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On the iterative methods for solving fractional initial value problems: new perspective

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Abstract

In this short communication, we introduce a new perspective for a numerical solution of fractional initial value problems (FIVPs). Basically, we split the considered FIVP into FIVPs on subdomains which can be solved iteratively to obtain the approximate solution for the whole domain.

Keywords: Fractional initial value problem, Fractional integral operator, Iterative methods.
2010 MSC: 34A08, 26A33, 65F10.

1. Introduction

In recent years, huge interests from scientists in modelling problems in the fields of fluid mechanics, electromagnetic, acoustics, chemistry, biology, physics and material sciences using fractional differential equations (FDEs); see, by way of example not exhaustive enumeration, [1, 2, 3, 4, 5, 6, 7, 8, 9]. However, unlike the integer differential equations (DEs), the determination of the exact solutions of FDEs is so complicated. Therefore, here is a race between researchers to discover accurate, simple and efficient numerical schemes to approximate the exact solutions. For instance, Kazem et al. [10] applied fractional-order Legendre Spectral Galerkin method, Bhrawy and Zaky [11] implemented the shifted fractional-order Jacobi orthogonal functions, and Rehman and Khan [12] implemented the Legendre wavelet method. It should be noted that the fractional-Legendre functions have been implemented in the so-called fractional-Legendre-Galerkin spectral method by few researchers in order to solve several types of fractional ordinary differential equations, see for example Kazem et al. [10], Klimek and Agrawal [13], Bhrawy and Alghamdi [14], Yiming et al. [15], Bhrawy et al. [16],

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Adams-Basforth method [17, 18] and Monotone iterative techniques [25, 24]. It is worth mentioning herein, that all the above numerical schemes are implemented on the whole domain which creates accumulative error that may significantly affects the accuracy and simplicity of numerical schemes. Consequently, our target is to handle this problem by proposing a new iterative technique that allow us to solve the FIDEs on subdomains. The explanation of this method will applied on the following fractional initial value problems of the form:

$$D_a^\alpha y(t) = f(t, y(t)) \quad t \in (a, T], \quad 0 < \alpha \leqslant 1, \tag{1.1}$$

subject to

$$y(a) = h_a, \tag{1.2}$$

where $h_a \in \mathbb{R}$ and $y \in L_1(a, T)$. The notation D_a^α denotes the fractional derivative. Without loss of generality, we assume D_a^α be the left-sided fractional derivative of Caputo sense.

2. Preliminary Results

In this section, we present some basic definitions and properties of fractional calculus theory.

Definition 2.1. The left-sided Riemann-Liouville fractional integral operator of order α is defined by

$$J_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \quad \alpha \in \mathbb{R}, \tag{2.1}$$

where, $t \in [a, T]$, y belongs to the Lebesgue space $L_1[a, T]$ and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds.$$

Lemma 2.2. *([19, 20, 21]) Let $\alpha, \beta > 0$, $t \in [a, T]$, $\gamma > -1$ and $y \in L_1[a, T]$. Then*

(i) $J_a^0 y(t) = y(t)$,

(ii) $J_a^\alpha J_a^\beta y(t) = J_a^{\alpha+\beta} y(t) = J_a^\beta J_a^\alpha\, y(t)$.

Definition 2.3. For $\alpha \in \mathbb{R}$, $m = \lceil \alpha \rceil$ and $t \in [a, T]$, the left-sided Caputo fractional derivatives operator is defined as:

$$D_a^\alpha y(t) = J_a^{m-\alpha} y^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau, \tag{2.2}$$

provided the integral exists. This operator was introduced by the Italian mathematician Caputo in 1967, see [22].

Lemma 2.4. *For $\alpha \in \mathbb{R}$, $m = \lceil \alpha \rceil$, $t \in [a, T]$ and $y \in L_1[a, T]$, we have*

1. $D_a^\alpha\, J_a^\alpha y(t) = y(t)$.
2. $J_a^\alpha\, D_a^\alpha y(t) = y(t) - \sum_{k=0}^{m-1} y^{(k)}(a) \frac{t^k}{k!}$.

3. Main results

In this section, for the sake of showing the motivation of our method, we solve the fractional differential equation .

$$D_a^{\alpha} y(t) = f(t, y(t)), \quad y(a) = h_a, \ t \in I := [a, T], \ \alpha \in (0, 1) \tag{3.1}$$

The domain $[a, T]$ is divided into $N-$subintervals with the grid points $t_n = a + nh$, $n = 0, \cdots, N$. Here h represents a uniform step size; $h = (T - a)/N$. Hence, the domain can be written as $I = \bigcup_{n=1}^{N} I_n$ where $I_n = [t_{n-1}, t_n]$. Write the solution in the domain decomposition form $y(t) = \sum_{n=1}^{N} y_n(t)$, where

$$y_n(t) = \begin{cases} y(t), & t \in I_n \\ 0, & o.w. \end{cases} \tag{3.2}$$

It should be noted herein that since $\alpha \in (0, 1]$, we assume the continuity of the solution at the nodes t_{n-1}, i.e., $y_n(t_{n-1}) = y_{n-1}(t_{n-1})$ for all $n \geqslant 2$. The following Lemma is the heart of the motivation in this work.

Lemma 3.1. *For $t \in I_n$, equation (3.1) reduces to*

$$D_{t_{n-1}}^{\alpha} y_n(t) = f(t, y_n(t)), \quad y_n(t_{n-1}) = \begin{cases} y_{n-1}(t_{n-1}), & n \geqslant 2 \\ y_0, & n = 1 \end{cases}. \tag{3.3}$$

Proof. Since $t \in I_n$, then based on the restrictions on $y(t)$ (3.2), $y(t) = y_n(t)$. In addition, applying the continuity condition at $t = t_{n-1}$, the results is obtained. □

As a result of Lemma 1, we may easily convert (3.3) to integral equation as shown in the next lemma.

Lemma 3.2. *For $t \in I_n$, the FIVP (3.3) on sub-domain I_n is equivalent to the following integral equation:*

$$y_n(t) = y_{n-1}(t_{n-1}) + \frac{1}{\Gamma(\alpha)} \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} f(\tau, y_n(\tau)) d\tau. \tag{3.4}$$

Proof. Applying the integral operator $J_{t_{n-1}}^{\alpha}$ to both sides of equation (3.3), one obtains

$$\begin{aligned} J_{t_{n-1}}^{\alpha} D_{t_{n-1}}^{\alpha} y_n(t) &= J_{t_{n-1}}^{\alpha} f(t, y_n(t)) \\ y_n(t) - y_n(t_{n-1}) &= \frac{1}{\Gamma(\alpha)} \int_{t_{n-1}}^{t} (t - \tau)^{\alpha - 1} f(\tau, y_n(\tau)) d\tau. \end{aligned}$$

Using the continuity condition at the node t_{n-1}, i.e., $y_n(t_{n-1}) = y_{n-1}(t_{n-1})$, the result is obtained. □

The benefits of the present algorithm can be easily observed especially in the large domains. Additionally, it can be applied for any numerical technique. In the next section we will present one example to show the efficiency of the present work.

A simple algorithm can be derived from the integral equation (3.4) is by approximating the function $f(\tau, y_n(\tau))$ at the point $t = t_{n-1}$; i.e., $f(\tau, y_n(\tau)) \approx f(t_{n-1}, y_{n-1}(t_{n-1}))$. Consequently, the integral equation (3.4) will be given by

$$\begin{aligned} y_n(t) &= y_{n-1}(t_{n-1}) + \frac{f(t_{n-1}, y_{n-1}(t_{n-1}))}{\Gamma(\alpha)} \int_{t_{n-1}}^{t} (t-\tau)^{\alpha-1} d\tau \\ &= y_{n-1}(t_{n-1}) + \frac{f(t_{n-1}, y_{n-1}(t_{n-1}))}{\alpha\Gamma(\alpha)} (t-t_{n-1})^{\alpha} \qquad (3.5) \\ &= y_{n-1}(t_{n-1}) + \frac{f(t_{n-1}, y_{n-1}(t_{n-1}))}{\Gamma(\alpha+1)} (t-t_{n-1})^{\alpha}. \qquad (3.6) \end{aligned}$$

Notice that replacing t by t_n will lead to the well-known algorithm, fractional explicit Euler method [23]. On the other hand, we can easily obtain a fractional implicit method at $t = t_n$ by approximating $f(\tau, y(\tau))$ in the integral equation (3.4) using the nodes t_{n-1} and t_n:

$$f(\tau, y_n(\tau)) \approx \frac{1}{h}[(\tau - t_{n-1}) f(t_n, y_n(t_n)) - (\tau - t_n) f(t_{n-1}, y_n(t_{n-1}))].$$

Therefore, (3.4) will be converted to

$$y_n(t) = y_{n-1}(t_{n-1}) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left(\alpha f(t_{n-1}, y_{n-1}(t_{n-1})) + f(t_n, y_n(t_n))\right). \qquad (3.7)$$

4. Concluding Remarks and Future work

The present work deals with numerical treatment of fractional initial value problems with large domains. we showed that this problem can be solved iteratively in selected sub-domains. Derivation of explicit and implicit fractional methods is illustrated.

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