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On a fractional differential inclusion involving a generalized Caputo type derivative with certain fractional integral boundary conditions

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Abstract

We study a class of fractional differential inclusions defined by Caputo-Katugampola fractional derivative involving a nonconvex set-valued map in the presence of certain fractional integral boundary conditions. Using a technique developed by Filippov we establish an existence result for the problem considered under the hypothesis that the set-valued map is Lipschitz in the state variable. Also, based on a result concerning the arcwise connectedness of the fixed point set of a class of set-valued contractions, we prove the arcwise connectedness of the solution set of the problem considered. The paper is the first in literature which contains such kind of results in the framework of the problem studied

Keywords: differential inclusion, fractional derivative, measurable selection.
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1. Introduction

In the last decades we can see a strong development of the theory of fractional differential equations and inclusions [1, 2, 3] etc. The explanation is that fractional differential equations are very useful tools in order to model several physical phenomena. Fractional calculus contains several fractional derivatives; from them, the fractional derivative introduced by Caputo allows to use Cauchy conditions which have physical meanings.

Recently, a generalized Caputo type fractional derivative was introduced in [4]. This Caputo-Katugampola fractional derivative unifies the well known Caputo and Caputo-Hadamard fractional derivatives into a single form. Even if Katugampola fractional integral operator looks similar to Erdélyi-Kober operator [5] it is argued [4] that is not possible to derive Hadamard equivalence operators from Erdélyi-Kober type operators. We also mention that in some recent papers [6, 7, 8, 9] were obtained certain qualitative properties of solutions of fractional differential equations and inclusions defined by Caputo-Katugampola.

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The present paper is concerned with the following problem

$$D_C^{\alpha,\rho}x(t) \in F(t,x(t)) \quad a.e.\ ([0,T]), \tag{1.1}$$

$$x(T) = \sum_{i=1}^{m} \lambda_i I^{\beta,\rho}x(\eta_i) + \xi, \quad \delta x(0) = 0, \tag{1.2}$$

where $\alpha \in (1,2]$, $\rho > 0, \beta > 0$, $D_C^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative, $I^{\beta,\rho}$ is the Katugampola type fractional integral, $\lambda_i \in \mathbf{R}$, $\eta_i \in [0,T]$, $i = \overline{1,m}$, $\xi \in \mathbf{R}$, $\delta = t^{1-\rho}\frac{d}{dt}$ and $F : [0,T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

In a recent paper [10], problem (1.1)–(1.2) was studied and two existence results for this problem are provided taking into account the situations when the values of the set-valued map F are convex and nonconvex. The results in [10] are obtained by using fixed point techniques.

Our goal is to continue the study in [10]. Namely, our aim is twofold: on one hand, we show that Filippov's technique [11] may be used in order to obtain the existence of solutions for problem (1.1)–(1.2). On the other hand, taking into account a result in [12] concerning the arcwise connectedness of the fixed point set of a class of set-valued contractions, we deduce the arcwise connectedness of the solution set of problem (1.1)–(1.2).

We note that even if similar results for other classes of fractional differential inclusions defined by Riemann-Liouville, Caputo or Caputo-Fabrizio fractional derivatives exists in the literature [13, 14, 15] etc., the theorems in the present paper are new in the framework of problem (1.1)–(1.2).

It is worth to remark that as papers in the literature closely related to the topic of the present article one may mention [16, 17, 18, 19, 20, 21, 22, 23, 24, 25] etc..

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, Section 3 is devoted to our existence theorem and in Section 4 we obtain the arcwise connectedness of the solution set.

2. Preliminaries

Let denote by I the interval $[0,T]$, $T > 0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|dt$. Denote $AC^n(I,\mathbf{R}) = \{f : I \to \mathbf{R};\quad f, f'^{(n-1)} \in C(I,\mathbf{R})$ and $f^{(n-1)}$ is absolutely continuous$\}$ and $AC^n_\delta(I,\mathbf{R}) = \{f : I \to \mathbf{R};\quad \delta^{(n-1)}f$ is absolutely continuous; $\delta = t^{1-\rho}\frac{d}{dt}\}$

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$. We denote by $\mathcal{D}(I,X)$ the family of all decomposable closed subsets of $L^1(I,X)$.

Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0,1\}$ denotes the characteristic function of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, where $d^*(A, B) = \sup\{d(a, B); a \in A\}$ and $d(x, B) = \inf_{y \in B} d(x, y)$.

Let $\rho > 0$. The next notions were introduced in [4].

a) The generalized left-sided fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $h : (0, \infty) \to \mathbf{R}$ is defined by

$$I^{\alpha,\rho}h(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s) ds, \tag{2.1}$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

b) The generalized fractional derivative, corresponding to the generalized left-sided fractional integral in (2.1) of a function $h : [0, \infty) \to \mathbf{R}$ is defined by

$$D^{\alpha,\rho}h(t) = (t^{1-\rho}\frac{d}{dt})^n (I^{n-\alpha,\rho}h)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} (t^{1-\rho}\frac{d}{dt})^n \int_0^t \frac{s^{\rho-1}h(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds$$

if the integral exists and $n = [\alpha] + 1$.

c) The Caputo-Katugampola generalized fractional derivative is defined by

$$D_c^{\alpha,\rho}h(t) = (D^{\alpha,\rho}[h(s) - \sum_{k=0}^{n-1} \frac{h^{(k)}(0)}{k!} s^k])(t),$$

with $n = [\alpha] + 1$.

If $\rho = 1$, the Caputo-Katugampola fractional derivative is the well known Caputo fractional derivative and if we pass to the limit with $\rho \to 0+$ in the above definition we get the Caputo-Hadamard fractional derivative.

In what follows $\rho > 0$, $\alpha \in (1, 2]$, $\beta > 0$ and $\Omega = 1 - \sum_{i=1}^m \lambda_i \frac{\eta_i^{\rho\beta}}{\rho^\beta \Gamma(\beta+1)}$.

Lemma 2.1. *Assume that $\Omega \neq 0$. For a given function $h \in C(I, \mathbf{R}) \cap L^1(I, \mathbf{R})$ the unique solution $x \in AC_\delta^2(I, \mathbf{R})$ of the problem $D_C^{\alpha,\rho}x(t) = h(t)$ a.e. (I) with boundary condition (1.2) is given by*

$$x(t) = I^{\alpha,\rho}h(t) + \frac{1}{\Omega}\{-I^{\alpha,\rho}h(T) + \sum_{i=1}^m \lambda_i I^{\alpha+\beta,\rho}h(\eta_i) + \xi\}, \quad t \in I., \tag{2.2}$$

For the proof of Lemma 2.1, see [10]; namely, Lemma 2.3.

Remark 2.2. If we define

$$\mathcal{G}(t, s) = \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)(t^\rho - s^\rho)^{1-\alpha}}\chi_{[0,t]}(s) - \frac{\rho^{1-\alpha}s^{\rho-1}}{\Omega\Gamma(\alpha)(T^\rho - s^\rho)^{1-\alpha}} + \sum_{i=1}^m \frac{\lambda_i \rho^{1-\alpha-\beta}s^{\rho-1}}{\Omega\Gamma(\alpha+\beta)(\eta_i^\rho - s^\rho)^{1-\alpha-\beta}}\chi_{[0,\eta_i]}(s)$$

then the solution in (2.2) may be written as

$$x(t) = \frac{\xi}{\Omega} + \int_0^T \mathcal{G}(t, s)h(s)ds. \tag{2.3}$$

Moreover, for any $s, t \in I$ we have the estimate

$$|\mathcal{G}(t,s)| \leqslant \frac{\rho^{1-\alpha}T^{\rho\alpha-1}}{\Gamma(\alpha)} + \frac{\rho^{1-\alpha}T^{\rho\alpha-1}}{|\Omega|\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{|\lambda_i|\rho^{1-\alpha-\beta}\eta_i^{\rho(\alpha+\beta)-1}}{|\Omega|\Gamma(\alpha+\beta)} =: M.$$

A function $x(.) \in AC^2_\delta(I, \mathbf{R})$ is called a solution of problem (1.1)–(1.2) if there exists a function $h(.) \in L^1(I, \mathbf{R})$ with $h(t) \in F(t, x(t))$ a.e. (I) and such that (2.2) is satisfied.

3. An existence result

The next lemma [26] contains a selection result for set-valued maps and is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem [27].

Lemma 3.1. *Consider X a separable Banach space, B is the closed unit ball in X, $F : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $c : I \to X, r : I \to \mathbf{R}_+$ are measurable functions. If*

$$F(t) \cap (c(t) + r(t)B) \neq \emptyset \quad \text{a.e. (I)},$$

then the set-valued map $t \to F(t) \cap (c(t) + r(t)B)$ has a measurable selection.

In the sequel we assume the following conditions on F.

Hypothesis H1. i) $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every $x \in \mathbf{R}$ $F(., x)$ is measurable.*

ii) *There exists $m \in L^1(I, \mathbf{R})$ such that for almost all $t \in I, F(t, .)$ is $m(t)$-Lipschitz in the sense that for all $x, y \in \mathbf{R}$*

$$d_H(F(t, x), F(t, y)) \leqslant m(t)|x - y|.$$

Theorem 3.2. *Assume that Hypothesis H1 is satisfied, assume that $M|m|_1 < 1$ and let $z(.) \in AC^2_\delta(I, \mathbf{R})$ be such that there exists $q(.) \in L^1(I, \mathbf{R})$ with $d(D_C^{\alpha,\rho} z(t), F(t, z(t))) \leqslant q(t)$ a.e. (I) and $z(T) = \sum_{i=1}^m \lambda_i I^{\beta,\rho} z(\eta_i) + \mu$, $\delta z(0) = 0$.*

Then there exists $x(.) : I \to \mathbf{R}$ a solution of problem (1.1)–(1.2) satisfying for all $t \in I$

$$|x(t) - z(t)| \leqslant \frac{1}{1 - M|m|_1} \frac{|\xi - \mu|}{|\Omega|} + \frac{M}{1 - M|m|_1}|q|_1. \tag{3.1}$$

Proof. We note first that the set-valued map $t \to F(t, z(t))$ is measurable with closed values and the fact that $d(D_C^{\alpha,\rho} z(t), F(t, z(t))) \leqslant q(t)$ a.e. (I) means

$$F(t, z(t)) \cap \{D_C^{\alpha,\rho} z(t) + q(t)[-1, 1]\} \neq \emptyset \quad \text{a.e. (I)}.$$

From Lemma 3.1 there exists a measurable selection $h_1(t) \in F(t, z(t))$ a.e. (I) such that

$$|h_1(t) - D_C^{\alpha,\rho} z(t)| \leqslant q(t) \quad \text{a.e. (I)}. \tag{3.2}$$

Define $x_1(t) = \frac{\xi}{\Omega} + \int_0^T \mathcal{G}(t,s)h_1(s)ds$ and one has

$$|x_1(t) - z(t)| = |\tfrac{\xi-\mu}{\Omega} + \int_0^T \mathcal{G}(t,s)(h_1(s) - D_C^{\alpha,\rho} z(s))ds| \leqslant \\ \tfrac{|\xi-\mu|}{|\Omega|} + \int_0^T |\mathcal{G}(t,s)|q(s)ds \leqslant \tfrac{|\xi-\mu|}{|\Omega|} + M|q|_1.$$

We construct next two sequences $x_n(.) \in C(I, \mathbf{R})$, $h_n(.) \in L^1(I, \mathbf{R})$, $n \geqslant 1$ such that

$$x_n(t) = \frac{\xi}{\Omega} + \int_0^T \mathcal{G}(t,s)h_n(s)ds, \quad t \in I, \tag{3.3}$$

$$h_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (I),\ n \geqslant 1, \tag{3.4}$$

$$|h_{n+1}(t) - h_n(t)| \leqslant m(t)|x_n(t) - x_{n-1}(t)| \quad a.e.\ (I), n \geqslant 1. \tag{3.5}$$

Assuming that this construction is done, then from (3.3)–(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leqslant \int_0^T |\mathcal{G}(t,t_1)|.|h_{n+1}(t_1) - h_n(t_1)|dt_1 \leqslant M\int_0^T m(t_1)|x_n(t_1)$$
$$-x_{n-1}(t_1)|dt_1 \leqslant M\int_0^T m(t_1)\int_0^T |\mathcal{G}(t_1,t_2)|.|h_n(t_2) - h_{n-1}(t_2)|dt_2 \leqslant$$
$$M^2\int_0^T m(t_1)\int_0^T m(t_2)|x_{n-1}(t_2) - x_{n-2}(t_2)|dt_2dt_1 \leqslant M^n\int_0^T m(t_1)\int_0^T m(t_2)...$$
$$\int_0^T m(t_n)|x_1(t_n) - z(t_n)|dt_n...dt_1 \leqslant (M|m|_1)^n(\frac{|\xi-\mu|}{|\Omega|} + M|q|_1).$$

It means that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence converging uniformly to some $x \in C(I, \mathbf{R})$. At the same time, by (3.4), for almost all $t \in I$, the sequence $\{h_n(t)\}_{n\in\mathbf{N}}$ is Cauchy in $\mathbf{R}$. Take h the pointwise limit of h_n. Moreover, one has

$$\begin{aligned} &|x_n(t) - z(t)| \leqslant |x_1(t) - z(t)| + \sum_{i=1}^{n-1}|x_{i+1}(t) - x_i(t)| \leqslant \frac{|\xi-\mu|}{|\Omega|} \\ &+M|q|_1 + \sum_{i=1}^{n-1}(\frac{|\xi-\mu|}{|\Omega|} + M|q|_1)(M|m|_1)^i = \frac{\frac{|\xi-\mu|}{|\Omega|} + M|q|_1}{1-M|m|_1}. \end{aligned} \tag{3.6}$$

Similarly, from (3.2), (3.5) and (3.6) we deduce for almost all $t \in I$

$$|h_n(t) - D_C^{\alpha,\rho}z(t)| \leqslant \sum_{i=1}^{n-1}|h_{i+1}(t) - h_i(t)| + |h_1(t) - D_C^{\alpha,\rho}z(t)| \leqslant$$
$$L(t)\frac{\frac{|\xi-\mu|}{|\Omega|} + M|q|_1}{1-M|m|_1} + q(t).$$

So, the sequence h_n is integrably bounded and therefore $h \in L^1(I, \mathbf{R})$.

With Lebesque's dominated convergence theorem, passing to the limit in (3.3), (3.4) we conclude that x is a solution of (1.1). Finally, taking the limit in (3.6) we deduce the estimate on x.

Next we construct the sequences x_n, h_n with the properties in (3.3)– (3.5). This will be achevied by induction.

At the beginning of the proof we already provided the first step of induction. We assume that for some $p \geqslant 1$ we already constructed $x_n \in C(I, \mathbf{R})$ and $h_n \in L^1(I, \mathbf{R}), n = 1, 2, ...p$ satisfying (3.3), (3.5) for $n = 1, 2, ...p$ and (3.4) for $n = 1, 2, ...p - 1$. The maps $t \to F(t, x_p(t))$ and $t \to L(t)|x_p(t) - x_{p-1}(t)|$ are measurable. But $F(t, .)$ is Lipschitz, hence for almost all $t \in I$

$$F(t, x_p(t)) \cap \{h_p(t) + m(t)|x_p(t) - x_{p-1}(t)|[-1, 1]\} \neq \emptyset.$$

Now we apply Lemma 3.1 in order to deduce the existence of a measurable selection $h_{p+1}(.)$ of $F(., x_p(.))$ such that

$$|h_{p+1}(t) - h_p(t)| \leqslant m(t)|x_p(t) - x_{p-1}(t)| \quad a.e.\ (I).$$

We put x_{p+1} like in (3.3) with $n = p + 1$. So, h_{p+1} verifies (3.4) and (3.5) and the proof is finished. □

The assumptions in Theorem 3.2 are satisfied, in particular, for $z(.) = 0$ and with $q(.) = m(.)$. We obtain the following consequence of Theorem 3.2.

Corollary 3.3. *Assume that Hypothesis H1 is satisfied, assume that* $M|m|_1 < 1$ *and* $d(0, F(t, 0)) \leqslant m(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.1)–(1.2) satisfying for all* $t \in I$

$$|x(t)| \leqslant \frac{1}{1 - M|m|_1} \frac{|\xi|}{|\Omega|} + \frac{M}{1 - M|m|_1}|q|_1. \tag{3.7}$$

Remark 3.4. A similar existence result to the one in Corollary 3.3 may be found in [10], namely Theorem 4.2. Its proof is performed by using the set-valued contraction principle. It is worth to mention that the approach in [10], apart from the requirement that the values of $F(.,.)$ are compact, does not provides a priori bounds as in (3.7).

Example 3.5. As an application of our above result one may consider the following example taken from [10]. Consider the following problem

$$D_C^{\frac{5}{4},\frac{1}{3}}x(t) \in [\frac{e^{-t}}{\sqrt{900+t}}(\tan^{-1}x(t) + \frac{1}{2}), \frac{1+t}{30}(\frac{|x(t)|}{|x(t)|+1} + \frac{1}{8})] \quad a.e.\ ([0, 2]), \tag{3.8}$$

$$x(2) = 2I^{\frac{3}{4},\frac{1}{3}}x(\frac{1}{2}) + \frac{1}{2}I^{\frac{3}{4},\frac{1}{3}}x(\frac{3}{2}) + \frac{1}{4}, \quad \delta x(0) = 0. \tag{3.9}$$

In this case, $T = 2$, $\alpha = \frac{5}{4}$, $\beta = \frac{3}{4}$, $\rho = \frac{1}{3}$, $m = 2$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{3}{2}$, $\xi = \frac{1}{4}$, $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$, $F(t, x) = [\frac{e^{-t}}{\sqrt{900+t}}(\tan^{-1}x + \frac{1}{2}), \frac{1+t}{30}(\frac{|x|}{|x|+1} + \frac{1}{8})]$ and $m(t) = \frac{1+t}{30}$. Since $M|m|_1 \approx 0,7572001 < 1$, we are able to apply Corollary 3.3 in order to deduce the existence of a solution for problem (3.8)–(3.9).

4. Arcwise connectedness of the solution set

In this section we are concerned with the more general problem

$$D_C^{\alpha,\rho}x(t) \in F(t, x(t), S(t, x(t))) \quad a.e.\ (I), \tag{4.1}$$

$$x(T) = \sum_{i=1}^{m} \lambda_i I^{\beta,\rho}x(\eta_i) + \xi, \quad \delta x(0) = 0. \tag{4.2}$$

We are working under the hypothesis that F and S are closed-valued Lipschitz with respect to the second variable and F is a contraction in the third variable. It is clear that the right-hand side of the differential inclusion in (4.1) is, in general, neither convex nor closed. However, we may prove that its solution set is the arcwise connectedness. Obviously, when F does not depend on the last variable (4.1) reduces to (1.1) and the result is still valid for problem (1.1).

Let Z be a metric space with the distance d_Z. In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces $Z_i, i = 1, 2$, is considered, it is assumed that Z is endowed with the metric $d_Z((z_1, z_2), (z_1', z_2')) = \sum_{i=1}^{2} d_{Z_i}(z_i, z_i')$.

Let X be a metric space and let $F : X \to \mathcal{P}(Z)$ be a set-valued map with nonempty closed values. F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x, x_0) < \delta$ implies $d_H(F(x), F(x_0)) < \epsilon$.

Consider $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |.|_X)$ be a Banach space.

We need two preliminary results in order to establish our result. To simplify the notation we write E in place of $L^1(T, X)$. The following lemmas are proved in [12].

Lemma 4.1. *$\Phi : T \times E \to \mathcal{P}(E)$ and $\Psi : T \times E \times E \to \mathcal{P}(E)$ are Hausdorff continuous multifunctions with nonempty, closed, decomposable values and satisfy*

a) There exists $Q \in [0, 1)$ such that, for every $t \in T$ and every $u, u' \in E$,

$$d_H(\Phi(t, u), \Phi(t, u')) \leqslant Q|u - u'|_E.$$

b) There exists $L \in [0, 1)$ such that $Q + L < 1$ and for every $t \in T$ and every (u, v), $(u', v') \in E \times E$,

$$d_H(\Psi(t, u, v), \Psi(t, u', v')) \leqslant L(|u - u'|_E + |v - v'|_E).$$

Define $\mathrm{Fix}(\Lambda(t, .)) = \{u \in E; u \in \Lambda(t, u)\}$, where $\Lambda(t, u) = \Psi(t, u, \Phi(t, u))$, $(t, u) \in T \times E$.

Then for every $t \in T$ the set $\mathrm{Fix}(\Lambda(t, .))$ is nonempty and arcwise connected.

Lemma 4.2. *Consider $A : T \to \mathcal{P}(X)$ and $B : T \times X \to \mathcal{P}(X)$ be two nonempty closed-valued set-valued maps verifying*

a) A is measurable and there exists $k \in L^1(T)$ such that $d_H(A(t), \{0\}) \leqslant k(t)$ for almost all $t \in T$.

b) The set-valued map $t \to B(t, x)$ is measurable for every $x \in X$.

c) The set-valued map $x \to B(t, x)$ is Hausdorff continuous for all $t \in T$.

Consider $b : T \to X$ a measurable selection from $t \to B(t, A(t))$.

Then there exists a selection $a \in L^1(T, X)$ of $A(.)$ such that $b(t) \in B(t, a(t))$, $t \in T$.

Hypothesis H2. $F : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $S : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ are set-valued maps with nonempty closed values and with the following properties

i) The set-valued maps $t \to F(t, u, v)$ and $t \to S(t, u)$ are measurable for all $u, v \in \mathbf{R}$.

ii) There exists $m \in L^1(I, \mathbf{R}_+)$ such that, for every $u, u' \in \mathbf{R}$,

$$d_H(S(t, u), S(t, u')) \leqslant m(t)|u - u'| \quad a.e. \ (I).$$

iii) There exist $p \in L^1(I, \mathbf{R}_+)$ and $\theta \in [0, 1)$ such that, for every $u, v, u', v \in \mathbf{R}$,

$$d_H(F(t, u, v), F(t, u', v')) \leqslant p(t)|u - u'| + \theta|v - v'| \quad a.e. \ (I).$$

iv) There exist $f, s \in L^1(I, \mathbf{R}_+)$ such that

$$d(0, F(t, 0, 0)) \leqslant f(t), \quad d(0, S(t, 0)) \leqslant s(t) \quad a.e. \ (I).$$

For fixed $\xi \in \mathbf{R}$ we denote by $\mathcal{S}(\xi)$ the solution set of (4.1)–(4.2).

In what follows $N(t) := \max\{m(t), p(t)\}$, $t \in I$.

Theorem 4.3. *Assume that Hypothesis H2 is satisfied and* $2M\int_0^T N(s)ds + \theta < 1$.

Then for every $\xi \in \mathbf{R}$, *the solution set* $\mathcal{S}(\xi)$ *of* (4.1)–(4.2) *is nonempty and arcwise connected in the space* $C(I, \mathbf{R})$.

Proof. For $\xi \in \mathbf{R}$ and $u \in L^1(I, \mathbf{R})$, set

$$u_\xi(t) = \frac{\xi}{\Omega} + \int_0^T \mathcal{G}(t,s)u(s)ds, \quad t \in I.$$

We prove first that $\Phi : \mathbf{R} \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ and $\Psi : \mathbf{R} \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ given by

$$\Phi(\xi, u) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in S(t, u_\xi(t)) \quad a.e.\ (I)\},$$

$$\Psi(\xi, u, v) = \{w \in L^1(I, \mathbf{R}); \quad w(t) \in F(t, u_\xi(t), v(t)) \quad a.e.\ (I)\},$$

meet the assumptions in Lemma 4.1.

Based on the fact that u_ξ is measurable and S satisfies Hypothesis H2 i) and ii), the set-valued map $t \to S(t, u_\xi(t))$ is measurable and nonempty closed valued, thus it has a measurable selection. By Hypothesis H2 iv), the set $\Phi(\xi, u)$ is nonempty. The fact that the set $\Phi(\xi, u)$ is closed and decomposable is a standard argument. Similarly, $\Psi(\xi, u, v)$ is a nonempty closed decomposable set.

Take $(\xi, u), (\xi_1, u_1) \in \mathbf{R} \times L^1(I, \mathbf{R})$ and consider $v \in \Phi(\xi, u)$. For every $\varepsilon > 0$ there exists $v_1 \in \Phi(\xi_1, u_1)$ such that, for every $t \in I$, one has

$$\begin{aligned}&|v(t) - v_1(t)| \leqslant d_H(S(t, u_\xi(t)), S(t, u_{\xi_1}(t))) + \varepsilon \\ &\leqslant N(t)[\tfrac{1}{|\Omega|}|\xi - \xi_1| + \textstyle\int_0^T |\mathcal{G}(t,s)|.|u(s) - u_1(s)|ds] + \varepsilon.\end{aligned}$$

Consequently,

$$|v - v_1|_1 \leqslant \frac{1}{|\Omega|}|\xi - \xi_1|.\int_0^T N(t)dt + M\int_0^T N(t)dt|u - u_1|_1 + T\varepsilon$$

for any $\varepsilon > 0$.

Thus,

$$d_{L^1(I,\mathbf{R})}(v, \Phi(\xi_1, u_1)) \leqslant \frac{1}{|\Omega|}|\xi - \xi_1|.\int_0^T N(t)dt + M\int_0^T N(t)dt|u - u_1|_1$$

for all $v \in \Phi(\xi, u)$. Therefore,

$$d_H(\Phi(\xi, u), \Phi(\xi_1, u_1)) \leqslant \frac{1}{|\Omega|}|\xi - \xi_1|.\int_0^T N(t)dt + M\int_0^T N(t)dt|u - u_1|_1$$

which means that Φ verifies the hypothesis of Lemma 4.1 and is Hausdorff continuous.

Take $(\xi, u, v), (\xi_1, u_1, v_1) \in \mathbf{R} \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and $w \in \Psi(\xi, u, v)$. As before, for each $\varepsilon > 0$ there exists $w_1 \in \Psi(\xi_1, u_1, v_1)$ such that for every $t \in I$

$$\begin{aligned}&|w(t) - w_1(t)| \leqslant d_H(F(t, u_\xi(t), v(t)), F(t, u_{\xi_1}(t), v_1(t))) + \varepsilon \\ &\leqslant N(t)|u_\xi(t) - u_{\xi_1}(t)| + \theta|v(t) - v_1(t)| + \varepsilon \\ &\leqslant N(t)[|\tfrac{\xi}{\Omega} - \tfrac{\xi_1}{\Omega}| + \textstyle\int_0^T |\mathcal{G}(t,s)|.|u(s) - u_1(s)|ds] + \theta|v(t) - v_1(t)| + \varepsilon \\ &\leqslant N(t)[\tfrac{1}{|\Omega|}|\xi - \xi_1| + M|u - u_1|_1] + \theta|v(t) - v_1(t)| + \varepsilon.\end{aligned}$$

Thus,

$$|w-w_1|_1 \leqslant \frac{1}{|\Omega|}|\xi-\xi_1|\int_0^T N(t)dt + M\int_0^T N(t)dt|u-u_1|_1+\theta|v-v_1|_1+T\varepsilon$$
$$\leqslant \frac{1}{|\Omega|}|\xi-\xi_1|\int_0^T N(t)dt + (M\int_0^T N(t)dt+\theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1))+T\varepsilon.$$

We deduce that

$$d_H(\Psi(\xi,u,v),\Psi(\xi_1,u_1,v_1)) \leqslant \frac{1}{|\Omega|}|\xi-\xi_1|\int_0^T N(t)dt+$$
$$(M\int_0^T N(t)dt+\theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1)),$$

i.e., Ψ verifies the hypothesis of Lemma 4.1 and is Hausdorff continuous.

We introduce now $\Lambda(\xi,u)=\Psi(\xi,u,\Phi(\xi,u))$, $(\xi,u)\in\mathbf{R}^2\times L^1(I,\mathbf{R})$. With Lemma 4.1 we find that the set $Fix(\Lambda(\xi,.))=\{u\in L^1(I,\mathbf{R}); u\in\Lambda(\xi,u)\}$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$.

Finally, we prove that

$$Fix(\Lambda(\xi,.))=\{u\in L^1(I,\mathbf{R});\ u(t)\in F(t,u_\xi(t),S(t,u_\xi(t)))\ a.e.\ (I)\}. \tag{4.3}$$

Let $\mathcal{Z}(\xi)$ be the right-hand side of (4.3). If $u\in Fix(\Lambda(\xi,.))$ then there is $v\in\Phi(\xi,v)$ such that $u\in\Psi(\xi,u,v)$. Hence, $v(t)\in S(t,u_\xi(t))$ and

$$u(t)\in F(t,u_\xi(t),v(t))\subset F(t,u_\xi(t),S(t,u_\xi(t)))\quad a.e.\ (I),$$

i.e., $Fix(\Lambda(\xi,.))\subset\mathcal{Z}(\xi)$.

Take $u\in\mathcal{Z}(\xi)$. By Lemma 4.2, there exists $v\in L^1(I,\mathbf{R})$ a selection of $t\to S(t,u_\xi(t)))$ such that $u(t)\in F(t,u_\xi(t),v(t))$ a.e. (I). Thus, $v\in\Phi(\xi,v)$, $u\in\Psi(\xi,u,v)$, hence $u\in\Lambda(\xi,u)$ and equality (4.3) is proved. We remark that the function $\mathcal{P}: L^1(I,\mathbf{R})\to C(I,\mathbf{R})$, $\mathcal{P}(u)(t):=\int_0^T \mathcal{G}(t,s)u(s)ds$, $t\in I$ is continuous and one has

$$\mathcal{S}(\xi)=\frac{\xi}{\Omega}+\mathcal{P}(Fix(\Gamma(\xi,.))),\quad \xi\in\mathbf{R}.$$

But $Fix(\Lambda(\xi,.))$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$, therefore the set $S(\xi)$ has the same properties in $C(I,\mathbf{R})$. □

5. Conclusions

In the present paper, we studied a class of fractional differential inclusions involving Caputo-Katugampola fractional derivative with certain fractional integral boundary conditions. We established an existence result for problem (1.1)–(1.2) when the set-valued map is Lipschitz in the state variable without any assumptions concerning the convexity of the values of the set-valued map. Our approach uses a technique due to Filippov ([11]) instead of an usual application of set-valued fixed point theorems. In this way we improved a similar existence result in the literature [10], obtained by using the set-valued contraction principle. An illustration of our result is provided by a numerical example. At the same time, we obtained a topological property of the solution set of the problem considered; namely it is proved the arcwise connectedness of the solution set of problem (1.1)–(1.2).

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