^{1}\left((1-t)^{s}\left(\ln\varphi^{*}\left(\tfrac{3a+b}{4}\right)\right)^{q}+t^{s}\left(\ln\varphi^{*}\left(\tfrac{a+b}{2}\right)\right)^{q}\right)dt\right)^{\frac{1}{q}}\right)$$

$$\times\exp\left(\tfrac{17(b-a)}{240}\left({}_2F_1\left(-p,1,\tfrac{\alpha+1}{\alpha};\tfrac{15}{17}\right)\right)^{\frac{1}{p}}\right.$$
$$\times\left.\left(\int_0^1\left((1-t)^s\left(\ln\varphi^*\left(\tfrac{a+b}{2}\right)\right)^q+t^s\left(\ln\varphi^*\left(\tfrac{a+3b}{4}\right)\right)^q\right)dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\tfrac{b-a}{16}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left(\ln\varphi^*\left(\tfrac{a+3b}{4}\right)\right)^q+t^s\left(\ln\varphi^*(b)\right)^q\right)dt\right)^{\frac{1}{q}}\right)$$
$$=\exp\left(\tfrac{b-a}{16}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\tfrac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln\left(\varphi^*(a)\right)\right)^q+\left(\ln\left(\varphi^*\left(\tfrac{3a+b}{4}\right)\right)\right)^q\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\tfrac{17(b-a)}{240}\left({}_2F_1\left(-p,1,\tfrac{\alpha+1}{\alpha};\tfrac{15}{17}\right)\right)^{\frac{1}{p}}\left(\tfrac{1}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\times\left.\left(\left(\ln\varphi^*\left(\tfrac{3a+b}{4}\right)\right)^q+\left(\ln\varphi^*\left(\tfrac{a+b}{2}\right)\right)^q\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\tfrac{17(b-a)}{240}\left({}_2F_1\left(-p,1,\tfrac{\alpha+1}{\alpha};\tfrac{15}{17}\right)\right)^{\frac{1}{p}}\left(\tfrac{1}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\times\left.\left(\left(\ln\varphi^*\left(\tfrac{a+b}{2}\right)\right)^q+\left(\ln\varphi^*\left(\tfrac{a+3b}{4}\right)\right)^q\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\tfrac{b-a}{16}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\tfrac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln\varphi^*\left(\tfrac{a+3b}{4}\right)\right)^q+\left(\ln\varphi^*(b)\right)^q\right)^{\frac{1}{q}}\right), \tag{3.10}$$

where we have used

$$\int_0^1\left((1-t)^\alpha+\tfrac{2}{15}\right)^p dt=\int_0^1\left(t^\alpha+\tfrac{2}{15}\right)^p dt=\frac{1}{\alpha}\int_0^1\left(u+\tfrac{2}{15}\right)^p u^{\frac{1}{\alpha}-1}du$$
$$=\frac{1}{\alpha}\int_0^1\left(\tfrac{17}{15}-x\right)^p(1-x)^{\frac{1}{\alpha}-1}dx$$
$$=\left(\tfrac{17}{15}\right)^p\int_0^1\left(1-\frac{15}{17}x\right)^p(1-x)^{\frac{1}{\alpha}-1}dx$$
$$=\left(\tfrac{17}{15}\right)^p{}_2F_1\left(-p,1,\frac{1}{\alpha}+1;\frac{15}{17}\right).$$

Using the fact that $\mathcal{A}^q+\mathcal{B}^q\leqslant(\mathcal{A}+\mathcal{B})^q$ for $\mathcal{A}\geqslant 0$, $\mathcal{B}\geqslant 0$ with $q\geqslant 1$, (3.10) gives

$$\left|\left[\left(\varphi\left(\tfrac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\tfrac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\tfrac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\mathcal{J}(a,b;\varphi)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}\right|$$
$$\leqslant\exp\left(\tfrac{b-a}{16}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\tfrac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln\varphi^*(a)+\ln\varphi^*\left(\tfrac{3a+b}{4}\right)\right)\right)$$

$$\times \exp\left(\frac{17(b-a)}{240}\left({}_2F_1\left(-p,1,\frac{\alpha+1}{\alpha};\frac{15}{17}\right)\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln\varphi^*\left(\frac{3a+b}{4}\right)+\ln\varphi^*\left(\frac{a+b}{2}\right)\right)\right)$$
$$\times \exp\left(\frac{17(b-a)}{240}\left({}_2F_1\left(-p,1,\frac{\alpha+1}{\alpha};\frac{15}{17}\right)\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln\varphi^*\left(\frac{a+b}{2}\right)+\ln\varphi^*\left(\frac{a+3b}{4}\right)\right)\right)$$
$$\times \exp\left(\frac{b-a}{16}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln\varphi^*\left(\frac{a+3b}{4}\right)+\ln\varphi^*(b)\right)\right)$$
$$=\left[\left(\varphi^*(a)\,\varphi^*\left(\frac{3a+b}{4}\right)\varphi^*\left(\frac{a+3b}{4}\right)\varphi^*(b)\right)^{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}\right.$$
$$\left.\times\left(\varphi^*\left(\frac{3a+b}{4}\right)\left(\varphi^*\left(\frac{a+b}{2}\right)\right)^2\varphi^*\left(\frac{a+3b}{4}\right)\right)^{\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{17}{15}\right)\left({}_2F_1\left(-p,1,\frac{\alpha+1}{\alpha};\frac{15}{17}\right)\right)^{\frac{1}{p}}}\right]^{\frac{b-a}{16}}.$$

This completes the proof. □

Corollary 3.7. *In Theorem 3.6, if we take $s=1$, then we get*

$$\left|\left[\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\frac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\frac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\mathcal{J}(a,b;\varphi)\right)^{-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|$$
$$\leqslant\left[\left(\varphi^*(a)\,\varphi^*\left(\frac{3a+b}{4}\right)\varphi^*\left(\frac{a+3b}{4}\right)\varphi^*(b)\right)^{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}}\right.$$
$$\left.\times\left(\varphi^*\left(\frac{3a+b}{4}\right)\left(\varphi^*\left(\frac{a+b}{2}\right)\right)^2\varphi^*\left(\frac{a+3b}{4}\right)\right)^{\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{17}{15}\right)\left({}_2F_1\left(-p,1,\frac{\alpha+1}{\alpha};\frac{15}{17}\right)\right)^{\frac{1}{p}}}\right]^{\frac{b-a}{16}},$$

where $\mathcal{J}$ is defined as (3.1).

Corollary 3.8. *If we take $\alpha=1$ in Theorem 3.6, we get*

$$\left|\left[\left(\varphi\left(\frac{3a+b}{4}\right)\right)^{\frac{8}{15}}\left(\varphi\left(\frac{a+b}{2}\right)\right)^{\frac{-1}{15}}\left(\varphi\left(\frac{a+3b}{4}\right)\right)^{\frac{8}{15}}\right]\left(\int_a^b(\varphi(x))^{dx}\right)^{\frac{1}{a-b}}\right|$$
$$\leqslant\left[\left(\varphi^*(a)\,\varphi^*\left(\frac{3a+b}{4}\right)\varphi^*\left(\frac{a+3b}{4}\right)\varphi^*(b)\right)^{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}\right.$$
$$\left.\times\left(\varphi^*\left(\frac{3a+b}{4}\right)\left(\varphi^*\left(\frac{a+b}{2}\right)\right)^2\varphi^*\left(\frac{a+3b}{4}\right)\right)^{\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}(p+1)}\right)^{\frac{1}{p}}}\right]^{\frac{b-a}{16}}.$$

Corollary 3.9. *By setting $\alpha=s=1$ in Theorem 3.6, we get*

$$\left|\left(\varphi(x)\,\varphi(a+b-x)\right)^{\frac{2(x-a)+\rho(a+b-2x)}{2(b-a)}}\left(\varphi\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\rho)(a+b-2x)}{b-a}}\left(\int_a^b(\varphi(x))^{dx}\right)^{\frac{1}{a-b}}\right|$$

$$\leqslant \left[\left(\varphi^*(a)\, \varphi^*\left(\tfrac{3a+b}{4}\right) \varphi^*\left(\tfrac{a+3b}{4}\right) \varphi^*(b) \right)^{\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}}} \right.$$

$$\left. \times \left(\varphi^*\left(\tfrac{3a+b}{4}\right) \left(\varphi^*\left(\tfrac{a+b}{2}\right)\right)^2 \varphi^*\left(\tfrac{a+3b}{4}\right) \right)^{\left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}(p+1)}\right)^{\frac{1}{p}}} \right]^{\frac{b-a}{16}}.$$

4. Applications to special means

Let us consider the following means for arbitrary real numbers a, b
The Arithmetic mean: $A(a_1, a_2, ..., a_n) = \frac{a_1+a_2+...+a_n}{n}$.
The Harmonic mean: $H(a_1, a_2, ..., a_n) = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}$
The logarithmic means: $L(a, b) = \frac{b-a}{\ln b - \ln a}$, $a, b > 0$ and $a \neq b$.
The p-Logarithmic mean: $L_p(a, b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 4.1. *Let* $a, b \in \mathbb{R}$ *with* $0 < a < b$, *then we have*

$$e^{\frac{A^8(a,a,a,b)+A^{-1}(a,b)+A^8(a,b,b,b)-15L_2^2(a,b)}{15}} \leqslant e^{\frac{34(a+b)}{15}}.$$

Proof. The assertion follows from Corollary 3.5 applied to the function $\varphi(t) = e^{t^2}$ whose $\varphi^*(t) = e^{2t}$ and $\left(\int_a^b \varphi(u)^{du}\right)^{\frac{1}{a-b}} = \exp\left\{-L_2^2(a, b)\right\}$. □

Proposition 4.2. *Let* $a, b \in \mathbb{R}$ *with* $0 < a < b$ *and* $n > 0$, *then we have*

$$H^{-\frac{8}{15}}(a, b, b, b)\, H^{-\frac{1}{15}}(a, b)\, H^{-15}(a, a, a, b)\, a^{\frac{nb}{a-b}} b^{-\frac{na}{a-b}} e^{-n} \leqslant e^{-\left(\frac{5(a+b)^2+28ab}{2(a+b)} + \frac{176ab(a+b)}{(3a+b)(a+3b)}\right)\frac{n}{15}}.$$

Proof. The assertion follows from Corollary 3.5 on the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$applied to the function $\varphi(t) = \frac{1}{t^n}$ whose $\varphi^*(t) = e^{-\frac{n}{t}}$ and $\left(\int_{\frac{1}{b}}^{\frac{1}{a}} \varphi(u)^{du}\right)^{\frac{ab}{a-b}} = a^{\frac{nb}{a-b}} b^{-\frac{na}{a-b}} e^{-n}$. □

5. Conclusion

This paper explores inequalities of the corrected dual Simpson-type for fractional multiplicative integrals using multiplicative s-convexity derived from a novel generalized identity. The findings hold the potential to inspire further investigations in this intriguing domain and extensions to other forms of convexity.

Conflict of interest The authors declare that they do not have any conflicts of interest.

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