Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2023)4(2) : 1-13
doi:10.48185/jfcns.v4i2.861

On Dirichlet Problem of Time-Fractional Advection-Diffusion Equation

Changdev P. Jadhav [a], Tanisha B. Dale[b], Vaijanath L. Chinchane [c,*]

[a,c] Department of Mathematics, Deogiri Institute of Engineering and Management Studies, Chhatrapati Sambhaji Nagar-431005, INDIA
[b] Department of Mathematics, Rajashri Shahu Science, Commerce and Arts Mahavidyalaya, Pathri

• *Received: 10 October 2023* • *Accepted: 04 December 2023* • *Published Online:27 December 2023*

Abstract

The significant motivation behind this research article is to utilize a technique depending upon a certain variant of the integral transform (Fourier and Laplace) to investigate the basic solution for the Dirichlet problem with constant boundary conditions. The time-fractional derivative of one-dimensional, the equation of advection-diffusion, and the Liouville-Caputo fractional derivative in a line fragment are presented. We also illustrate the results using graphical representations.

Keywords: Laplace transform and its variant, non-Fickian diffusion, Liouville-Caputo derivative, fractional advection-diffusion equation.
2010 MSC: 26A33, 43K05, 39B62.

1. Introduction

Fractional calculus (FC) is becoming one of the most popular topics in applied mathematics due to its expanding applications in various scientific and technical domains. Because fractional-order derivatives differ from classical derivatives in that they have non-local quality, they are an excellent choice for capturing memory and hereditary traits in various real-world events.

During last decade, Many Mathematicians have studied numerical methods for different types of fractional partial differential equations involving time and/or space derivative such as time-fractional diffusion equations, space-time fractional diffusion equations, Cauchy reaction diffusion equations, Time Fractional Convection-Diffusion Equations, etc., see [2, 3, 10, 11]. Kamran and et al. provided a method for the numerical simulation of

*Corresponding author: chinchane85@gmail.com

time-fractional diffusion equations. Also, they proposed scheme combines the local meshless method based on a radial basis function (RBF) with Laplace transform, see [10] Two- and three-dimensional numerical solutions for the heat transfer problems in absorptive media with a two-condition model are presented, and thermal dispersion tensors are investigated [12]. In [15], the authoritative equation for flux matter is given below:

$$\mathbf{j} = -a \operatorname{grad} q + vq. \tag{1.1}$$

The diffusivity coefficient is addressed by a and v represents the speed vector. Taking into account the equation of balance, particularly for mass outcomes in the equation for normal advection-diffusion (by using the condition v = constant), we get

$$\frac{\partial q}{\partial t} = a\triangle q - v \cdot \operatorname{grad} q. \tag{1.2}$$

As far as dissemination or heat conduction, including an extra velocity field, is concerned, (1.2) can be deciphered as a transport phenomenon in permeable media, Brownian movement, or groundwater geology [14]. The advection-diffusion equation (1.2) has the following form for a single spatial coordinate x:

$$\frac{\partial q}{\partial t} = a\frac{\partial^2 q}{\partial x^2} - v\frac{\partial q}{\partial x}. \tag{1.3}$$

The study of fractional-order derivatives, useful in physics, geophysics, geology, viscoelasticity, engineering, and biotechnology, has recently attracted greater attention. See [16, 17, 29] for more information. Here, we recall some fractional derivative definitions used in our present study. It is tacitly assumed that all of the presented quantities exist in the mathematical sense. The readers are advised to refer to the associated literature for the exact conditions of the transformed functions.

2. Preliminaries

Here, we give some definitions and lemmas that will be used in this paper.

Definition 2.1 (see [4, 21]). The fractional derivative of Riemann Liouville type for a function $q(t)$ of order θ is as follow

$$D^{\theta}_{RL}\{q(t)\} = \frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\theta)}\int_0^t (t-\tau)^{n-\theta-1} q(\tau)d\tau\right), \quad (n-1<\theta<n), \tag{2.1}$$

where n denotes a positive integer. Henceforth, we denote by $\Gamma(\theta)$ the familiar (Euler's) gamma function of argument θ.

Definition 2.2 (see [21, 23, 24]). The fractional derivative of Liouville-Caputo type for a function $q(t)$ of order θ is given by

$$\frac{d^{\theta}q(t)}{dt^{\theta}} = \frac{1}{\Gamma(n-\theta)}\int_0^t (t-\tau)^{n-\theta-1}\frac{d^{\theta}q(\tau)}{d\tau^{\theta}}d\tau, \quad (n-1<\theta<n), \tag{2.2}$$

where n denotes a positive integer.

Definition 2.3 (see [1, 22]). Finite sin Fourier transform of a function $u(x)$, defined on $[0, L]$, considering spatial co-ordinate x, is given by

$$\mathbb{F}\left[u(x)\right] = \bar{u}(p_n) = \int_0^L u(x)\sin(p_n x)dx, \tag{2.3}$$

with $p_n = \frac{n\pi}{L}$ and n being a positive integer. Moreover, the inverse finite-sin-Fourier transform of a function $u(x)$ is as follows:

$$u(x) = \frac{2}{L}\sum_{n=1}^{\infty}\bar{u}(p_n)\sin(p_n x). \tag{2.4}$$

Definition 2.4. [22] The second-order derivative finite-sin-Fourier transform of a function $u(x)$ defined on $[0, L]$ is computed by using the following formula:

$$\mathbb{F}\left[\frac{d^2u(x)}{dx^2}\right] = -p_n^2\bar{u}(p_n) + p_n[u(0) - (-1)^n u(L)]. \tag{2.5}$$

Definition 2.5 (see [6, 8, 23]). In the present-day literature, one of the many obvious parametric and argument variations of the classical Laplace transform:

$$\mathcal{L}\left\{f(t) : s\right\} := \int_0^\infty e^{-st}\, f(t)\ \ dt =: F_{\mathcal{L}}(s) \tag{2.6}$$

and it's s-multiplied version of the Laplace transform (or, in other words; the Laplace-Carson transform):

$$\mathcal{LC}\left\{f(t) : s\right\} := s\int_0^\infty e^{-st}\, f(t)\ \ dt =: F_{\mathcal{LC}}(s) \tag{2.7}$$

is the alleged "Sumudu transform" which, for a function $f(t)$, is characterized by

$$\begin{aligned}\mathcal{S}[f(t) : s] := \hat{f}(s) &= \int_0^\infty f(st)e^{-t}dt \\ &= \frac{1}{s}\,\mathcal{L}\left\{f(t) : \frac{1}{s}\right\} \\ &= \mathcal{LC}\left\{f(t) : \frac{1}{s}\right\}.\end{aligned}$$

Lemma 2.6 (see [6, 8]). *The above-defined variants of the Laplace transform, and the Laplace-Carson transform when applied to the fractional derivative of Liouville-Caputo type for a function $f(t)$ of order θ, yields*

$$\mathcal{S}\left[\frac{d^\theta f(t)}{dt^\theta} : s\right] = s^{-\theta}\left[\hat{f}(s) - \sum_{k=0}^{n-1} s^k\left[f^k(0)\right]\right], \quad (n-1 < \theta \leqq n), \tag{2.8}$$

where n is a positive integer.

Definition 2.7 (see [25]). Mittag-Leffler function:

$$E_{\theta}(z) := E_{\theta,\lambda}(z)\big|_{\lambda=1},$$

where

$$E_{\theta,\lambda}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\theta n + \lambda)}, \quad (\theta > 0;\ \lambda \in \mathbb{R}). \tag{2.9}$$

In [18], (1.1) was investigated. One may compare this to the works of expanded Fick or Fourier law, see [5, 17, 19, 20] and several recent developments reported in [9, 26, 27, 28]. The generalized constitutive equation for mass flux defined using long-tail power kernel is given by the citation [18] and is as follows:

$$j = \mathcal{D}_{\mathcal{RL}}^{\infty - \subseteq}[-a \operatorname{grad} q(t) + vq(t)]. \tag{2.10}$$

By combining the constitutive equation (2.10), mass balance equation and time-fractional advection-diffusion equation is created:

$$\frac{\partial^{\theta} q}{\partial t^{\theta}} = a \triangle q - v \cdot \operatorname{grad} q \tag{2.11}$$

along with the order $theta$ of the fractional derivative of Liouville-Caputo. An extensive survey of works on fractional advection-diffusion equation and the numerical methods used to solve it is given in the article cited in [13]. There are just a few studies in the literature that examine the analytical answers of fractional advection-diffusion equation, see [13]. The Dirichlet problem for (2.11) in a line segment $0 < x < L$ is examined in this study. The sought-after function is subject to constant boundary conditions, whereas the basic solution is subject to the Dirac delta boundary condition.

The following is how the paper has been set up. In Section 1, the definitions of integral transformations and fractional calculus core concepts are covered . In Section 3, we solve the advection-diffusion equation in a line segment for the desired function with zero conditions. We present a discussion concerning our findings in this article in Section 4.

3. Main Results

Here, we look at the equation for advection-diffusion on a line segment. The Dirichlet problem for the desired function is taken into consideration to find a solution.

A. Basic Solution of the Problem

The equation of time-fractional advection-diffusion in the line segment $0 < x < L$ is written as follows

$$\frac{\partial^{\theta} q}{\partial t^{\theta}} = a \frac{\partial^2 q}{\partial x^2} - v \frac{\partial q}{\partial x}, \quad (0 \leqq \theta \leqq 1). \tag{3.1}$$

The equation (3.1) is considered under zero initial conditions, as it is generally done, for any $c > 0, v > 0, 0 < t < \infty$, for

$$t = 0: \quad q(x,t) = 0 \tag{3.2}$$

and at the end of the segment with the following Dirichlet boundary conditions:

$$x=0: \quad q(x,t)=m_0\delta(t) \tag{3.3}$$

and

$$x=L: \quad q(x,t)=0, \tag{3.4}$$

where the Dirac delta function is represented by $\delta(t)$. To obtain the non-dimensional quantity, the constant multiplier m_0 is introduced. It is worth seeing that the new sought-for function is given by

$$q(x,t)=u(x,t)\ \exp\left(\frac{v\,x}{2a}\right) \tag{3.5}$$

minimizes the initial-boundary value problem under consideration to the subsequent equation:

$$\frac{\partial^\theta u}{\partial t^\theta}=a\frac{\partial^2 u}{\partial x^2}-\frac{v^2}{4a}u, \tag{3.6}$$

where

$$u(x,0)=0, \tag{3.7}$$

$$u(0,t)=m_0\delta(t), \tag{3.8}$$

and

$$u(L,t)=0. \tag{3.9}$$

By using (3.8), (3.9) and (2.5), we can solve the equation (3.6) by means of finite Sin-Fourier transform (3.6), so that

$$\frac{\partial^\theta \bar{u}(p_n,t)}{\partial t^\theta}=\left(-ap_n^2-\frac{v^2}{4a}\right)\bar{u}(p_n,t)+ap_n m_0\delta(t) \tag{3.10}$$

with

$$t=0: \quad \bar{u}(p_n,t)=0. \tag{3.11}$$

Under the initial condition (3.11), by applying the above-defined variant of the Laplace transform to (3.10), we get

$$\frac{\hat{u}(p_n,s)}{s^\theta}=\left(-ap_n^2-\frac{v^2}{4a}\right)\hat{u}(p_n,s)+\frac{ap_n m_0}{s}, \tag{3.12}$$

$$\hat{u}(p_n,s)=s^\theta\left(-ap_n^2-\frac{v^2}{4a}\right)\hat{u}(p_n,s)+ap_n m_0 s^{\theta-1}, \tag{3.13}$$

$$\hat{u}(p_n,s)-s^\theta\left(-ap_n^2-\frac{v^2}{4a}\right)\hat{u}(p_n,s)=ap_n m_0 s^{\theta-1}, \tag{3.14}$$

$$\hat{u}(p_n,s)\left[1-s^\theta\left(-ap_n^2-\frac{v^2}{4a}\right)\right]=ap_n m_0 s^{\theta-1}, \tag{3.15}$$

and

$$\hat{u}(p_n,s)=\frac{ap_n m_0 s^{\theta-1}}{\left[1-s^\theta\left(-ap_n^2-\frac{v^2}{4a}\right)\right]}. \tag{3.16}$$

Now, by using the inverse transform of (3.16), we can see that

$$u(x,t) = \frac{2am_0 t^{\theta-1}}{L} \sum_{n=1}^{\infty} p_n \sin(p_n x) E_{\theta,1}\left[-\left(ap_n^2 + \frac{v^2}{4a}\right)t^{\theta}\right]. \tag{3.17}$$

We eventually achieve the fundamental approach to the Dirichlet issue by turning to the subject of quantity $q(x,t)$ in accordance with (3.5):

$$q(x,t) = \frac{2am_0 t^{\theta-1}}{L} \exp\left(\frac{v\,x}{2a}\right) \sum_{n=1}^{\infty} p_n \sin(p_n x) E_{\theta,1}\left[-\left(ap_n^2 + \frac{v^2}{4a}\right)t^{\theta}\right]. \tag{3.18}$$

Thus, by using the non-dimensional quantities:

$$\bar{x} = \frac{x}{L}, \bar{p}_n = Lp_n = n\pi, \bar{v} = \frac{vL}{a}, k = \frac{\sqrt{a}t^{\frac{\theta}{2}}}{L}, \bar{q}(\bar{x},k) = \frac{L^2 q(\bar{x},k)}{am_0 t^{\theta-1}}, \tag{3.19}$$

we obtain

$$\bar{q}(\bar{x},k) = 2\exp\left(\frac{\bar{v}\bar{x}}{2}\right) \sum_{n=1}^{\infty} \bar{p}_n \sin(\bar{p}_n \bar{x}) E_{\theta}\left[-k^2\left(\bar{p}_n^2 + \frac{\bar{v}^2}{4}\right)\right], \tag{3.20}$$

which is the fundamental solution of (3.1).

The Mittag-Leffler functions $E_{\theta}(z)$ and $E_{\theta,\lambda}(z)$ are implemented in most of the mathematical software, allowing a computational analysis of the solution which we have derived here. One can use, for instance, the `library(MittagLefflerR)` of the R-software. As some illustrative examples, figures 1 shows how the fundamental solution (3.20), which depends on the fractional derivative's order $(\alpha = 0.2)$ and distance $\bar{x}$. Similarly, 2 and 3 depict the solution $\bar{q}(\bar{x},k)$ for varying values of $\bar{x}$ and θ and for fixed values for $(\bar{v},k)$, with

$$(\bar{v},k) = (15, 0.1), \quad (5,1) \quad \text{and} \quad (1,5),$$

respectively.

B. A function's constant boundary value

Currently, the Dirichlet boundary condition, zero initial condition, time-fractional mode of (1.1) and the constant boundary-values of sought-after function, that is

$$q(0,t) = q_0, \tag{3.21}$$

and

$$q(L,t) = 0. \tag{3.22}$$

As explained previously, the new function u is included in (3.5), and the result in the transform domain is given by finite Sin-Fourier transform concerning spatial coordinate x and variant of the Laplace transform about time t. Therefore, we have

$$\frac{\partial^{\theta}\bar{u}(p_n,t)}{\partial t^{\theta}} = \left(-ap_n^2 - \frac{v^2}{4a}\right)\bar{u}(p_n,t) + ap_n p_0, \tag{3.23}$$

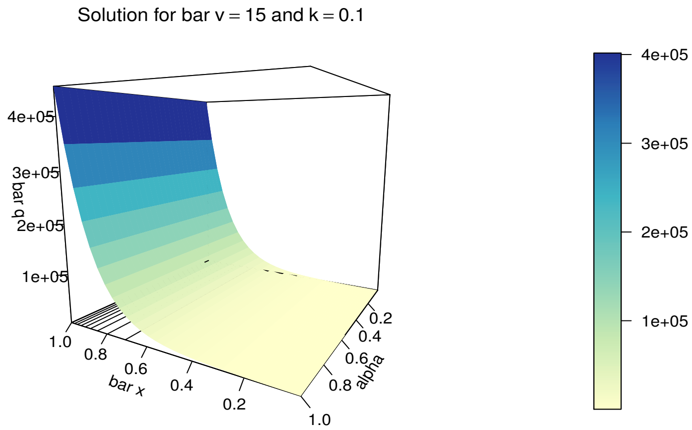

Figure 1: Plot of the solution $\bar{q}(\bar{x}, k)$ for $(\bar{v}, k) = (15, 0.1)$.

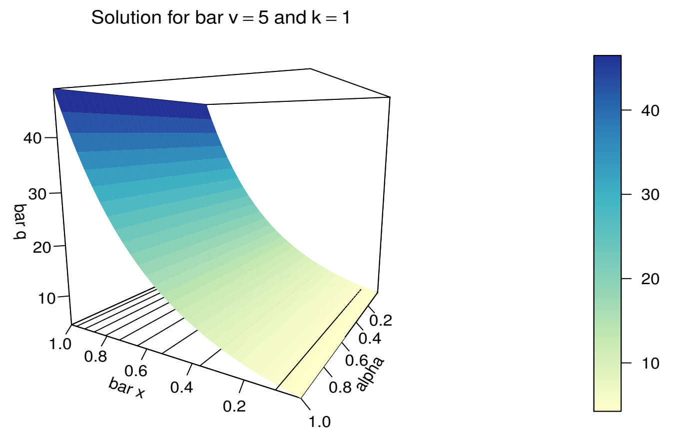

Figure 2: Plot of the solution $\bar{q}(\bar{x}, k)$ for $(\bar{v}, k) = (5, 1)$.

$$\frac{\hat{u}(p_n, s)}{s^\theta} = \left(-ap_n^2 - \frac{v^2}{4a}\right)\hat{u}(p_n, s) + ap_n q_0, \tag{3.24}$$

$$\hat{u}(p_n, s) = s^\theta\left(-ap_n^2 - \frac{v^2}{4a}\right)\hat{u}(p_n, s) + ap_n q_0 s^\theta, \tag{3.25}$$

$$\hat{u}(p_n, s) - s^\theta\left(-ap_n^2 - \frac{v^2}{4a}\right)\hat{u}(p_n, s) = ap_n q_0 s^\theta, \tag{3.26}$$

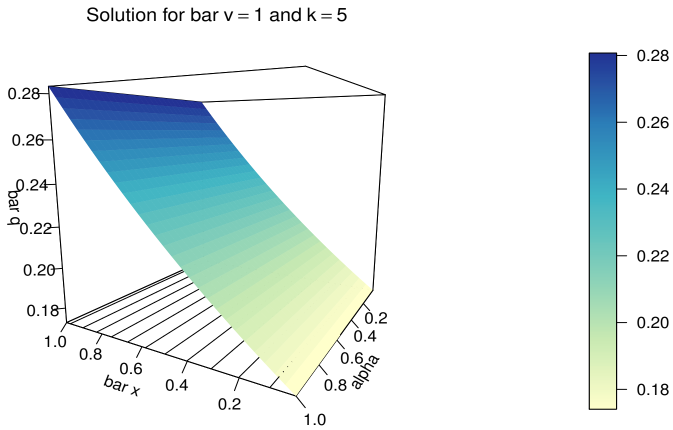

Figure 3: Plot of the solution $\bar{q}(\bar{x}, k)$ for $(\bar{v}, k) = (1, 5)$.

$$\hat{u}(p_n, s)\left[1 - s^{\theta}\left(-ap_n^2 - \frac{v^2}{4a}\right)\right] = ap_n q_0 s^{\theta}, \tag{3.27}$$

and

$$\hat{u}(p_n, s) = \frac{ap_n q_0 s^{\theta}}{\left[1 - s^{\theta}\left(-ap_n^2 - \frac{v^2}{4a}\right)\right]}. \tag{3.28}$$

Thus, by taking into account the fact that

$$\frac{1}{\left[1 - s^{\theta}(-ap_n^2 - \frac{v^2}{4a})\right]} = \frac{1}{\left(ap_n^2 + \frac{v^2}{4a}\right)}\left\{1 - \frac{1}{s^{\theta}\left[\frac{1}{s^{\theta}} - (-ap_n^2 - \frac{v^2}{4a})\right]}\right\}, \tag{3.29}$$

we obtain

$$\hat{u}(p_n, s) = \frac{ap_n q_0}{\left(ap_n^2 + \frac{v^2}{4a}\right)}\left\{1 - \frac{1}{s^{\theta}\left[\frac{1}{s^{\theta}} - (-ap_n^2 - \frac{v^2}{4a})\right]}\right\}, \tag{3.30}$$

which after the inversion of the integral transform, yields

$$u(x, t) = \frac{2q_0 t^{\theta}}{L}\sum_{n=1}^{\infty}\frac{p_n \sin(p_n x)}{\left(p_n^2 + \frac{v^2}{4a}\right)}\left\{1 - E_{\theta,1}\left[-\left(ap_n^2 + \frac{v^2}{4a}\right)t^{\theta}\right]\right\}. \tag{3.31}$$

We now consider the following series expansion [**?**]:

$$\sum_{n=1}^{\infty}\frac{n\sin(nz)}{n^2 + b^2} = \frac{\pi}{2}\frac{\sinh[(\pi - z)b]}{\sinh[(\pi b)]}. \tag{3.32}$$

Going back to the factor $q(x,t)$ from (3.5), we thus find that

$$q(x,t) = q_0 t^{\theta} \exp\left(\frac{vx}{2a}\right)\left\{\frac{\sinh\left[\frac{v}{2a}(L-x)\right]}{\sinh(\frac{vL}{2a})} - \frac{2}{L}\sum_{n=1}^{\infty}\frac{p_n \sin(p_n x)}{p_n^2+\frac{v^2}{4a}} E_{\theta}\left[-\left(ap_n^2+\frac{v^2}{4a}\right)t^{\theta}\right]\right\}, \tag{3.33}$$

and the non-dimensional form is given by

$$\bar{q}(\bar{x},k) = \exp\left(\frac{\bar{v}\bar{x}}{2}\right)\left\{\frac{\sinh\left[\frac{\bar{v}}{2}(1-\bar{x})\right]}{\sinh(\frac{\bar{v}}{2})} - 2\sum_{n=1}^{\infty}\frac{\bar{p}_n \sin(\bar{p}_n \bar{x})}{\bar{p}_n^2+\frac{\bar{v}^2}{4}} E_{\theta}\left[-k^2\left(\bar{p}_n^2+\frac{\bar{v}^2}{4}\right)\right]\right\}, \tag{3.34}$$

where

$$\bar{q}(\bar{x},k) = \frac{q(\bar{x},kk)}{q_0 t^{\theta}}. \tag{3.35}$$

The other non-dimensional quantities are the same as in (3.19). To end this section, we observe that figures 4 show how the fundamental solution (3.34) which depends on the fractional derivative's order ($\alpha = 0.2$) and distance $\bar{x}$. Similarly, 5 and 6 depict the solution $\bar{q}(\bar{x}, qk)$ for varying values of $\bar{x}$ and θ and for fixed values for $(\bar{v}, k)$, with

$$(\bar{v}, k) = (15, 0.1), \ (5, 1) \ \text{and} \ (1, 5),$$

respectively.

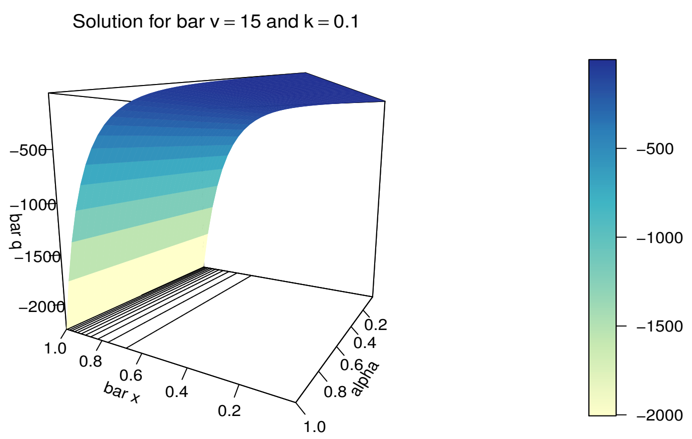

Figure 4: Plot of the solution $\bar{q}(\bar{x}, k)$ for $(\bar{v}, k) = (15, 0.1)$.

4. Conclusion

The time-fractional advection-diffusion problem was studied using the conventional Laplace transform by Povstenko et al. [15]. In this study, we have effectively addressed

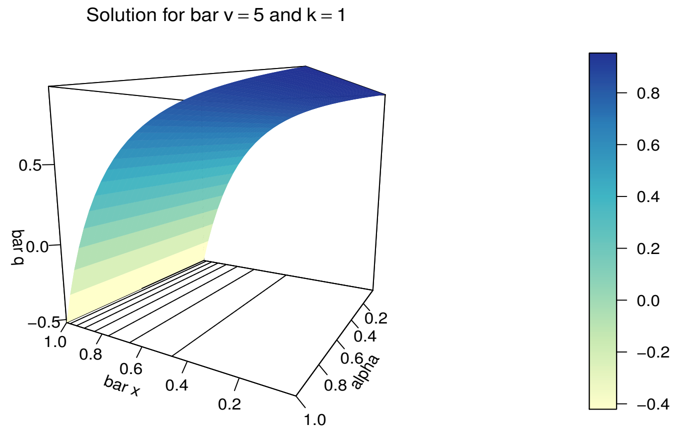

Figure 5: Plot of the solution $\bar{q}(\bar{x}, k)$ for $(\bar{v}, k) = (5, 1)$.

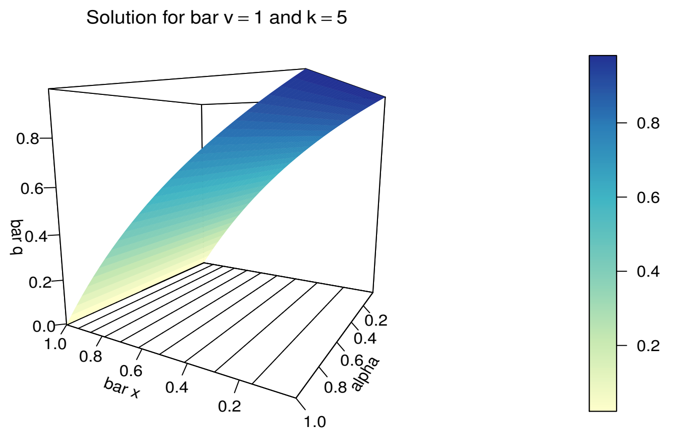

Figure 6: Plot of the solution $\bar{q}(\bar{x}, k)$ for $(\bar{v}, k) = (1, 5)$.

the solution of the time-fractional advection diffusion problem with the Liouville-Caputo approach using the Fourier transform and the specified variant of the Laplace transform. In order to carry out the desired function, we have used the integral transform technique to arrive at a solution when the boundary conditions are constant. Moreover, we have presented (1.1) in time-fractional mode along with the initial condition (3.2) and the boundary condition of Dirichlet type with the desired function's constant boundary values. We have displayed some graphics to illustrate our theoretical findings in this article. This graphical illustration shows the relationship between the diffusion concerning the distance, keeping k and θ as constant. As per the solution obtained by the presented method, the dependency of concentration $\bar{q}$ with different values of θ and $\bar{x}$ from 0.2 to 1, keeping $\bar{v}$ and k constant as (15, 0.1), (5,1) and (1,5) considering zero initial condition as well as constant boundary condition. Since the advection-diffusion equation can be used in various real-world situations, its variable coefficients will mean different things depending on the context in which it is used. As a result, the solute dispersion problem will provide the basis for the mathematical formulation of advection diffusion in this context. Since the medium in this issue is considered inhomogeneous, the velocity is thought to depend on the location variable. The mathematical formulation is the same; the second instance merely adds two additional equations.

Acknowledgements

We especially thank Prof. Christophe Chesneau for his guidance for the graphical representations included in this article. The authors would like to thank the worthy referees and editors for their valuable suggestions for our paper in Mathematics.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] Bracewell, R. N. (1986). *The Fourier Transform and Its Applications*, McGraw-Hill, New York.

[2] Birajdar, G.A. (2020). *An Implicit Numerical Method for Semilinear Space-Time Fractional Diffusion Equation*, Dyn. Contin. Discret. Impuls. Syst. Ser. A Math. Anal. 27, 209–217.

[3] Birajdar, G.A. (2014). emphAn Implicit Numerical Method for Semilinear Space-time Fractional Diffusion Equation, Walailak Journal of Science and Technology (WJST), Walailak Journal of Science and Technology (WJST). 12, 11, 983–994. doi:10.14456/WJST.2015.40.

[4] Chinchane, V. L. and Pachpatte, D. B. (2013). *Uniquenss of fractional differential equation using the inequalities,* International Journal of Pure and Applied Mathematics, 88(4), 529-536. doi:http://dx.doi.org/10.12732/ijpam.v88i4.7

[5] Gorenflo, R. & Mainardi, F. (1997). *Fractional calculus: Integral and differential equations of fractional order*, In: Carpinteri, A, Mainardi, F (eds.) Fractals and Fractional Calculus in Continuum Mechanics, Springer, Wien, 223–276.

[6] Goswami, P and Alqahtani R. T. (2016). *Solutions of fractional differential equations by Sumudu transform and variational iteration methods*, *J. Nonlinear Sci. Appl.* 9, 1944–1951.

[7] Gul H. , Alrabaiah H., Ali S., Shah K. and Muhammad S. (2020). *Computation of solution to fractional order partial reaction diffusion equations*, Journal of Advanced Research, 25, 31-38, https://doi.org/10.1016/j.jare.2020.04.021.

[8] Gupta, V. G., Shrama, B. & Kilicman, A. (2010). *A note on fractional Sumudu transform, J. Appl. Math.* 2010, Article ID 154189, 1–9. doi:10.1155/2010/154189.

[9] Izadi, M. & Srivastava, H. M. (2022). *An optimized second order numerical scheme applied to the non-linear Fisher's reaction-diffusion equation*, *J. Interdisciplinary Math.* 25, 471–492.

[10] Kamran Irfan, M. Alotaibi, F.M. Haque, S. Mlaiki, N. Shah, K. (2023). *RBF-based local meshless method for fractional diffusion equations*, Fractal Fract., 7, 143. https://doi.org/10.3390/fractalfract7020143.

[11] Kamran I, Raheel K, Gul R and Kamal S. (2023). *On the numerical approximation of three-dimensional time fractional convection-diffusion equations*, Fractal Fract., 7, 143. https://doi.org/10.3390/fractalfract7020143.

[12] Kaviany, M. (1995). *Principles of Heat Transfer in Porous Media*, 2nd edn., Springer, New York.

[13] Liu, F., Anh, V. V., Turner, I. W.& Zhuang, P. (2003). *Time-fractional advection-dispersion equation J. Appl. Math. Comput.* 123, 233-245.

[14] Metzler, R. Klafter, J. (2000). *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, *Phys. Rep.* 339, 1–77.

[15] Povstenko, Y. & Klekot, J. (2016). *The Dirichlet problem for the time-fractional advection-diffusion equation in a line segment*, *Boundary Value Problems*, Article ID 89, *2016*. https://doi.org/10.1186/s13661-016-0597-4

[16] Povstenko, Y. (2015). *Fractional Thermoelasticity*, Springer, New York.

[17] Povstenko, Y. (2015). *Theory of diffusive stresses based on the fractional advection-diffusion equation*,In: Abi Zeid Daou, R.; Moreau, X. (eds.) Fractional Calculus: Applications, Nova Science Publishers, New York, 227–241 .

[18] Povstenko, Y. (2005). *Fractional heat conduction equation and associated thermal stress J. Therm. Stresses* 28, 83–102.

[19] Povstenko, Y. (2009). *Thermoelasticity which uses fractional heat conduction equation*, *J. Math. Sci.* 162, 296–305.

[20] Povstenko, Y. (2014). *Fundamental solutions to time-fractional advection diffusion equation in a case of two space variables*, *Math. Probl. Engrg.* 2014, 705364. https://doi.org/10.1155/2014/705364.

[21] Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, New York.

[22] Sneddon, I. N. *Use of Integral Transform*, New York, McGraw-Hill, 1972

[23] Srivastava, H. M. (2021). *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, *J. Nonlinear Convex Anal.* 22, 1501–1520.

[24] Srivastava, H. M. (2021). *An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions*, *J. Adv. Engrg. Comput.* 5, 135–166. http://dx.doi.org/10.55579/jaec.202153.340.

[25] Srivastava. H. M. (2021). *A survey of some recent developments on higher transcendental functions of*

analytic number theory and applied mathematics, *Symmetry* 13 (12), Article ID 2294, 1–22. https://doi.org/10.3390/sym13122294.

[26] Srivastava, H. M. & Izadi, M.(2022). *The Rothe-Newton approach to simulate the variable coefficient convection-diffusion equations J. Mahani Math. Res. Cent.* 11, 141–157.

[27] Srivastava, H.M., Ahmad, H., Ahmad, I., P. Thounthong, P. & Khan, M. N. (2021) *Numerical simulation of 3-D fractional-order convection-diffusion PDE by a local meshless method*, *Thermal Sci.* 25(1A), 347–358.

[28] Tripathi, V. M., Srivastava, H. M., Singh, H. Swarup, C. & Aggarwal, S. (2021) *Mathematical analysis of non-isothermal reaction-diffusion models arising in spherical catalyst and spherical biocatalyst, Appl. Sci.* 11, Article ID 10423, 1–14. https://doi.org/10.3390/app112110423.

[29] Uchaikin, V. V. (2013). *Fractional Derivatives for Physicists and Engineers*: *Background and Theory*, Springer, Berlin.