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Nabla generalized fractional Riemann-Liouville calculus on time scales with an application to dynamic equations

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Abstract

We introduce more general concepts of nabla Riemann-Liouville fractional integrals and derivatives on time scales. Such generalizations on time scales help us to study relations between fractional difference equations and fractional differential equations. Sufficient conditions for the existence and uniqueness of the solution to an initial value problem are described by nabla derivatives on time scales. Some properties of the new operator are proved and illustrated with examples.

Keywords: Time scales, fractional derivatives, dynamic equations, initial value problems
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1. Introduction

In recent decades, fractal calculus and fractional calculus have been becoming hot topics in both mathematics and engineering for nondifferential solutions [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. So the use of the fractional derivative has gained noticeable improvement and attention in many branches of sciences [13, 14, 15]. Another important area of study is dynamic equations on time scales, which goes back to 1988 and the work of Aulbach and Hilger, and has been used with success to unify differential and difference equations [20]. In 2016, Benkhettou et al. [7], introduced a concept of fractional derivative of Riemann-Liouville on time scales. Several authors have obtained important results about different subjects on time scales. See for instance M. Rchid et al [8], A. Abdeljawad et al [9], T. Gülsen et al [10].

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This article is organized as follows, we define a Riemann–Liouville generalized fractional nabla derivative on time scales (Definition 3.1) and Riemann–Liouville generalized fractional nabla integrations on time scales (Definition 3.2) help us to study of their important properties. This is the first step in this direction. On the other hand, we consider the following initial value problem:

$$\begin{cases} \beta x^{\nabla}(t) + {}_{a}^{\mathbb{T}}D_{y,t,\nabla}^{\alpha}(x(t)) = f(t, x(t)), \text{ for } t \in \mathcal{J}_{a,b}, \\ x(a) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$, ${}_{a}^{\mathbb{T}}D_{y,t,\nabla}^{\alpha}$ is the Riemann–Liouville generalized fractional nabla derivative with respect to function y and $\mathcal{J}_{a,b} = [a, \rho(b)]_{\mathbb{T}}$. The problem (1.1) will be studied under the following assumptions $f \in \mathcal{C}(\mathcal{J}_{a,b} \times \mathbb{R}, \mathbb{R})$ and $\beta \neq 0$. Our main results give necessary and sufficient conditions for the existence and uniqueness of solution to problem (1.1).

The rest of the paper is structured as follows: Section 2 contains some defnitions and facts of time scale calculus. In Section 3, we establish some new properties of the Riemann-Liouville fractional nabla operator and, we investigate some IVPs for some classes fractional dynamic equations. In Section 4, we illustrate our results with examples. Some conclusions are discussed in Section 5.

2. Preliminaries

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ by :

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also, if $t < \sup \mathbb{T}$ and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by : $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$, otherwise, $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ is bounded, then $\mathbb{T}_0 \subseteq \mathbb{T}_k$ where $\mathbb{T}_0 = \mathbb{T} \backslash \{\min \mathbb{T}\}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$.

The function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and has a right-sided limits exist at right-dense points in $\mathbb{T}$, write $f \in \mathcal{C}_{ld}(\mathbb{T}, \mathbb{R})$. For a function $f : \mathbb{T} \to \mathbb{R}$, the function f^{ρ} denotes $f \circ \rho$. The ∇-derivative of $f : \mathbb{T} \to \mathbb{R}$ at a left dense point t is defined by :

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

If t is left scattered, then the $\Delta-$derivative is defined by :

$$f^{\nabla}(t) = \frac{f(t) - f^{\rho}(t)}{\nu(t)}.$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by : $\mathcal{C}^1_{ld}(\mathbb{T}, \mathbb{R})$.

In what follows, with $\mathcal{C}([a,b]_{\mathbb{T}}, \mathbb{R})$ we denote the Banach space of all continuous functions from $[a,b]_{\mathbb{T}}$ into $\mathbb{R}$, where $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$, with the norm

$$\|x\|_{\infty} := \sup\{|x(t)| : t \in [a,b]_{\mathbb{T}}\}.$$

Let $[a,b]_{\mathbb{T}}$ denote a closed bounded interval in $\mathbb{T}$. A function $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called a delta antiderivative function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ provided F is continuous on $[a,b]_{\mathbb{T}}$, delta differentiable on $[a,b)$, and $F^{\nabla}(t) = f(t)$, for all $t \in [a,b)$. Then, we define the ∇-integral of f from a to b by :

$$\int_a^b f(t)\,\nabla t = F(b) - F(a).$$

3. Main Results

We introduce a new notion of generalized fractional nabla derivative on time scales. Before that, we define the generalized fractional nabla integral on a time scale $\mathbb{T}$.

Definition 3.1 (Riemann-Liouville generalized fractional nabla integral on time scales)**.** Suppose $\mathbb{T}$ is a time scale, $[a,b]$ is an interval of $\mathbb{T}$, h is an integrable function on $[a,b]$, and g is monotone having a nabla derivative g with $g^{\nabla}(t) \neq 0$ for any $t \in [a,b]$. Let $0 < \alpha < 1$. Then the (left) generalized fractional nabla integral of order α of h is defined by :

$${}_a^{\mathbb{T}}I^{\alpha}_{g,t,\nabla} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t g^{\nabla}(s)\,(g(t) - g(s))^{\alpha-1}\, h(s)\,\nabla s,$$

where Γ is the gamma function.

We introduce Riemann–Liouville generalized fractional nabla derivative on time scales the following definition.

Definition 3.2. Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, $h : \mathbb{T} \to \mathbb{R}$ and g is monotone having a nabla derivative g with $g^{\nabla}(t) \neq 0$ for any $t \in [a,b]$. The (left) Riemann–Liouville generalized fractional nabla derivative of order α of h is defined by :

$${}_a^{\mathbb{T}}D^{\alpha}_{g,t,\nabla} h(t) = \frac{1}{\Gamma(1-\alpha)\, g^{\nabla}(t)} \left(\int_a^t g^{\nabla}(s)\,(g(t) - g(s))^{-\alpha}\, h(s)\,\nabla s \right)^{\nabla}.$$

Remark 3.3. If $\mathbb{T} = \mathbb{R}$, then Definitions 3.1 and 3.2 give, respectively, the well-known generalized fractional integral and derivative of Riemann–Liouville.

Lemma 3.4. *Let $\alpha \in (0,1)$ and $f : \mathcal{J}_{a,b} \times \mathbb{R} \to \mathbb{R}$. Function $x \in \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$ is a solution of the problem (1.1) if and only if it is a solution of the following integral equation:*

$$\beta x(t) = \int_a^t f(s, x(s))\,\nabla s - \frac{1}{\Gamma(1-\alpha)} \times \int_a^t y^{\nabla}(s)\,(y(t) - y(s))^{-\alpha}\, x(s)\,\nabla s.$$

Proof. By Definition 3.1, we have

$$\begin{aligned} \beta x^{\nabla}(t) &= f(t,x(t)) - \frac{1}{\Gamma(1-\alpha)} \times \left(\int_a^t y^{\nabla}(s)(y(t)-y(s))^{-\alpha} x(s)\nabla s \right)^{\nabla} \\ &= f(t,x(t)) - \left({}_a^{\mathbb{T}} I_{y,t,\nabla}^{1-\alpha} x(t) \right)^{\nabla} \\ &= f(t,x(t)) - {}_a^{\mathbb{T}} D_{y,t,\nabla}^{\alpha}(x(t)). \end{aligned}$$

The proof is complete. □

Our main result is based on the Banach fixed point theorem [18].

Theorem 3.5. *Let* $\alpha \in (0,1)$ *and* $f \in \mathcal{C}(\mathcal{J}_{a,b} \times \mathbb{R}, \mathbb{R})$, *there exists a positive and continuous function* $r : \mathcal{J}_{a,b} \to \mathbb{R}$, *such that*

$$|f(t,x) - f(t,y)| \leqslant r(t)|x-y|, \text{ for } (x,y,t) \in \mathbb{R}^2 \times \mathcal{J}_{a,b}. \tag{3.1}$$

If

$$\sup_{t\in\mathcal{J}_{a,b}} \int_a^t \left|y^{\nabla}(s)\right| |y(t)-y(s)|^{-\alpha} \nabla s < |\beta| - \int_a^b r(s)\nabla s, \tag{3.2}$$

then the problem (1.1) *has a unique solution on* $\mathcal{J}_{a,b}$.

Proof. We transform the problem (1.1) into a fixed point problem. Consider the operator $T : \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R}) \to \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$ defined by :

$$\begin{aligned} \mathcal{T}x(t) &= \frac{1}{\beta} \int_a^t f(s,x(s))\nabla s \\ &\quad - \frac{1}{\beta\Gamma(1-\alpha)} \int_a^t y^{\nabla}(s)(y(t)-y(s))^{-\alpha} x(s)\nabla s. \end{aligned} \tag{3.3}$$

We need to prove that $\mathcal{T}$ has a fixed point, which is a unique solution of (1.1) on $\mathcal{J}_{a,b}$. For that, we show that $\mathcal{T}$ is a contraction. Let $x_1, x_2 \in \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$. For $t \in \mathcal{J}_{a,b}$, we have

$$\begin{aligned} |\mathcal{T}x_1(t) - \mathcal{T}x_2(t)| &\leqslant \frac{1}{|\beta|} \int_a^t |f(s,x_1(s)) - f(s,x_2(s))| \nabla s \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_a^t \left|y^{\nabla}(s)\right| |y(t)-y(s)|^{-\alpha} |x_1(s) - x_2(s)| \nabla s \\ &\leqslant \frac{1}{|\beta|} \left(\int_a^b r(s)\nabla s + \int_a^t \left|y^{\nabla}(s)\right| |y(t)-y(s)|^{-\alpha} \nabla s \right) \\ &\quad \times \|x_1 - x_2\|. \end{aligned}$$

By (3.2), $\mathcal{T}$ is a contraction and thus, by the contraction mapping theorem, we deduce that $\mathcal{T}$ has a unique fixed point. This fixed point is the unique solution of (1.1). □

Now, we give our second main result guarantees the existence of at least one solution of the problem (1.1). This result is based on the Schauder's fixed point theorem [18].

Theorem 3.6. *Let* $\alpha \in (0,1)$ *and* $f \in \mathcal{C}(\mathcal{J}_{a,b} \times \mathbb{R}, \mathbb{R})$, *there are two functions* $r \in \mathcal{C}(\mathcal{J}_{a,b}, [0,\infty))$ *and* $\varphi \in \mathcal{C}(\mathbb{R}, [0,\infty))$, *such that*

$$|f(t,x)| \leqslant r(t), \quad \textit{for all } (y,t) \times \mathcal{J}_{a,b}.$$

Then the problem (1.1) *has a solution on* $\mathcal{J}_{a,b}$.

Proof. We use Schauder's fixed point theorem to prove that $\mathcal{T}$ defined by (3.3) has a fixed point. The proof is given in several steps.
Step 1*:* $\mathcal{T}$ is continuous. Let x_n be a sequence such that $x_n \to x$ in $\mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$. Then, for each $t \in \mathcal{J}_{a,b}$,

$$\begin{aligned}
|\mathcal{T}x_n(t) - \mathcal{T}x(t)| &\leqslant \frac{1}{|\beta|}\int_a^t |f(s,x_n(s)) - f(s,x(s))| \nabla s \\
&\quad + \frac{1}{|\beta|\Gamma(1-\alpha)}\int_a^t \left|y^{\nabla}(s)\right| (y(t)-y(s))^{-\alpha} \times |x_n(s) - x(s)| \nabla s \\
&\leqslant \frac{b-a}{|\beta|} \sup_{t\in\mathcal{J}_{a,b}} |f(s,x_n(s)) - f(s,x(s))| \\
&\quad + \sup_{t\in\mathcal{J}_{a,b}} \left(\int_a^t \frac{1}{\Gamma(1-\alpha)} \left|y^{\nabla}(s)\right| (y(t)-y(s))^{-\alpha} \nabla s\right) \times \|x_n - x\|.
\end{aligned}$$

Since f is a continuous function, we have $\mathcal{T}x_n \to \mathcal{T}x$ in $\mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$.
Step 2*:* the map $\mathcal{T}$ maps bounded sets into bounded sets in $\mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$. Indeed, it is enough to show that for any ε there exists a positive constant δ such that, for each $x \in B(0,\varepsilon)$, we have $\mathcal{T}x \in B(0,\delta)$. By hypothesis, for each $t \in \mathcal{J}_{a,b}$, we get

$$\begin{aligned}
|\mathcal{T}x(t)| &\leqslant \frac{1}{|\beta|}\int_a^t r(s)\, x \nabla s + \frac{1}{|\beta|\Gamma(1-\alpha)}\int_a^t \left|y^{\nabla}(s)\right| (y(t)-y(s))^{-\alpha} |x(s)| \nabla s \\
&\leqslant \frac{1}{|\beta|}\int_a^b r(s) \nabla s + \frac{\varepsilon}{|\lambda|} \sup_{t\in\mathcal{J}_{a,b}} \int_a^t \left|y^{\nabla}(s)\right| (y(t)-y(s))^{-\alpha} |x(s)| \nabla s \\
&= \delta.
\end{aligned}$$

Step 3*:* the map $\mathcal{T}$ maps bounded sets into equicontinuous sets of $\mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$. Let $t_1, t_2 \in \mathcal{J}_{a,b}$, $t_1 < t_2$ and $B(0,\varepsilon)$ be a bounded set of $\mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$. For all $x \in B(0,\varepsilon)$, we get

$$\begin{aligned}
|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &\leqslant \frac{1}{|\beta|}\int_{t_1}^{t_2} r(s)\, \varphi \nabla s \\
&\quad + \frac{1}{|\beta|\Gamma(1-\alpha)}\int_a^{t_2} \left|y^{\nabla}(s)\right| (y(t_2)-y(s))^{-\alpha} |x(s)| \nabla s \\
&\quad - \frac{1}{|\beta|\Gamma(1-\alpha)}\int_a^{t_1} \left|y^{\nabla}(s)\right| (y(t_1)-y(s))^{-\alpha} |x(s)| \nabla s.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we conclude that $\mathcal{T} : \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R}) \to \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$ is completely continuous. As a consequence of Schauder's fixed point theorem, we conclude that $\mathcal{T}$ has a fixed point, which is solution of the problem (1.1). □

4. Example

In this section, we give an example to illustrate our main result.

Example 4.1. Let $\mathbb{T} = \mathbb{Z}$, $a = 1$ and $b = m$, where $m \in \mathbb{N}$. We consider the following initial value problem:

$$\begin{cases} \beta x^{\nabla}(t) + \left({}_{1}^{\mathbb{Z}}D_{t,\nabla}^{\frac{1}{2}} \right) x(t) = x(t), \text{ for } t \in \mathcal{J}_{1,m-1}, \\ x(1) = 0. \end{cases} \tag{4.1}$$

Here, $|\beta| \geqslant 3$, $=\frac{1}{2}$, $y(t) = t+1$ and $f(t,x) = x$, for $t \in [1, m-1]_{\mathbb{Z}}$ and $x \in \mathbb{R}$, we find that the problem (4.1) is a private case of the problem (1.1) in $\mathbb{T} = \mathbb{Z}$. Then (3.1) holds. If $m \in \left[1, \frac{|\beta|}{3}\right]$, then (3.2) holds, Thus, the conditions of Theorem 3.5 are satisfied, and we conclude that there is a function $x \in \mathcal{C}(\mathcal{J}_{1,m-1}, \mathbb{R})$ the unique solution of (4.1).

5. Conclusion

We take $\mathbb{T} = \mathbb{N}$, $a = 1$ and $h, g : \mathbb{T} \to \mathbb{R}$, $h(t) = 1$, $g(t) = t$. Let $\alpha > 0$ and $b > 0$, then by Definition 3.1, we have

$${}_{a}^{\mathbb{N}}I_{g,t,\nabla}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} g^{\nabla}(s) \left(g(t) - g(s)\right)^{\alpha-1} h(s) \nabla s,$$

$$\begin{aligned} {}_{a}^{\mathbb{N}}I_{g,t,\nabla}^{\alpha} h(t) &= \frac{1}{\Gamma(\alpha)} \int_{1}^{t} (t-s)^{\alpha-1} \nabla s = \frac{1}{\Gamma(\alpha)} \sum_{s=2}^{s=t} (t-s)^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{s=t-2} s^{\alpha-1} = \psi(t). \end{aligned}$$

By, the last equality, deduce

$${}_{a}^{\mathbb{N}}I_{g,t,\nabla}^{\alpha+\beta} h(t) = \frac{1}{\Gamma(\alpha+\beta)} \sum_{s=1}^{s=t-2} s^{\alpha+\beta-1}.$$

On the other hand, we have

$$\begin{aligned} \left({}_{a}^{\mathbb{N}}I_{g,t,\nabla}^{\beta} \circ {}_{a}^{\mathbb{N}} I_{g,t,\nabla}^{\alpha} \right) h(t) &= {}_{a}^{\mathbb{N}}I_{g,t,\nabla}^{\beta} \varphi(t) = \frac{1}{\Gamma(\beta)} \int_{1}^{t} (t-s)^{\beta-1} \varphi(s) \nabla s \\ &= \frac{1}{\Gamma(\beta)} \sum_{s=2}^{s=t} (t-s)^{\beta-1} \varphi(s). \end{aligned}$$

Thus

$$\left({}_{a}^{\mathbb{N}}I_{g,t,\nabla}^{\beta} \circ {}_{a}^{\mathbb{N}} I_{g,t,\nabla}^{\alpha} \right) h(3) \neq {}_{a}^{\mathbb{N}} I_{g,t,\nabla}^{\alpha} h(3),$$

we conclude that ${}_{a}^{\mathbb{T}}I_{g,t,\nabla}^{\beta} \circ {}_{a}^{\mathbb{T}} I_{g,t,\nabla}^{\alpha} = {}_{a}^{\mathbb{T}} I_{g,t,\nabla}^{\alpha}$, for $\alpha > 0$, $\beta > 0$ are not always correct on the time scales.

If we take $\beta = 0$ in the problem (1.1), we get

$$\begin{cases} {}_{a}^{\mathbb{T}}D_{y,t,\nabla}^{\alpha}\left(x\left(t\right)\right) = f\left(t, x\left(t\right)\right), & \text{for } t \in \mathcal{J}_{a,b}, \\ x\left(a\right) = 0. \end{cases} \tag{5.1}$$

We consider the following integral equation

$$x\left(t\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \int_{a}^{t} y^{\nabla}\left(s\right)\left(y\left(t\right) - y\left(s\right)\right)^{-\alpha} x\left(s\right) \nabla s, \quad \text{for } t \in \mathcal{J}_{a,b}. \tag{5.2}$$

Since ${}_{a}^{\mathbb{T}}D_{y,t,\nabla}^{\alpha} \circ_{a}^{\mathbb{N}} I_{y,t,\nabla}^{\beta} = \mathrm{Id}$, and ${}_{a}^{\mathbb{N}}I_{y,t,\nabla}^{\beta} \circ D_{y,t,\nabla}^{\alpha} = \mathrm{Id}$ for $\alpha > 0$ are not always correct defined on the time scales. Then, if x is a solution to the problem (5.1) it has no permanent relationship the solution of integral equation 5.2.

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