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A Method to solve ordinary fractional differential equations using Elzaki and Sumudu transform

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Abstract

The main objective of the paper is to solve ordinary fractional differential equations using Elzaki and Sumudu transform. Moreover some ordinary fractional differential equations are solved by using the presented method. Using different types of fractional differential operators existing methods have been extended and applied for ordinary fractional differential equations.

Keywords: " Caputo-Fabrizio, " "Sumudu transform, " "Elzaki transform, " "Fractional Differential Equations ".
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1. Introduction

Differential equations play a significant role in many disciplines, including engineering, physics, biology, and economics. Classical calculus contains derivatives, integrals, and differential equations of orders other than integers. Fractional calculus is a generalization of this discipline (fractional). The Adomian decomposition technique and the Elzaki integral transform were utilised by Ira Sumiati et al. [1] to resolve fractional ordinary differential equations. The results demonstrate that the Elzaki-Adomian decomposition approach is particularly efficient, practical, and user-friendly for solving linear and nonlinear fractional

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ordinary differential equations. Aytac Arikoglu et al. [2] applied the differential transform method (DTM), a well-known transformation technique, to the domain of fractional differential equations. By presenting new theorems and their supporting arguments, numerical examples are done for a variety of issues, including the application of the approach to the Bagley-Torvik, Ricatti, and composite fractional oscillation equations. The acquired results are in strong agreement with those found in the open literature, demonstrating the robustness, accuracy, and simplicity of the technique presented here. The analytical solution for the logistic differential equation of fractional order and non-singular kernel was presented by Juan J. Nieto along with the solution [3].
A multi-step Adams-type method was proposed by Zabidi et al. [4] to solve differential equations of fractional order. The Adams-Moulton method for fractional case is used in the method's development together with the Lagrange interpolation. Further, a fractional derivative is used in the Caputo derivative operator. The suggested method's analysis is offered in terms of method order, order of accuracy, and convergence analysis, with the proposed method's convergence being demonstrated. The method's stability is further investigated, and for different values of, it appears that the stability zones are symmetric to the real axis.
Numerous numerical examples for solving linear and nonlinear fractional differential equations are presented to demonstrate the effectiveness of the suggested approach. Thange et al. [5] introduced Aboodh transform introduced for different fractional differential operators. These results are used to illustrate some exemplary examples of the initial value fractional order differential equation utilizing this transform solution. To resolve some fractional differential equations. Aruldoss and et al. [6] used the Aboodh transform. The methodology is based on some general theorems on special Aboodh transform and binomial series expansion coefficient solutions to some fractional differential equations. Recently, The Caputo and Caputo-Fabrizio fractional operators have recently become quite popular among mathematicians in applied fields. We cite for additional information [7, 8, 9, 10, 11, 12].

2. Preliminaries

We look at a few definitions needed for writing in this paper.

Definition 2.1. [13] The fractional derivative of order α by Caputo-Fabrizio is

$$ {}_{a}^{CF}D_t^{\alpha}f(t) = \frac{M(\alpha)}{(1-\alpha)}\int_a^t f'(\tau)\exp\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right]d\tau, 0<\alpha\leqslant 1.$$

Where $f'\epsilon H'(a,b), b>a$ and $M(\alpha)$ is a normalization constant depending on α such that $M(0)=M(1)=1$.

Definition 2.2. [14, 15] The Atangana–Baleanu Caputo (ABC) fractional derivative with the Mittag–Leffler non-singular kernel of order α is defined by

$$ {}_{a}^{ABC}D_t^{\alpha}f(t) = \frac{B(\alpha)}{(1-\alpha)}\int_a^t f'(\tau)E_{\alpha}\left[-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau, \tau>0$$

where E_α is the Mittag–Leffler function and $B(\alpha)$ is a normalizing positive function satisfying $B(0)=B(1)=1$.

Definition 2.3. [16, 17] Let n be a natural integer such that $n-1 < \alpha < n$, and let α be any positive real number. For the given the continuous function $f(t)$ a range of values between $[a, T]$, and $T > a$. Then Riemann-Liouville derivative of order α is then given by,

$$ {}_{a}^{RL}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau. $$

Definition 2.4. [16, 17] Let n be a natural integer such that $n-1 < \alpha < n$, and let α be any positive real number. For the given the continuous function $f(t)$ a range of values between $[a, T]$, and $T > a$. Then Caputo fractional derivative of order α is then given by,

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) = \int_a^t (t-\tau)^{n-\alpha-1} f^n(\tau)d\tau. $$

Definition 2.5. [18] If f has exponential growth, is locally integrable on $[0, \infty)$ and has the properties of $|f(t)| \leqslant Me^{\omega t}, t > 0, (\omega \epsilon R)$ then Laplace transform is defined as

$$ F(s) =< f(t), e^{-st} >= \int_0^\infty e^{-st}f(t)dt, s\epsilon C, Rs > \omega $$

Definition 2.6. Laplace transform of Caputo-Fabrizio fractional derivative is provided by

$$ L\left\{{}_{0}^{CF}D_{t}^{\alpha+n}f(t)\right\}(s) = \frac{s^{n+1}L\{f(t)\} - f(0)s^n - f'(0)s^{n-1} .... - f^n(0)}{s+(1-\alpha)}, 0 < \alpha \leqslant 1 $$

$$ L\left\{{}_{0}^{CF}D_{t}^{\alpha}f(t)\right\}(s) = \frac{sL\{f(t)\} - f(0)}{s+(1-\alpha)}. $$

Definition 2.7. [19] A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A defined by

$$ A = f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{-\nu t}. $$

In the case of a provided function in the set A, the constant M must be finite number, $k_1 k_2$, may be finite or infinite. The Aboodh transform denoted by the operator $A(.)$ Defined by the integral equations

$$ A\{f(t)\} = K(\nu) = \frac{1}{\nu}\int_0^\infty f(t)e^{-\nu t}dt, \ t \geqslant 0, k_1 \leqslant \nu \leqslant k_2. $$

Definition 2.8. [20] The Elzaki transform is a novel transform defined for functions of exponential order. We assume functions in the set A defined by:

$$ A = f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, ift\epsilon(-1)^j x[0, \infty) $$

For a certain function in the set A, the constant M must have a finite value, while k_1, k_2 may be finite or infinite. The integral equations define the Elzaki transform, which is denoted by the operator $E(.)$.

$$ E\{f(t)\} = T(\nu) = \nu\int_0^\infty f(t)e^{-\frac{t}{\nu}}dt, t \geqslant 0, k_1 \leqslant \nu \leqslant k_2. $$

3. Main Results

In this section, we solve ordinary fractional differential equations using Elzaki and Sumudu transform.

Theorem 3.1. *If Aboodh transform of function* $f(t)$ *is* $A(f)(s)$ *then, Aboodh transform of Caputo-Fabrizio fractional derivative of order* α *is given by*

$$A\left[{}^{CF}_{0}D^{\alpha}_{t}f(t)\right](s)=\frac{sA(f)(s)-\frac{f(0)}{s}}{\alpha+s(1-\alpha)}. \tag{3.1}$$

Proof: *Suppose that Adoodh transform of function* $f(t)$ *is* $q(s)$ *and Caputo Fabrizio fractional derivative of order* $\alpha > 0$ *is*

$${}^{CF}_{a}D^{\alpha}_{t}f(t)=\frac{M(\alpha)}{(1-\alpha)}\int_{a}^{t}f'(\tau)\exp\frac{-\alpha(t-\tau)}{1-\alpha}d\tau, 0<\alpha\leqslant 1$$

where $f'\epsilon H'(a,b), b > a$ *and* $M(\alpha)$ *is a normalization constant depending on* α *such that* $M(0) = M(1) = 1$
Applying Laplace transform on both sides of this equation

$$L\left\{{}^{CF}_{a}D^{\alpha}_{t}f(t)\right\}=\frac{sL(f)(s)-f(0)}{\alpha+(1-\alpha)}.$$

Using Aboodh Laplace Duality $L\{f(t)\} = sA\{f(t)\}$, *we get [8]*

$$sA\left\{{}^{CF}_{a}D^{\alpha}_{t}f(t)\right\}=\frac{ssA(f)(s)-f(0)}{\alpha+(1-\alpha)}.$$

$$A\left\{{}^{CF}_{a}D^{\alpha}_{t}f(t)\right\}=\frac{sA(f)(s)-\frac{f(0)}{s}}{\alpha+(1-\alpha)}.$$

This is required result.

Theorem 3.2. *If Aboodh transform of function* $f(t)$ *is* $A(f)(s)$ *then Aboodh transform of Atangana-Baleanu-Caputo derivative of order* α *is given by*

$$A\left[{}^{ABC}_{a}D^{\alpha}_{t}f(t)\right](s)=\frac{B(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\left[s^{\alpha}A(f)(s)-s^{\alpha-2}f(0)\right]. \tag{3.2}$$

Proof: *Suppose that Laplace transform of function* $f(t)$ *is* $L[f(t)]$ *using Atangana-Baleanu-Caputo fractional derivative of order* $\alpha > 0$ *is*

$${}^{ABC}_{a}D^{\alpha}_{t}f(t)=\frac{B(\alpha)}{(1-\alpha)}\int_{a}^{t}f'(\tau)E_{\alpha}\left[-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau, \tau>0.$$

Laplace transform is used on both sides of this equation.

$$L\left\{{}^{ABC}_{a}D^{\alpha}_{t}\right\}f(t)=\frac{B(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\left[s^{\alpha}L\{f(t)\}-s^{\alpha-1}f(0)\right].$$

By Aboodh Laplace Duality $L\{f(t)\} = sA\{f(t)\}$, *we get [21]*

$$sA\left\{{}^{ABC}_{a}D^{\alpha}_{t}f(t)\right\} = \frac{B(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\left[s^{\alpha}sA\{f(t)\} - s^{\alpha-1}f(0)\right].$$

$$A\left\{{}^{ABC}_{a}D^{\alpha}_{t}f(t)\right\} = \frac{B(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\left[s^{\alpha}A\{f(t)\} - s^{\alpha-2}f(0)\right].$$

This is required result.

Theorem 3.3. *If Sumudu transform of function* $f(t)$ *is* $S[f(t)]$ *then Sumudu transform of Riemann Liouville fractional derivative of order* α *is*

$$S\left[{}^{RL}_{a}D^{\alpha}_{t}f(t)\right](u) = u^{-\alpha}S[f(t)] - \sum_{k=1}^{n} u^{-k}\left[D^{\alpha-k}f(t)\right]_{t=0}. \tag{3.3}$$

Proof: The Riemann Liouville fractional derivative of order α *is given by*

$${}^{RL}_{a}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f(\tau)d\tau.$$

Applying Elzaki transform on both sides of this equation is [22]

$$E\left\{{}^{RL}_{a}D^{\alpha}f(t)\right\} = u^{-\alpha}\left[T(u) - \sum_{k=1}^{n}u^{\alpha-k+2}\left(D^{\alpha-k}f(t)\right)\right]_{t=0}.$$

$$E\left\{{}^{RL}_{a}D^{\alpha}f(t)\right\} = u^{-\alpha}T(u) - u^{-\alpha}\sum_{k=1}^{n}u^{\alpha-k+2}\left(D^{\alpha-k}f(t)\right)_{t=0}.$$

$$E\left\{{}^{RL}_{a}D^{\alpha}f(t)\right\} = u^{-\alpha}T(u) - \sum_{k=1}^{n}u^{2-k}\left(D^{\alpha-k}f(t)\right)_{t=0}.$$

By using Elzaki-Sumudu duality, $Ef(t) = u^{2}S[f(t)]$, *we have*

$$u^{2}S\left\{{}^{RL}_{a}D^{\alpha}f(t)\right\} = u^{-\alpha}u^{2}S[f(t)] - \sum_{k=1}^{n}u^{2-k}\left(D^{\alpha-k}f(t)\right)_{t=0}.$$

$$S\left\{{}^{RL}_{a}D^{\alpha}f(t)\right\} = u^{-\alpha}u^{2}S[f(t)] - \sum_{k=1}^{n}u^{-k}\left(D^{\alpha-k}f(t)\right)_{t=0}.$$

which is required result.

Theorem 3.4. *If Sumudu transform of function* $f(t)$ *is* $S[f(t)]$ *then Sumudu transform of Caputo fractional derivative of order* α *is*

$$S\left[{}^{C}_{a}D^{\alpha}_{t}f(t)\right](u) = \frac{S[f(t)]}{u^{\alpha}} - \sum_{k=1}^{n}u^{k-\alpha-1}\left[f^{k}(0)\right], n-1 < \alpha \leqslant n \tag{3.4}$$

Proof: *Caputo fractional derivative of order* α *is then given by,*

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) = \int_{a}^{t} (t-\tau)^{n-\alpha-1} f^{n}(\tau)d\tau. $$

Suppose that Elzaki transform of function $f(t)$ *is* $E(u)$ *considering Caputo fractional derivative of order* α *is [1]*

$$ E\left\{{}_{a}^{C}D^{\alpha}f(t)\right\} = \frac{E(u)}{u^{\alpha}} - \sum_{k=0}^{n-1} u^{k-\alpha+2}\left(f^{k}(0)\right), n-1<\alpha\leqslant n $$

By using Elzaki-Sumudu duality, $Ef(t) = u^{2}Sf(t)$*, we have*

$$ u^{2}S\left\{{}_{a}^{C}D^{\alpha}f(t)\right\} = u^{2}\frac{S[f(t)]}{u^{\alpha}} - \sum_{k=0}^{n-1} u^{k-\alpha+2}f^{k}(0), n-1<\alpha\leqslant n $$

$$ S\left\{{}_{a}^{C}D^{\alpha}f(t)\right\} = \frac{S[f(t)]}{u^{\alpha}} - \sum_{k=1}^{n} u^{k-\alpha-1}f^{k}(0), n-1<\alpha\leqslant n $$

This is required result.

4. Applications

Here, we presents some examples of applying the Aboodh and Sumudu transform to solve fractional ordinary differential equations.

Example. 1 Solve

$$ D^{\frac{1}{2}}f(t) = t, with\ f(0) = f_{0}. $$

Taking Aboodh transform on both sides and using Caputo-Fabrizio fractional derivative

$$ \frac{1}{\frac{1}{2}+s\left(1-\frac{1}{2}\right)}\left[sA(f)(s) - \frac{f_{0}}{s}\right] = \frac{1}{s^{3}}. $$

$$ \frac{1}{\frac{1}{2}+\frac{1}{2}s}\left[sA(f)(s) - \frac{f_{0}}{s}\right] = \frac{1}{s^{3}}. $$

$$ \frac{2}{1+s}\left[sA(f)(s) - \frac{f_{0}}{s}\right] = \frac{1}{s^{3}}. $$

$$ sA(f)(s) - \frac{f_{0}}{s} = \frac{1+s}{2s^{3}}. $$

$$ sA(f)(s) = \frac{1}{2}\left[\frac{1}{s^{3}} + \frac{1}{s^{2}}\right] + \frac{f_{0}}{s}. $$

$$ A(f)(s) = \frac{1}{2}\left[\frac{1}{s^{4}} + \frac{1}{s^{3}}\right] + \frac{f_{0}}{s^{2}}. $$

Taking inverse Aboodh transform, we get

$$f(t) = \frac{1}{2}\left[\frac{t^2}{2} + t\right] + f_0.$$

Example. 2 Solution of the type of problems

$$D^{\alpha}y(t) = ay(t), y(0) = c.$$

Taking Sumudu transform on both sides and using Caputo fractional derivative,

$$u^{-\alpha}S\{y(t)\} - \sum_{k=1}^{n} u^{-k}\left[D^{\alpha-k}(y(t))\right]_{t=0} = aSy(t).$$

$$u^{-\alpha}y(u) - ay(u) = \sum_{k=1}^{n} u^{-k}D^{\alpha-k}(y(t)).$$

$$y(u)\left[u^{-\alpha} - a\right] = \sum_{k=1}^{n} u^{-k}D^{\alpha-k}(y(t)).$$

$$y(u) = \frac{1}{u^{-\alpha} - a}\sum_{k=1}^{n} u^{-k}D^{\alpha-k}(y(t)).$$

$$y(u) = S^{-1}\left[\frac{1}{u^{-\alpha} - a}\sum_{k=1}^{n} u^{-k}D^{\alpha-k}(y(t))\right].$$

Example. 3 Solve

$$D^{\frac{1}{2}}y(t) = t, with\ y(0) = m.$$

Taking Sumudu transform on both sides and using Caputo fractional derivative,

$$u^{\frac{-1}{2}}y(u) - u^{\frac{-1}{2}}y(0) = u.$$

$$u^{\frac{-1}{2}}y(u) - mu^{\frac{-1}{2}} = u.$$

$$u^{\frac{-1}{2}}y(u) = u + mu^{\frac{-1}{2}}.$$

$$y(u) = u^{\frac{3}{2}} + m.$$

$$y(t) = \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} + m.$$

Example. 4 Solve
$\left[{}^{C}D_t^{\alpha}y(t)\right] + y = 0, y(0) = 1.$
Taking Sumudu transform on both sides,

$$S\left[{}^{C}D_t^{\alpha}y(t)\right] + S\{y(t)\} = 0.$$

$$\frac{y(u)}{u^{\alpha}} - \sum_{k=1}^{n} u^{k-\alpha-1}y^{k}(0) + y(u) = 0.$$

$$\frac{y(u)}{u^{\alpha}} - u^{-\alpha}y(0) + y(u) = 0.$$

$$y(u)\left[\frac{1}{u^{\alpha}} + 1\right] = u^{-\alpha}.$$

$$y(u) = \frac{1}{u^{\alpha}}\frac{u^{\alpha}}{1+\frac{u}{\alpha}} = \frac{1}{1+u^{\alpha}}.$$

$$y(t) = e^{-t^{\alpha}}.$$

Example. 5 Solve

$$\left[{}^{C}D_{t}^{\alpha}y(t)\right] = 1 + 2y(t), with\ y(0) = 0.$$

Taking Sumudu transform on both sides

$$S\left[{}^{C}D_{t}^{\alpha}y(t)\right] = S\{1\} + 2S\{y(t)\}.$$

$$\frac{y(u)}{u^{\alpha}} - \sum_{k=1}^{n} u^{k-\alpha-1}y^{k}(0) = 1 + 2y(u).$$

$$\frac{y(u)}{u^{\alpha}} - u^{\alpha}y(0) = 1 + 2y(u).$$

$$\frac{y(u)}{u^{\alpha}} - 2y(u) = 1.$$

$$y(u)\left[\frac{1}{u^{\alpha}} - 2\right] = 1.$$

$$y(u)\left[\frac{1-2u\alpha}{u^{\alpha}}\right] = 1.$$

$$y(u) = \left[\frac{u\alpha}{1-2u^{\alpha}}\right].$$

Taking inverse Sumudu transform, we get,

$$y(t) = \frac{-1}{2}\exp\left[\left(\frac{1}{2}\right)t^{-\alpha}\right].$$

5. Conclusion

In this study it is concluded that Sumudu transform is effective and very convenient tool to solve homogeneous and non homogeneous initial value problems. Different types of engineering problems where ordinary fractional differential equations arises, can be easily solved with the help of theorems in connection with Elzaki and Sumudu transform.

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