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Application of Elzaki's Method on Fractional Differential Equations

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Abstract

This paper studies a new version of integral transform so-called Elzaki's transform and some of its properties. The transform is applied to obtain a solution of a class of nonlinear fractional differential equations. Some examples are presented to illustrate the method.

Keywords: Elzaki's transform, Fractional calculus.

1. Introduction

Fractional calculus generalizes the theory of derivatives of integer orders to non-integer orders. Fractional derivatives are non-local and capture the history of dynamics. Hence, it is an important tool in the study of dynamical systems. It is very useful in dealing with properties of memory effects in many problems of sciences and technology. For more, see [1, 2, 3, 4, 5, 6]. As a result of its vast areas of applications, a lot of authors carried out fractional differential equations. In a bid to solve the equations, a lot of transforms such as Sumudu [7], Shehu [8], Laplace [9], Fourier transforms [10] etc were used. For more transforms of fractional differential equations, see ([14],[15],[11],[18],[16],[17],[19]) and the references therein. Recently, Triag Elzaki [13] developed a new integral transform called the Elzaki transform in solving ordinary and partial differential equations. In this paper, we are interested in applying the Elzaki transform method to the class of nonlinear fractional differential equations given by

$$ {}^{c}D^{\alpha}_{0_+} x(t) + k\, {}^{c}D^{\beta}_{0_+} x(t) + g(t, x(t)) = h(t),\ t \in [0, 1], \tag{1.1} $$

$$ x(0) = 0, \tag{1.2} $$

$$ x'(0) = \frac{\beta}{\alpha}, \tag{1.3} $$

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where $0 < \beta < 1 < \alpha < 2$, are real constants, $x \in C^2[0,1]$ and ${}^cD^{\alpha}_{0_+}x(t)$ is the Caputo fractional derivative of a function x of order α, k is a negative constant, $g : [0,1] \times [0,\infty)$ is an $L_\infty-$ Caratheódory function, $h \in L^{\frac{1}{\beta}}[0,1]$.

2. Definition of terms

Let,

$$A = \left\{ f(t) : \exists M, K_1, K_2 > 0 :\mid f(t) \mid < Me^{\frac{|t|}{K_j}}, t \in (-i)^j \times [0,\infty) \right\} \tag{2.1}$$

be a class of functions.

Definition 2.1.[13] The Elzaki transform of a function $f \in A$ denoted by $E[f(t)] = T(u)$ is defined as

$$T(u) = u\int_0^\infty f(t)e^{\frac{-t}{u}}dt \tag{2.2}$$

where $f(t)$ is denoted by

$$E[f(t)] = T(u)$$

defined in [1,2] as

$$T(u) = u^2\int_0^\infty f(ut)e^{-t}dt \; K_1, K_2 > 0. \tag{2.3}$$

Definition 2.2.[8] Let x be $n-th$ times continuously differentiable function. The Caputo fractional derivative of a function x, of order α, with lower limit 0 is defined as,

$${}^cD^{\alpha}_{0_+}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds, \tag{2.4}$$

with $n-1 < \alpha < n,\ n = [\alpha]+1$.

Definition 2.3. The Riemann-Liouville fractional integral of a function x, of order $\alpha > 0$, and denoted by $I^{\alpha}_{0_+}x(t)$ is defined by,

$$I^{\alpha}_{0_+}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds. \tag{2.5}$$

3. Manuscript Content

From the definition of terms in the previous section, we deduce from equations (2.4) and (2.5) that,

$${}^cD^{\alpha}_{0_+}x(t) = I^{n-\alpha}_{0_+}x^{(n)}(t). \tag{3.1}$$

Also, we can see from equation (2.3) that if $f(t) = 1$ then,

$$E[1] = T(u) = u\int_0^\infty e^{\frac{-t}{u}}dt$$

$$= -u \times \frac{u}{1} e^{\frac{-t}{u}} |_0^\infty$$
$$= u^2.$$

When $f(t) = t$ then,

$$E(t) = T(u) = u \int_0^\infty t e^{\frac{-t}{u}} dt$$
$$= u \left[t e^{\frac{-t}{u}} \times -u \right]_0^\infty + u^2 \int_0^\infty e^{\frac{-t}{u}} dt$$
$$= u^3.$$

Generally, for $n \geqslant 1$,

$$E[t^n] = n! u^{n+2}$$

Theorem 2.4.[19] Let f be an $n-$ continuously differentiable function on $[0, \infty)$, then

(i)

$$E[f'(t)] = \frac{T(u)}{u} - uf(0)$$

(ii)

$$E[f^{(n)}(t)] = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^k(0) \; , n \geqslant 1$$

Theorem 2.5 The Elzaki Transform of the Riemann-Liouville fractional integral of a function f, of order $\alpha > 0$, and denoted by $E(I_{0_+}^\alpha f(t))$ is given as

$$E(I_{0_+}^\alpha f(t)) = u^\alpha E(f(t)). \qquad (3.2)$$

Proof: By definition,

$$E[I_{0_+}^\alpha x(t)] = u \int_0^\infty I_{0_+}^\alpha x(t) e^{-\frac{t}{u}} dt$$
$$= u \int_0^\infty \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds \; e^{-\frac{t}{u}} dt$$
$$= \frac{u}{\Gamma(\alpha)} \int_0^\infty x(s) ds \int_s^\infty (t-s)^{\alpha-1} e^{-\frac{t}{u}} dt.$$

By setting $t - s = p$, then we have,

$$E[I_{0_+}^\alpha x(t)] = \frac{u}{\Gamma(\alpha)} \int_0^\infty x(s) ds \int_0^\infty p^{\alpha-1} e^{-\frac{(p+s)}{u}} dp$$
$$= \frac{u}{\Gamma(\alpha)} \int_0^\infty x(s) e^{-\frac{s}{u}} ds \int_0^\infty p^{\alpha-1} e^{-\frac{p}{u}} dp$$

Again, on setting $\frac{p}{u} = w$ on the preceding, we obtain,

$$E[I_{0_+}^{\alpha} x(t)] = \frac{u}{\Gamma(\alpha)} \int_0^{\infty} x(s) e^{-\frac{s}{u}} ds \int_0^{\infty} w^{\alpha-1} u^{\alpha} e^{-w} dw$$
$$= u^{\alpha+1} \int_0^{\infty} x(s) e^{-\frac{s}{v}} ds$$
$$= u^{\alpha} E(x(t)).$$

Theorem 2.6. The Elzaki Transform of the Caputo fractional derivative of a function f, of order $n-1 < \alpha < n$ is given as

$$E[{}^cD_{0_+}^{\alpha} x(t)] = u^{-\alpha} E[f(t)] - \sum_{k=0}^{n-1} u^{2-\alpha+k} f^{(k)}(0) \tag{3.3}$$

Proof: By making use of equation (3.1) and making use of Theorem 2.5, we have,

$$E[{}^cD_{0_+}^{\alpha} f(t)] = E[I_{0_+}^{n-\alpha} f^{(n)}(t)]$$
$$= u^{n-\alpha} E[f^{(n)}(t)]$$

By Theorem (2.4ii), we have that

$$E[{}^cD_{0_+}^{\alpha} f(t)] = u^{n-\alpha} \left(\frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^k(0) \right) ,n \geqslant 1$$
$$= u^{-\alpha} \left(T(u) - \sum_{k=0}^{n-1} u^{2+k} f^{(k)}(0) \right)$$
$$= u^{-\alpha} \left(E[f(t)] - \sum_{k=0}^{n-1} u^{2+k} f^{(k)}(0) \right)$$
$$= u^{-\alpha} E[f(t)] - \sum_{k=0}^{n-1} u^{2-\alpha+k} f^{(k)}(0).$$

Thus,

$$E[{}^cD_{0_+}^{\alpha} x(t)] = u^{-\alpha} E[f(t)] - \sum_{k=0}^{n-1} u^{2-\alpha+k} f^{(k)}(0)$$

Corollary 2.7 For $n = 2$,, The Elzaki transform of the caputo fractional derivative of a function f, of order $1 < \alpha < 2$ is given as

$$E[{}^cD^{\alpha}_{0_+}x(t)] = u^{-\alpha}E[f(t)] - u^{2-\alpha}f(0) - u^{1-\alpha}f'(0) \tag{3.4}$$

Proof:

$$\begin{aligned} E[{}^cD^{\alpha}_{0_+}f(t)] &= E[I^{2-\alpha}_{0_+}x''(t)] \\ &= u^{2-\alpha}E(f''(t)) \\ &= u^{2-\alpha}\left(\frac{E[f(t)]}{u^2} - x(0) - uf'(0)\right) \\ & u^{-\alpha}E[f(t)] - u^{2-\alpha}f(0) - u^{1-\alpha}f'(0) \end{aligned}$$

Therefore,

$$E[{}^cD^{\alpha}_{0_+}x(t)] = u^{-\alpha}E[f(t)] - u^{2-\alpha}f(0) - u^{1-\alpha}f'(0) \tag{3.5}$$

Equally,

$$\begin{aligned} E[{}^cD^{\beta}_{0_+}x(t)] &= E[I^{1-\beta}_{0_+}x'(t)] \\ &= u^{1-\beta}E[x'(t)] \\ &= u^{1-\beta}\left[\frac{E[x(t)]}{u} - ux(0)\right] \end{aligned}$$

Thus,

$$E[{}^cD^{\beta}_{0_+}x(t)] = u^{1-\beta}\left[\frac{E[x(t)]}{u} - ux(0)\right] \tag{3.6}$$

Applying Elzaki transform to equations (1.1), we have

$$E\left[{}^cD^{\alpha}_{0_+}x(t) + k\,{}^cD^{\beta}_{0_+}x(t) + g(t, x(t)) = h(t)\right], \ t \in [0, 1]$$

Making use of equations (1.2) and (1.3), we have

$$u^{2-\alpha}\left(\frac{E[x(t)]}{u^2} - x(0) - ux'(0)\right) + k(u^{1-\beta}\left[\frac{E[x(t)]}{u} - ux(0)\right]) = E[h(t) - g(t, x(t))]$$

$$\Rightarrow E[x(t)]\left[u^{-\alpha} + ku^{-\beta}\right] = E[h(t) - g(t, x(t))] + \frac{\beta}{\alpha}u^{3-\alpha}$$

$$\Rightarrow E[x(t)] = \frac{E[h(t) - g(t, x(t))]}{u^{-\alpha} + ku^{-\beta}} + \frac{\beta u^{3-\alpha}}{\alpha(u^{-\alpha} + ku^{-\beta})}$$

Taking the inverse Elzaki transform of the the above, we have

$$x(t) = E^{-1}\left[\frac{E[h(t) - g(t, x(t))]}{u^{-\alpha} + ku^{-\beta}} + \frac{\beta u^{3-\alpha}}{\alpha(u^{-\alpha} + ku^{-\beta})}\right] \tag{3.7}$$

4. Some Examples

Example 1

$$ {}^{c}D_{0_+}^{\frac{3}{2}}x(t) + 0.8\,{}^{c}D_{0_+}^{\frac{1}{2}}x(t) + \sin^3 x(t) = t^2,\ t \in [0,1], \tag{4.1}$$

$$ x(0) = 0, \tag{4.2}$$

$$ x'(0) = 3 \tag{4.3}$$

Proof: Taking the Elzaki transform of equation (4.1), we have

$$E[{}^{c}D_{0_+}^{\frac{3}{2}}x(t) + 0.8\,{}^{c}D_{0_+}^{\frac{1}{2}}x(t) + \sin^3 x(t) = t^2]$$

From equation (4.2) and (4.3), we have that

$$u^{\frac{1}{2}}\left(\frac{E[x(t)]}{u^2} - x(0) - ux'(0)\right) + 0.8u^{\frac{1}{2}}\left[\frac{E[x(t)]}{u} - ux(0)\right] = E[t^2 - \sin x^3(t)]$$

$$\implies u^{\frac{-3}{2}}E[x(t)] - 3u^{\frac{3}{2}} + 0.8u^{\frac{-1}{2}}E[x(t)] = E[t^2 - \sin x^3(t)]$$

$$\implies E[x(t)]\left(u^{\frac{-3}{2}} + 0.8u^{\frac{-1}{2}}\right) = E[t^2 - \sin x^3(t)] + 3u^{\frac{3}{2}}$$

$$\implies E[x(t)] = \frac{E[t^2 - \sin x^3(t)] + 3u^{\frac{3}{2}}}{u^{\frac{-3}{2}} + 0.8u^{\frac{-1}{2}}}.$$

Taking the inverse Elzaki transform of the preceding equation , we obtain

$$x(t) = E^{-1}\left(\frac{E[t^2 - \sin x^3(t)] + 3u^{\frac{3}{2}}}{u^{\frac{-3}{2}} + 0.8u^{\frac{-1}{2}}}\right)$$

Example 2 Using the Elzaki transform, obtain the solution of the following equations

$$ {}^{c}D_{0_+}^{\frac{3\pi}{5}}x(t) + \gamma\,{}^{c}D_{0_+}^{\frac{e}{\pi}}x(t) + x^2(t) = t,\ t \in [0,1], \tag{4.4}$$

$$ x(0) = 0, \tag{4.5}$$

$$ x'(0) = \frac{5e}{3} \tag{4.6}$$

where γ is a nonzero real number

Proof As usual, we take the Elzaki transform of the equation (4.4). That is,

$$E\left({}^{c}D_{0_+}^{\frac{3\pi}{5}}x(t) + \gamma\,{}^{c}D_{0_+}^{\frac{e}{\pi}}x(t) + x^2(t) = t\right)$$

$$\implies u^{\frac{10-3\pi}{5}}\left(\frac{E[x(t)]}{u^2} - x(0) - ux'(0)\right) + \gamma u^{\frac{\pi-e}{\pi}}\left(\frac{E[x(t)}{u} - ux(0)\right) = E[t - x^2(t)]$$

$$\implies u^{\frac{10-3\pi}{5}}\left(\frac{E[x(t)]}{u^2} - u\frac{5e}{3}\right) + \gamma u^{\frac{\pi-e}{\pi}}\left(\frac{E[x(t)}{u}\right) = E[t - x^2(t)]$$

$$\Longrightarrow u^{\frac{-3\pi}{5}}E[x(t)] - \frac{5}{3e}u^{\frac{5-3\pi}{5}} + \gamma u^{1-e-\frac{e}{\pi}}E[x(t)] = E[t - x^2(t)]$$

$$\Longrightarrow E[x(t)]\left(u^{\frac{-3\pi}{5}} + \gamma u^{1-e-\frac{e}{\pi}}\right) = \frac{5}{3e}u^{\frac{5-3\pi}{5}} + E[t - x^2(t)]$$

$$\Longrightarrow E[x(t)] = \frac{5}{3e}\left(\frac{u^{\frac{5-3\pi}{5}}}{u^{\frac{-3\pi}{5}} + \gamma u^{1-e-\frac{e}{\pi}}}\right) + \frac{E[t - x^2(t)]}{u^{\frac{-3\pi}{5}} + \gamma u^{1-e-\frac{e}{\pi}}}$$

Taking the inverse transform of the preceding, we obtain

$$x(t) = E^{-1}\left(\frac{5}{3e}\left(\frac{u^{\frac{5-3\pi}{5}}}{u^{\frac{-3\pi}{5}} + \gamma u^{1-e-\frac{e}{\pi}}}\right) + \frac{E[t - x^2(t)]}{u^{\frac{-3\pi}{5}} + \gamma u^{1-e-\frac{e}{\pi}}}\right)$$

5. Conclusion

We applied a new integral transform called Elzaki transform on a class of fractional differential equations and also studied some of its properties. Some examples were used to demonstrate the application.

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